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A Simple Algorithm for Generating Random Variates with a Log-Concave Density

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Abstract — Zusammenfassung

A Simple Algorithm for Generating Random Variates with a Log-Concave Density. We present a short algorithm for generating random variates with log-concave density f on R and known mode in average number of operations independent of f. Included in this class are the normal, gamma, Weibull, beta and exponential power (all with shape parameters at least 1), logistic, hyperbolic secant and extreme value distributions. The algorithm merely requires the presence of a uniform [0, 1] random number generator and a subprogram for computing f. It can be implemented in about 10 lines of FORTRAN code.

Key words and phrases: Random variate generation, simulation, log concavity, inequalities.

Ein einfacher Algorithmus zur Erzeugung zufälliger Veränderlicher mit log-konkaver Dichte. Wir legen einen kurzen Algorithmus zur Erzeugung von Zufallsveränderlichen mit log-konkaver Dichte f auf R mit bekanntem Median-Wert vor. Die mittlere Anzahl der erforderlichen Operationen ist unabhängig von f. Die log-konkaven Dichtefunktionen beschreiben u. a. die Normal-, Gamma-, Weibull-, Beta-, Potenzexponential- (alle mit Formparameter mindestens 1), Perks- und Extremwert-Verteilung.

1. Introduction

This note is motivated by the need for developing algorithms for generating random variates from large families of densities. Often the density can be computed but not the distribution function, so that the inversion method is not applicable. When the family of densities is suitable restricted, a general algorithm is sometimes within reach. For example, in Devroye (1984), such a general algorithm is given for the class of all bounded monotonically decreasing densities with support [0, 1]. In this note, we assume that f is available for computation and that it is log-concave, i. e. $\log f$ is a concave function on its support. We also assume that a mode is known. Without loss of generality, we can assume that a mode is located at 0 (otherwise, apply a translation), and that $f(0)=1$ (otherwise, rescale).

The importance of this class is clear from the partial list of members:

(i) the *normal* density $(2\pi)^{-1/2}\exp\left(-\frac{x^2}{2}\right)$;

(ii) the *gamma* density $\dfrac{x^{a-1}e^{-x}}{\Gamma(a)}$, $x>0$, for $a\geq 1$;

(iii) the *Weibull* density $a x^{a-1}\exp(-x^a)$, $x>0$, for $a\geq 1$;

(iv) the *beta* density $\dfrac{x^{a-1}(1-x)^{b-1}}{B(a,b)}$, $0\leq x\leq 1$, for $a,b\geq 1$;

(v) the *exponential power density* (EPD) $\dfrac{\exp(-|x|^a)}{2\Gamma\left(1+\frac{1}{a}\right)}$, for $a\geq 1$;

(vi) the *Perks distribution* (Perks (1932)) with density of the form $c/(e^x+e^{-x}+a)$, for $a>-2$ (c is a normalization constant); for $a=2$, this yields the *logistic* density; for $a=0$, we obtain the *hyperbolic secant* distribution (Talacko (1956));

(vii) the *extreme value distribution* with density $\dfrac{k^k}{(k-1)!}\exp(-kx-ke^{-x})$ for $k=1,2,3,\ldots$ (Gumbel (1958)),

(viii) the *generalized inverse gaussian density* $c x^{a-1}\exp(-bx-b^*/x)$, $x\geq 0$, for $a\geq 1$, $b,b^*>0$ (Jorgensen, 1982, p. 2.3).

The uniform density on [0, 1] is a member of sub-class (iv), while the Laplace density is a member of the EPD family. The class of log-concave densities is closed under convolutions, i.e. the density of a sum of two random variables with log-concave densities is again log-concave (Ibragimov (1965); Lekkerkerker (1953)).

The algorithm for generating random variates from this general class of densities should be reasonably fast, although we should not expect the speed obtained by algorithms that are tailored to a specific density. The small size of the algorithm, when implemented, is also very important. It could be at the basis of built-in generators for large families of densities in microcomputers and pocket calculators.

Assume that the basic operations take a unit of time, i.e. the basic arithmetic operations, compare, move, exp, log, generate a uniform random variate, compute f. Then a good algorithm should use up on the average a number of time units that is uniformly bounded over the class of all log-concave densities. The algorithm given here does more: its average time is bounded and is independent of the density f within the given class when measured in terms of these fundamental "time units". One should note that it was not until 1974 that uniformly fast algorithms were found for the gamma family (Ahrens and Dieter (1974); see recent survey by Tadikamalla and Johnson (1981)).

2. Development of the Algorithm

The algorithm is based upon the rejection method. It uses the following inequalities in crucial places:

Inequality 1: *Let f be the log-concave on* $[0,\infty)$ with mode at 0 and $f(0)=1$. *Then* $f(x)\leq g(x)$ *where*

$$g(x)=\begin{cases}1, \ 0\le x\le 1,\\ \text{the unique solution } t<1 \text{ of } t=\exp(-x(1-t)), \ x>1.\end{cases} \tag{1}$$

The bound cannot be improved because g is the supremum over all f in the family.

Proof: We need only consider the case $x>1$. The essential observation is that among all log-concave densities with mode at 0 and $f(0)=1$, the one maximizing $f(x)$ is of the form determined by

$$\log f(u)=\begin{cases}-au, \ 0\le u\le x\\ -\infty, \ x<u\end{cases}$$

for some $a>0$. Thus, $f(u)=e^{-au}$, $0\le u\le x$. Here a is chosen for the sake of normalization; thus,

$$1=\frac{1-e^{-ax}}{a}.$$

Replace $1-a$ by t. This concludes the proof of inequality 1.

Inequality 2: *The function g given in* (1) *can be bounded by two sequences of functions* $y_n(x)$, $z_n(x)$ *for* $x>1$:

$$0=z_0(x)\le z_1(x)\le\ldots\le g(x)\le\ldots\le y_1(x)\le y_0(x)=\frac{1}{x}.$$

The sequences are defined recursively by

$$y_{n+1}(x)=\exp(-x(1-y_n(x)))$$

and

$$z_{n+1}(x)=\exp(-x(1-z_n(x))).$$

They converge for all $x\ge 1$: $\lim_n y_n(x)=\lim_n z_n(x)=g(x)$.

Furthermore,

$$g(x)\le y_1(x)=\exp(1-x), \ x\ge 1, \tag{2}$$

and

$$g(x)\le y_2(x)=\exp(-x+x e^{1-x}), \ x\ge 1. \tag{3}$$

Proof: Fix $x>1$. Consider the functions $f_1(u)=u$ and $f_2(u)=\exp(-x(1-u))$ for $0\le u\le 1$. We have $f_1(1)=f_2(1)=1$, $f_2'(1)=x>1=f_1'(1)$, $f_2'(0)=x e^{-x}<1=f_1'(0)$, f_2 is convex and increases from e^{-x} at $u=0$ to 1 at $u=1$. This shows that for $x>1$, there is exactly one solution in (0, 1) for $f_1(u)=f_2(u)$. It can be obtained by functional iteration: if we start with $z_0(x)=0$, and use $z_{n+1}(x)=f_2(z_n(x))$, the unique solution is approached from below in a monotone manner. If we start with $y_0(x)$ and $y_0(x)$ is guaranteed to be at least equal to the value of the solution, then the functional iteration $y_{n+1}(x)=f_2(y_n(x))$ can be used to approach the solution from above in a monotone way. As initial overestimate $y_0(x)$ we can take $y_0(x)=\frac{1}{x}$ because $f_2\left(\frac{1}{x}\right)\le f_1\left(\frac{1}{x}\right)$.

Let $c=f(m)$, and let f be log-concave on $[m,\infty)$ with mode at m. By Inequality 2, we have

$$\frac{1}{c} f\left(m+\frac{x}{c}\right) \le h(x)=\min(1, e^{1-x}). \qquad (4)$$

Inequality (4) allows us to use the rejection method in a straightforward way:

Algorithm LCU (*L*og-*C*oncave densities, *U*niform version.)

Step 0: [*Set-up*. To be done once for each density.] Compute $c \leftarrow f(m)$.

Step 1: Generate a random variate X with density proportional to h, and prepare for the acceptance/rejection step:

1. Generate U uniform on $[0,2]$, and V uniform on $[0,1]$.
2. **If** $U \le 1$ **then** $X \leftarrow U$, $T \leftarrow V$;
   else $X \leftarrow 1-\log(U-1)$, $T \leftarrow V(U-1)$.

Step 2: [*Acceptance/rejection step.*]

1. $X \leftarrow m+\dfrac{X}{c}$.
2. **If** $T \le \dfrac{1}{c} f(X)$ **then** exit
   else go to 1.

At the end of step 1, (X,T) is distributed as $(X, Vh(X))$ where X has density $\frac{h(x)}{2}$ on $[0,\infty)$ (this follows from the fact that the integral under h is exactly 2), and V is independent of X and uniformly distributed on $[0,1]$. Sinde the area under f is 1, we accept in step 2 with probability $\frac{1}{2}$, *independent of f*. The *average number of iterations is exactly* 2, for all log-concave densities on $[m,\infty)$ with mode at m.

3. Modifications and Extensions

If f is log-concave with mode at m, then

$$\frac{1}{c} f\left(m+\frac{x}{c}\right) \le h(|x|). \qquad (5)$$

The integral of $h(|x|)$ over R is 4. Thus, algorithm LCU with the appropriate modification $\left(\text{i.e., replace step 2.1 by: "Generate } W \text{ uniform on } [0,1]. \textit{ If } W \le 0 \textit{ then } X \leftarrow -X.\ X \leftarrow m+\frac{X}{c}\text{"}\right)$ executes steps 1 and 2 four times on the average. Thus, we pay rather heavily for the presence of two tails. A quick fix-up is not possible because of the fact that the sum of two log-concave functions is not necessarily log-concave. Thus, we could not "add" the left portion of f to the right portion suitably mirrored and apply the algorithm LCU to the sum. However, when f is *symmetric* about mode m, we can achieve the performance of the original one-sided version of LCU: just replace step 2.1 by the new step "Generate W uniform on $[0,1]$. *If* $W \le 0.5$ *then* $X \leftarrow -X$. $X \leftarrow m + \frac{X}{2c}$".

If exponential random variates can be generated very cheaply, and the computation of $\log f$ poses no time problems (in most examples, it is faster to compute $\log f$ than to compute f), then the following exponential version can be useful:

Algorithm LCE (Log-Concave densities, Exponential version.)

Step 0: [Set-up. To be done once for each density.] Compute $c \leftarrow f(m)$, $r \leftarrow \log c$.

Step 1: Generate U uniform on $[0,2]$, and E independent of U and exponential.
If $U \le 1$ **then** $X \leftarrow U$, $T \leftarrow -E$
else $X \leftarrow 1 + E^*$, $T \leftarrow -E - E^*$ (E^* is a new exponential random variate).

Step 2: **Case** f log-concave on $[m, \infty)$: $X \leftarrow m + \frac{X}{c}$
f log-concave on $(-\infty, \infty)$: generate W uniform on $[0,1]$;
if $W \le 0.5$ **then** $X \leftarrow -X$;
case f symmetric: $X \leftarrow m + \frac{X}{2c}$;
otherwise: $X \leftarrow m + \frac{X}{c}$.

If $T \le \log f(X) - r$ **then** exit
else go to 1.

Log-concave densities also occur in R^d, e.g. the multivariate normal density is log-concave. The closure under convolutions also holds in R^d (Davidovic et al. (1969)), and marginals of log-concave densities are again log-concave (Prekopa (1973)). Unfortunately, it is useless to try to look for a generalization of the present inequalities to R^d for $d \ge 2$ because the class of log-concave functions in R^d with mode at 0 and $f(0) = 1$ includes the class of functions

$$I_A(x)$$

where A is any convex set containing the origin, and I is the indicator function. Thus, the only universal upper bound over this class of functions is 1.

Other approaches could be based upon the combined use of f and the distribution function F (when this is easy to compute, say). Since log-concave functions are unimodal, we refer the reader to Devroye (1984) where general algorithms are given for unimodal densities when both F and f can be computed. The dominating function found here is rather crude, especially in the two-tailed case. Perhaps faster times can be obtained, at the expense of simplicity, when the derivative f' is given in a subprogram: this will allow one to obtain better dominating curves for the rejection method.

4. FORTRAN Programs

```
C     REAL FUNCTION GENLCE (RMODE, OLD, ALOGF)
C
C GENLCE PRODUCES RANDOM VARIATES WITH LOG-CONCAVE
C DENSITY F
C
C RMODE: POSITION OF THE MODE OF F
C OLD:   SET TO .TRUE. WHEN GENCLE WAS CALLED BEFORE
C               FOR THE SAME F
C               .FALSE. WHEN GENCLE IS CALLED FOR THE
C               FIRST TIME
C ALOGF: A FUNCTION SUBPROGRAM FOR COMPUTING LOG F
C GENLCE ASSUMES A UNIFORM [0,1] GENERATOR (UNI)
C                 AN EXPONENTIAL GENERATOR (REXP)
C                 A PROGRAM FOR COMPUTING LOG F (ALOGF)
C
      DATA CINV, R/2*0.0/
      IF (OLD) GO TO 1
        R=ALOGF (RMODE)
        CINV=EXP (-R)
1     X=2*UNI (0)
      T=-REXP (0)
      IF (X.LT.1.0) GO TO 2
      X=1.0+REXP (0)
      T=1.0-X+T
2     IF (UNI (0) .LT.0.5) X=-X
3     GENLCE=RMODE+CINV*X
      IF (T.GT.ALOGF (GENCLE)-R) GO TO 1
      RETURN
      END
```

In GENCLE, statement 2 should be replaced by "2 CONTINUE" when f is log-concave on $[m,\infty)$, and the definition of CINV should be replaced by "CINV=0.5*EXP(−R)" when f is log-concave on $(-\infty,\infty)$ and symmetric about m.

5. A Small Experiment

One small experiment was carried out on an AMDAHL V7 computer at McGill University for three one-parameter densities, the gamma, Weibull and EPD densities. One thousand random variates were generated for each density by two methods:

(i) A typical robust tailor-designed algorithm of comparable size. For the gamma distribution, we took Cheng's algorithm GB (Cheng, 1977). We should note that Best's algorithm XG (Best, 1978) is about equally fast and short, and that most other methods of comparable complexity are slower (see for example the

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(ii) Algorithm LCE (subprogram GENLCE). For the EPD distribution, the modification for symmetric distributions is implemented.

The average times were computed for parameter values $a = 1.5, 3.3, 9.9, 16.2, 99.9$. They do not include set-up times, i.e. we are not including the time needed to compute a mode (as a function of a), and we are not including the time needed to compute the value of f at this mode. However, the time spent passing parameters is not subtracted: three parameters are passed in GENLCE, one of which is the name of a subprogram; only one parameter is passed in GB. The slight variation in the time of GENLCE for the gamma and Weibull densities as a function of a is due to the variable times of several service subprograms (logarithm, exponent, power, logarithm of the gamma function) with respect to their arguments.

The figures in Table 1 show the difference between LCE and its modifications. For example, it is important to implement the modification for symmetric distributions if the density is known to be symmetric. For the EPD, the performance of LCE approaches that of the specialized algorithm.

Table 1. *Average times in microseconds per random variate*

Parameter a	1.5	3.3	9.9	16.2	99.9
SPECIAL ALGORITHMS					
Gamma	50.3	47.1	44.7	44.0	43.4
Weibull	37.1	37.2	37.2	36.8	37.4
EPD	83.7	87.4	88.5	88.2	88.0
ALGORITHM LCE (PROGRAM GEMLCE)					
Gamma	127.0	136.0	132.0	141.0	145.0
Weibull	174.0	203.0	238.0	225.0	222.0
EPD	101.0	99.4	99.3	98.8	98.5

One drawback of the given method is the requirement that $\log f$ or f must be computed. This often involves the computation of a complicated normalization factor. The contribution to the total time is not negligible when only a small number of random variates are needed for the same distribution. The problem can be avoided when f is given analytically, by using a subprogram for computing $\log \frac{f(x)}{f(m)}$. The acceptance/rejection step in the algorithm LCE would then read

If $T \leq \log \frac{f(X)}{f(m)}$ **then** exit
else go to 1.

This time-saving modification was not used in our experiment. It offers also numerical advantages because $f(x)/f(m)$ is "normalized": all its values are comparable to 1 for x near m. In some cases, there are even questions about the computability of $f(x)$, that are eliminated when one considers $f(x)/f(m)$. For example, for the generalized inverse gaussian distribution, the normalization constant c depends upon a Bessel function of the third kind evaluated at a. In $f(x)/f(m)$, the normalization factor is canceled.

6. The Optimal Rejection Algorithm

In this section we assume (without loss of generality) that the density f is normalized as in Inequality 1. The optimal rejection algorithm (in terms of expected number of iterations) would use the function g defined in (1) as the dominating curve. Although g is only defined implicitly, we can nevertheless generate random variates with density $g/\int g$:

Property 1: *If E_1, E_2, U, D are independent random variables where E_1, E_2 are exponentially distributed, U is uniformly distributed on $[0,1]$ and D is integer-valued with $P(D=j)=6/(\pi^2 j^2)$, $j\geq 1$, then*

$$(X,Y)=\left(U\,\frac{(E_1+E_2)/D}{1-\exp(-(E_1+E_2)/D)},\ \exp(-(E_1+E_2)/D)\right)$$

is uniformly distributed in $\{(x,y): x\geq 0,\ 0\leq y\leq g(x)\}$. In particular, X has density $g/\int g$, and Y is distributed as $Vg(X)$ where V is a uniform $[0,1]$ random variable independent of X.

Proof: We flip the axes around and note that the desired Y should have a density proportional to $-\log(y)/(1-y)$, $0\leq y\leq 1$, and that X should be distributed as $U(-\log(Y)/(1-Y))$ where U is independent of Y. By the transformation $y=\exp(-z)$, $Y=\exp(-Z)$, we see that Z has density proportional to

$$\frac{z\,e^{-z}}{1-e^{-z}}=\sum_{j=0}^{\infty} z\,e^{-(j+1)z}=\frac{\pi^2}{6}\left(\sum_{j=1}^{\infty}(j^2 z\,e^{-jz})\left(\frac{6}{\pi^2 j^2}\right)\right),\ z\geq 0,$$

i.e. Z is distributed as $(E_1+E_2)/D$ (since E_1+E_2 has density $z\,e^{-z}$, $z\geq 0$). Thus, the couple $(UZ/(1-\exp(-Z)),\ \exp(-Z))$ has the correct uniform distribution.

Algorithm:

Step 1: Generate the following independent random variates: U is uniformly distributed on $[0,1]$, E_1, E_2 are exponentially distributed, and D is integer-valued with $P(D=j)=6/(\pi^2 j^2)$, $j\geq 1$. Note that D can be obtained as follows:

Repeat Generate (U^*, V^*) uniformly in $[0,1]^2$.
  If $U^*\leq\frac{1}{2}$ **then** $D\leftarrow 1$
    else $D\leftarrow\lfloor 1/(2(1-U^*))\rfloor$.
Until $DV^*\geq 1$.
$Z\leftarrow(E_1+E_2)/D$, $Y\leftarrow\exp(-Z)$, $X\leftarrow UZ/(1-Y)$.

Step 2: **If** $Y \le f(X)$ **then** exit.
else go to 1.

In the proof of property 1, we have implicitly shown that $\int g = \pi^2/6 = 1.6433\ldots$, which is the expected number of iterations. If D is generated as suggested in the algorithm, then there is also some hidden rejection, since the expected number of iterations to obtain D is $12/\pi^2$.

7. The Mirror Principle

Consider now a normalized log-concave f with two tails $(m=0, f(0)=1)$. As we have seen, in this case the rejection algorithms LCU, LCE are not particularly efficient, the area under the dominating curve being 4. The expected number of iterations can be improved considerably by two observations suggested to the author by Richard Brent.

Inequality 3: *If f is a log-concave density with known mode $m=0$, and $F(0)=p$ is known (F is the distribution function for f), then, if $f(0)=1$,*

$$f(x) \le \begin{cases} \min(1, \exp(1-|x|/(1-p))), & x \ge 0, \\ \min(1, \exp(1-|x|/p)), & x < 0. \end{cases} \tag{6}$$

The area under the bounding curve in (6) is 2.

Proof: Note that $f(x)/(1-p)$ is a log-concave density on $(0, \infty)$, and that $f(x)/p$ is a log-concave density on $(-\infty, 0)$. Since $f(x(1-p))$ is log-concave on $(0, \infty)$, we have

$$f(x(1-p)) \le \min(1, \exp(1-x)), \quad x \ge 0.$$

Inequality (6) and the statement about the area now follow without work.

The details of the rejection algorithm based on (6) are left to the reader. Even if $p=F(m)$ is not known, we can reduce the number of iterations, based on

Inequality 4: *Let f be a log-concave density with mode at 0, and $f(0)=1$. Then, for $x \ge 0$,*

$$\begin{aligned} f(x)+f(-x) \le g(x) &= \sup_{p \in (0,1)} \left(\min\left(1, e^{1-\frac{x}{1-p}}\right) + \min\left(1, e^{1-\frac{x}{p}}\right)\right) \\ &= \begin{cases} 2, & 0 \le x \le 1/2, \\ 1+\exp(2-1/(1-x)), & 1/2 \le x \le 1, \\ \exp(1-x), & 1 \le x. \end{cases} \end{aligned} \tag{7}$$

Furthermore,

$$\int g = \frac{5}{2} + \frac{1}{4} \int_0^\infty \left(1+\frac{u}{2}\right)^{-2} e^{-u}\, du < \frac{5}{2} + \frac{1}{4} \int_0^\infty \frac{e^{-u}}{1+u}\, du = 2.6491\ldots.$$

If $g^* = g$ except on $\left(\frac{1}{2}, 1\right)$, where g^* is linear with $g^*\left(\frac{1}{2}\right)=2$, $g^*(1)=1$, then $g^* \ge g$, and

$$\int g^* = \frac{11}{4}.$$

Proof: The first inequality in (7) follows from (6) (consider p as $F(0)$). Thus, it suffices to establish the equality on the right-hand-side of (7). Let us write $g(x) = \sup_{0 \le p \le \frac{1}{2}} h_p(x)$ where

$$h_p(x) = \begin{cases} 2, \ x \le p, \\ 1 + \exp\left(1 - \frac{x}{p}\right), \ p \le x \le 1-p, \\ \exp\left(1 - \frac{x}{p}\right) + \exp\left(1 - \frac{x}{1-p}\right), \ 1-p \le x < \infty. \end{cases}$$

First, we observe that $g(x) \ge$ right-hand side of (7), because for $x \le \frac{1}{2}$, we have $h_{1/2}(x) = 2$, for $\frac{1}{2} \le x \le 1$, we have $h_{1-x}(x) = 1 + \exp(2 - 1/(1-x))$, and for $x \ge 1$, we have $h_0(x) = \exp(1-x)$. We now show that $g(x) \le$ right-hand-side of (7). Decompose h_p as $h_{p1} + h_{p2} + h_{p3}$ where $h_{p1} = h_p$ on $[0, p]$ and 0 elsewhere, $h_{p2} = h_p$ on $[p, 1-p]$ and 0 elsewhere, and $h_{p3} = h_p$ on $[1-p, \infty)$ and 0 elsewhere. Clearly, $h_{p1} \le g$, all $p \le \frac{1}{2}$, $x \ge 0$. Since $[p, 1-p]$ is a subinterval of $[0, 1]$, we have $h_{p2} \le g$, all $p \le \frac{1}{2}$, $x \ge 0$. Thus, $h_p \le g$ if $h_{p3} \le \exp(1-x)$, all $x \ge 1$, $p \le 1/2$. For this, it suffices in turn to show (8):

$$\exp\left(-\frac{1}{p}\right) + \exp\left(-\frac{1}{1-p}\right) \le e^{-1}, \ 0 \le p \le \frac{1}{2}, \tag{8}$$

because this would yield

$$e\left((e^{-\frac{1}{p}})^x + (e^{-\frac{1}{1-p}})^x\right) \le e(e^{-\frac{1}{p}} + e^{-\frac{1}{1-p}})^x \le e^{1-x}, \ x \ge 1.$$

Putting $u = (1-p)/p$, we have

$$(\exp(-1/p) + \exp(-1/(1-p)))e = \exp(-u) + \exp(-1/u).$$

The function $\exp(-u) + \exp(-1/u)$ can be shown to have equal maxima at $u = 0$ and $u \uparrow \infty$, and a minimum at $u = 1$, with maximal and minimal values 1 and $2/e$ respectively. The first integral ($\int g$) can be written as

$$\frac{5}{2} + e^2 \int_{1/2}^{1} \exp\left(-\frac{1}{1-x}\right) dx = \frac{5}{2} + \frac{1}{4} \int_0^\infty \left(1 + \frac{u}{2}\right)^{-2} e^{-u} du$$

(set $u = 1/(1-x) - 2$). The rest follows easily. In the computation, we made use of a formula for the exponential integral given in Abramowitz and Stegun (1970, p. 231). The statement about g^* follows directly from the convexity of the function h_{p2} on $\left[\frac{1}{2}, 1\right]$.

Based on Inequality 4, we can now give the corresponding algorithm (which requires on the average 2.75 iterations, and 5.5 evaluations of f, and should be used only when the number of uniform random variates per generated random variate is to be reduced):

Algorithm:

Step 1: Generate four independent uniform [0, 1] random variates, U, V, W, Z.
If $U \leq 4/11$ **then** $X \leftarrow W/2,\ T \leftarrow 2V$.
else if $U \leq 7/11$ **then** $X \leftarrow \frac{1}{2} + \frac{1}{2}\min(W, 2W^*),\ T \leftarrow V(1+2(1-X))$
(where W^* is another uniform [0, 1] random variate).
else $X \leftarrow 1 - \log(W),\ T \leftarrow VW$.

Step 2: **If** $T \leq f(X) + f(-X)$ **then if** $Z \leq f(X)/(f(X)+f(-X))$, **then** exit with X.
else exit with $-X$.
else go to 1.

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