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Upper and Lower Class Sequences for Minimal Uniform Spacings

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Summary. Let K_n be the k-th smallest spacing defined by $X_1, \ldots, X_n$, independent random variables uniformly distributed on $[0,1]$, where $k \geqq 1$ is a fixed integer. Let u_n be a sequence of positive numbers satisfying $u_n/n^2 \downarrow$. We show that

(i) $P(n^2 K_n \leqq u_n \text{ i.o.}) = 0(1)$ when $\sum_n u_n^k/n < \infty \; (=\infty)$.

(ii) If in addition $u_n \uparrow$, then $P(n^2 K_n \geqq u_n \text{ i.o.}) = 0(1)$ when

$$\sum_n (u_n^k/n) \exp(-u_n) < \infty \; (=\infty).$$

1. Introduction

In this paper we investigate the asymptotic behavior of the k-th smallest uniform spacing. Among other things, a complete characterization of upper and lower class sequences is obtained. The asymptotic behavior is similar in many respects to that of the minimum of independent uniformly distributed random variables. Let $X_1, \ldots, X_n$ be independent identically distributed uniform (0, 1) random variables with order statistics $0 < X_1(n) < \ldots < X_n(n) < 1$, and let u_n be any sequence of positive numbers. Geffroy (1958/1959) has shown that if $u_n/n\downarrow$, then

$$P(nX_1(n) \leqq u_n \text{ i.o.}) = \begin{matrix} 0 \\ 1 \end{matrix} \quad \text{when } \sum_1^\infty \frac{u_n}{n} \begin{matrix} < \infty \\ = \infty \end{matrix}.$$

Robbins and Siegmund (1972) have shown that if $u_n/n\downarrow$ and $u_n\uparrow$, then

$$P(nX_1(n) \geqq u_n \text{ i.o.}) = \begin{matrix} 0 \\ 1 \end{matrix} \quad \text{when } \sum_1^\infty \frac{u_n}{n} e^{-u_n} \begin{matrix} < \infty \\ = \infty \end{matrix}.$$

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The latter result is a slight variation of a similar result by Barndorff-Nielsen (1961): if $u_n/n\downarrow$ and $(1-u_n/n)^n\downarrow$, then

$$P(nX_1(n)\geqq u_n \text{ i.o.})=\begin{matrix}0\\1\end{matrix} \quad \text{when } \sum_3^\infty \frac{\log_2 n}{n}\left(1-\frac{u_n}{n}\right)^n \begin{matrix}<\infty\\=\infty\end{matrix}.$$

Here $\log_j$ denotes the j times iterated logarithm. Deheuvels (1974) gives another proof of essentially the same result.

The spacings are defined by

$$S_i(n)=X_i(n)-X_{i-1}(n), \qquad 1\leqq i\leqq n+1,$$

where $X_0(n)=0$ and $X_{n+1}(n)=1$ by convention. The order statistics are $S_{(1)}(n)<S_{(2)}(n)<\ldots<S_{(n+1)}(n)$. For fixed integer $k\leqq n+1$, we define $K_n=S_{(k)}(n)$. The purpose of this paper is to show that under very weak regularity conditions on the sequence u_n,

$$P(n^2K_n\leqq u_n \text{ i.o.})=\begin{matrix}0\\1\end{matrix} \quad \text{when } \sum_1^\infty \frac{u_n^k}{n} \begin{matrix}<\infty\\=\infty\end{matrix}, \tag{1.1}$$

and

$$P(n^2K_n\geqq u_n \text{ i.o.})=\begin{matrix}0\\1\end{matrix} \quad \text{when } \sum_1^\infty \frac{u_n^k}{n}e^{-u_n} \begin{matrix}<\infty\\=\infty\end{matrix}. \tag{1.2}$$

The asymptotic behavior of the maximal uniform spacing $(S_{(n+1)}(n))$ is treated in Devroye (1981). Section 2 contains some very simple lemmas, a derivation of the asymptotic distribution of K_n, and large deviation results for $P(n^2K_n\leqq u_n)$ and $P(n^2K_n\geqq u_n)$. Section 3 deals with (1.1), and Sect. 4 treats (1.2).

2. Asymptotic Distribution. Inequalities

Lemma 2.1. [Tail of the gamma distribution] (Devroye, 1981). *If X is gamma (n) distributed and $u>0$, then*

$$P\left(\frac{X}{n}-1\geqq u\right)\leqq \exp(-nu^2(1-u)/2)$$

and

$$P\left(\frac{X}{n}-1\leqq -u\right)\leqq \exp(-nu^2/2).$$

Lemma 2.2. [Property of spacings] (see, e.g., Pyke (1965, p. 403)). *$(S_1(n),\ldots,S_{n+1}(n))$ is distributed as $(E_1/T,\ldots,E_{n+1}/T)$ where $T=\sum_{i=1}^{n+1}E_i$ and $E_1,\ldots,E_{n+1}$ are independent exponentially distributed random variables.*

Theorem 2.1. [Limit distribution of K_n]. *$n^2K_n \xrightarrow{\mathscr{L}} E_1+\ldots+E_k$, where $E_1,\ldots,E_k$ are independent exponentially distributed random variables. In other words, n^2K_n has a gamma (k) limit distribution. (The case $k=1$ goes back to Levy (1939).)*

Proof of Theorem 2.1. Let $0<u<1$, $x>0$. Then

$$\begin{aligned} P(n^2 K_n \leqq x) &= P\left(E_{(k)} \leqq \frac{x}{n^2} \sum_{i=1}^{n+1} E_i\right) \\ &\leqq P\left(E_{(k)} \leqq \frac{x}{n}(1+u)\right) + P\left(\frac{1}{n} \sum_{i=1}^{n+1} E_i \geqq 1+u\right) \\ &\geqq P\left(E_{(k)} \leqq \frac{x}{n}(1-u)\right) - P\left(\frac{1}{n} \sum_{i=1}^{n+1} E_i \leqq 1-u\right), \end{aligned} \tag{2.1}$$

where $E_1, \ldots, E_{n+1}$ are independent exponential random variables, and $E_{(k)}$ is the k-th smallest among the E_i's. For fixed $u \in (0,1)$, Lemma 2.1 implies that the last terms in both the upper and lower bounds in (2.1) are not greater than $\exp(-a n)$ for some $a(u)>0$. Let us now look at the probability $P(E_{(k)} \leqq v/n)$ for $v>0$. If Y is a binomial (n,p) random variable and Z is a Poisson (v) random variable, where $p=1-\exp\left(-\frac{v}{n}\right)$, then $P(E_{(k)} \leqq v/n) = P(Y \geqq k) \to P(Z \geqq k)$ because $n p \to v$ as $n \to \infty$. Thus,

$$\lim_{n\to\infty} P\left(E_{(k)} \leqq \frac{v}{n}\right) = 1 - \sum_{j=0}^{k-1} \frac{v^j}{j!} e^{-v}.$$

The right-hand side of the last expression is continuous in v. Replace v once by $x(1+u)$ and once by $x(1-u)$ and let $u \downarrow 0$. By (2.1) we may then conclude that

$$\lim_{n\to\infty} P(n^2 K_n \leqq x) = 1 - \sum_{j=0}^{k-1} \frac{x^j}{j!} e^{-x}, \qquad x>0.$$

This concludes the proof of the theorem.

Lemma 2.3. [Binomial tail inequalities]. *Let X be a binomial (n,p) random variable, and let k be a positive integer not exceeding n. Then*

$$\left(1-\frac{k}{n}\right)^k (1-n p) \leqq \frac{P(X \geqq k)}{\frac{(n p)^k}{k!}} \leqq \frac{e^{pk}}{1-n p e^p} \tag{2.2}$$

where the upper bound is valid for $n p e^p < 1$. Also,

$$\left(1-\frac{k}{n}\right)^{k-1} \exp\left(-\frac{n p^2}{2(1-p)}\right) \leqq \frac{P(X<k)}{e^{-np}\frac{(n p)^{k-1}}{(k-1)!}} \leqq e^{p(k-1)}\left(1+\frac{(k-1)!}{n p-1}\right) \tag{2.3}$$

where the upper bound is valid for $n p>1$. Finally,

$$P(X-n p>n \varepsilon) \leqq \left(\frac{p e}{\varepsilon}\right)^{n\varepsilon}, \qquad \varepsilon \in [p, 1-p). \tag{2.4}$$

Proof of Lemma 2.3. The upper bounds in (2.2) and (2.3) follow from the following inequalities:

$$P(X \geqq k) = \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j} \leqq \sum_{j=k}^{n} \frac{(n\,p)^j}{j\,!} e^{-p(n-j)}$$

$$\leqq \frac{(n\,p)^k}{k\,!} \mathrm{e}^{-np} \mathrm{e}^{pk} \sum_{j=0}^{\infty} (\mathrm{n\,p\,e}^p)^j;$$

$$P(X<k) \leqq \sum_{j=0}^{k-1} \frac{(n\,p)^j}{j\,!} e^{-p(n-j)} \leqq e^{p(k-1)} e^{-np} \frac{(n\,p)^{k-1}}{(k-1)\,!} \left(1 + \sum_{j=1}^{\infty} \frac{(k-1)\,!}{(n\,p)^j}\right).$$

The lower bounds in (2.2) and (2.3) follow from

$$P(X \geqq k) \geqq \binom{n}{k} p^k (1-p)^{n-k} \geqq \frac{(n-k)^k}{k!} p^k (1-p)^{n-k}$$

$$\geqq \frac{(np)^k}{k!} \left(1 - \frac{k}{n}\right)^k (1-np)$$

and

$$P(X<k) \geqq \binom{n}{k-1} p^{k-1} (1-p)^{n-k+1} \geqq \frac{(n-k)^{k-1}}{(k-1)!} p^{k-1} (1-p)^n$$

$$\geqq \frac{(np)^{k-1}}{(k-1)!} \left(1 - \frac{k}{n}\right)^{k-1} \exp\left(-np - \frac{np^2}{2(1-p)}\right)$$

where we used the inequality $\log(1-u) \geqq -u - u^2//(2(1-u))$, $1 > u > 0$.

Finally, we recall Okamoto's inequality (Okamoto, 1958) obtained easily by Chernoff's technique (Chernoff, 1952):

$$P(X - np \geqq n\varepsilon) \leqq \left(\left(\frac{p}{p+\varepsilon}\right)^{p+\varepsilon} \left(\frac{1-p}{1-p-\varepsilon}\right)^{1-p-\varepsilon}\right)^n, \qquad \varepsilon \in (0, 1-p).$$

When $\varepsilon \geqq p$, the expression on the right-hand side can be further bounded from above by $((p/\varepsilon)^\varepsilon e^\varepsilon)^n$.

Theorem 2.2. [Large deviation inequalities for K_n]. *Let $u_n \to 0$ be a positive number sequence satisfying* $\log(1/u_n) = o(n)$, *then*

$$P(n^2 K_n \leqq u_n) \sim u_n^k / k\,!\,. \tag{2.5}$$

Let $u_n \to \infty$ be a positive number sequence satisfying $u_n = o(n^{1/3})$. Then

$$P(n^2 K_n \geqq u_n) \sim u_n^{k-1} e^{-u_n} / (k-1)\,!\,. \tag{2.6}$$

Proof of Theorem 2.2. Let $E_1, \ldots, E_{n+1}$, T be as in Lemma 2.2, and let $E_{(k)}$ be as in (2.1). Define $b_n = \sqrt{3k \log(1/u_n)/n}$ and $c_n = \sqrt{3u_n/n}$. To prove (2.5), note that by (2.1),

$$P\left(E_{(k)} \leqq \frac{u_n}{n}(1+b_n)\right) + P\left(\frac{T}{n} \geqq 1 + b_n\right) \geqq P(n^2 K_n \leqq u_n)$$

$$\geqq P\left(E_{(k)} \leqq \frac{u_n}{n}(1-b_n)\right) - P\left(\frac{T}{n} \leqq 1 - b_n\right). \tag{2.7}$$

Note that $P\left(E_{(k)} \leqq \frac{u_n}{n}(1 \pm b_n)\right) = P(X \geqq k)$ where X is a binomial (n, p) random variable, and $p = 1 - \exp(-u_n(1 \pm b_n)/n)$. Note that as $n \to \infty$, $u_n \to 0$, $p \to 0$, $np \sim u_n$, $b_n \to 0$. Thus, by inequality (2.2), we conclude that

$$P\left(E_{(k)} \leqq \frac{u_n}{n}(1 \pm b_n)\right) \sim (np)^k/k! \sim u_n^k/k!.$$

Since by Lemma 2.1

$$P\left(\left|\frac{T}{n} - 1\right| \geqq b_n\right) \leqq \exp\left(-\mathrm{n}b_n^2\left(\frac{1}{2} + o(1)\right)\right) = u_n^{(3k + o(1))/2} = o(u_n^k),$$

we see that (2.5) follows from (2.7). To prove (2.6) we proceed similarly:

$$\begin{aligned} P\left(E_{(k)} \geqq \frac{u_n}{n}(1 - c_n)\right) + P\left(\frac{T}{n} \leqq 1 - c_n\right) &\geqq P(n^2 K_n \geqq u_n) \\ &\geqq P\left(E_{(k)} \geqq \frac{u_n}{n}(1 + c_n)\right) - P\left(\frac{T}{n} \geqq 1 + c_n\right). \end{aligned} \tag{2.8}$$

Note that $P\left(E_{(k)} \geqq \frac{u_n}{n}(1 \pm c_n)\right) = P(X < k)$ where X is a binomial (n, p) random variable, and $p = 1 - \exp(-u_n(1 \pm c_n)/n)$. As $n \to \infty$, we have $u_n \to \infty$, $c_n \to 0$, $p \to 0$, $u_n^2/n \to 0$, $u_n c_n \to 0$, $np \sim u_n$ and $\exp(-np) \sim \exp(-u_n)$. By inequality (2.3), we conclude that

$$P\left(E_{(k)} \geqq \frac{u_n}{n}(1 \pm c_n)\right) \sim e^{-np}\frac{(np)^{k-1}}{(k-1)!} \sim e^{-u_n}\frac{u_n^{k-1}}{(k-1)!}.$$

By Lemma 2.1,

$$P\left(\left|\frac{T}{n} - 1\right| \geqq c_n\right) \leqq \exp(-nc_n^2(\tfrac{1}{2} + o(1))) = \exp(-\tfrac{3}{2}u_n(1 + o(1))) = o(e^{-u_n}u_n^{k-1}).$$

Thus, (2.6) follows from (2.8).

Lemma 2.4. [Inequalities for uniform spacings]. *Let $\mathscr{F}_n$ be the σ-algebra generated by $X_1, \ldots, X_n$, let $X_0 = 0$, $X_{-1} = 1$, $\delta > 0$ and $A \in \mathscr{F}_n$. Then*

$$P\left(\bigcup_{j=-1}^{n}[|X_j - X_{n+1}| < \delta], A\right) \leqq 2\delta(n+1)P(A).$$

If $S(x, n)$ is the length of the spacing among $S_i(n)$, $1 \leqq i \leqq n+1$, covering the point $x \in [0, 1]$, then

$$P(S(X_{n+1}, n) < \delta, A) \leqq P\left(\sum_{i=1}^{n+1} S_i(n) I_{[S_i(n) < \delta]} \geqq \theta\right) + \theta P(A),$$

for all $\theta \in (0, 1)$, where I is the indicator function of an event.

Proof of Lemma 2.4. Immediate.

Lemma 2.5. [Conditional inequalities for K_n]. *Let u_n be a positive number sequence satisfying $u_n/n^2\downarrow$, and let $n>m$ be two positive integers. Then*

$$P(n^2K_n\geqq u_n|m^2K_m\geqq u_m)\leqq P(X<k)$$

and

$$P(n^2K_n\geqq u_n|m^2K_m\geqq u_m)\leqq P(n^2K_{n-m}\geqq u_n)$$

where X is a binomial $(n-m,p)$ random variable and $p=\max(0,\min(1,(m+1-k)u_n/n^2))$.

Proof of Lemma 2.5. The second inequality follows from the fact that if K^*_{n-m} is the k-th smallest spacing defined by $X_{m+1},\dots,X_n$, then

$$[K_n\geqq u_n/n^2, K_m\geqq u_m/m^2]\subseteq[K^*_{n-m}\geqq u_n/n^2, K_m\geqq u_m/m^2],$$

and thus, by independence,

$$P(K_n\geqq u_n/n^2, K_m\geqq u_m/m^2)\leqq P(K^*_{n-m}\geqq u_n/n^2)\,P(K_m\geqq u_m/m^2).$$

The first inequality follows by a similar argument. Let $X_i(m)$, $1\leqq i\leqq m$ be the order statistics of $X_1,\dots,X_n$, and let $X_0(m)=0$, $X_{m+1}(m)=1$. Let A_m be the union of all sets of the form $[X_i(m), X_i(m)+u_n/n^2]$, $0\leqq i\leqq m$, for which $[X_i(m), X_{i+1}(m)]$ is not among the k smallest intervals (spacings). It is clear that the length of A_m is $(m+1-k)u_n/n^2$ for $m+1\geqq k$, on the set $[K_m\geqq u_m/m^2]$. Let N^* be the number of points among $X_{m+1},\dots,X_n$ that fall in A_m. Clearly, the first inequality now follows from the implication

$$[K_n\geqq u_n/n^2, K_m\geqq u_m/m^2]\subseteq[N^*<k, K_m\geqq u_m/m^2].$$

3. Lower Class Sequences

Theorem 3.1. *If u_n is a positive number sequence satisfying $u_n/n^2\downarrow$, then*

$$P(n^2K_n\leqq u_n \text{ i.o.})=\begin{matrix}0\\1\end{matrix}\quad\text{when}\quad\sum_1^\infty\frac{u_n^k}{n}\begin{matrix}<\infty\\=\infty\end{matrix}.$$

Remark 3.1. If u_n is a positive number sequence satisfying $u_n/n^2\downarrow$ and $\sum u_n^k/n<\infty$, then necessarily $u_n\to 0$. Indeed, if this is not true, then we can find $a>0$ and $n_1<n_2<\dots$ with $(n_i/n_{i+1})^{2k}\leqq 1/2$, $u_{n_i}\geqq a$. Now,

$$\begin{aligned}\sum_n u_n^k/n&=\sum_i\sum_{j=n_i+1}^{n_{i+1}}u_j^k/j\geqq\sum_i(u_{n_{i+1}}^k/n_{i+1}^{2k})\sum_{j=n_i+1}^{n_{i+1}}j^{2k-1}\\&\geqq\sum_i(u_{n_{i+1}}^k/n_{i+1}^{2k})(n_{i+1}^{2k}-n_i^{2k})/(2k)\geqq(1/(4k))\sum_i a=\infty,\end{aligned}$$

which leads to a contradiction.

Remark 3.2. In the proof that $P(n^2K_n\leqq u_n \text{ i.o.})=0$, we can without loss of generality assume that $nu_n\to\infty$ (and thus that $\log(1/u_n)=o(n)$). Assume that we have proved the claim with these additional assumptions, and that we are given a sequence u_n that merely satisfies $u_n/n^2\downarrow$ and $\sum u_n^k/n<\infty$. Define $u_n^* =\max(u_n, 1/\sqrt{n})$, and check that $u_n^*/n^2\downarrow$, $u_n^*\to 0$ (Remark 3.1) and $nu_n^*\to\infty$. Also, $\sum u_n^{*k}/n\leqq\sum 1/n^{1+k/2}+\sum u_n^k/n<\infty$. Thus, by our assumption, $P(n^2K_n\leqq u_n^*$ i.o.$)=0$. Therefore, it is certainly true that $P(n^2K_n\leqq u_n$ i.o.$)=0$.

Remark 3.3. In the proof that $P(n^2K_n\leqq u_n$ i.o.$)=1$, we can without loss of generality assume that $\log(1/u_n)=o(n)$. Indeed, assume that we have proved the claim under theses additional assumptions, and that we are given a positive number sequence u_n merely satisfying $u_n/n^2\downarrow$ and $\sum u_n^k/n=\infty$. Define $u_n^* =\max(u_n, 1/\sqrt{n})$, and verify that $u_n^*/n^2\downarrow$, and that $\log(1/u_n^*)=o(n)$. Also, $\sum u_n^{*k}/n\geqq\sum u_n^k/n=\infty$. Thus, by our assumption, $P(n^2K_n\leqq u_n^*$ i.o.$)=1$. But clearly,

$$[n^2K_n\leqq u_n \text{ i.o.}]\supseteq[n^2K_n\leqq u_n^* \text{ i.o.}]\cap[n^2K_n\leqq 1/\sqrt{n} \text{ f.o.}]$$

and $P(n^2K_n\leqq 1/\sqrt{n}$ f.o.$)=1$ (by the first part of the theorem), so that $P(n^2K_n\leqq u_n$ i.o.$)=1$.

Remark 3.4. The condition that u_n/n^2 is monotone $\downarrow$ may of course be relaxed to the condition that u_n/n^2 is eventually monotone $\downarrow$. Theorem 3.1 also implies the following: when $p\geqq 1$ is an integer, then

$$P(n^2K_n\leqq(\log n\log_2 n\ldots\log_p^{1+\delta} n)^{-1/k} \text{ i.o.})=\begin{matrix}0\\1\end{matrix} \text{ when } \begin{matrix}\delta>0\\ \delta\leqq 0\end{matrix}.$$

Proof of Theorem 3.1. First we prove that $P(n^2K_n\leqq u_n$ i.o.$)=0$ under the assumptions that $u_n/n^2\downarrow$, $\sum u_n^k/n<\infty$, $u_n\to 0$ (Remark 3.1) and $nu_n\to\infty$ (which implies $\log(1/u_n)=o(n)$, see Remark 3.2). We will apply a modified version of the Borel-Cantelli lemma, i.e. if A_n is a sequence of events, then $P(A_n$ i.o.$)=0$ when $P(A_n)\to 0$ and $\sum P(A_n^c A_{n+1})<\infty$ (see e.g. Barndorff-Nielsen, 1961). Let $A_n =[n^2K_n\leqq u_n]$. By Theorem 2.2, we have

$$P(A_n)\sim u_n^k/k!\to 0.$$

Let K_n' and K_n'' be the $(k-1)st$ and $(k-2)d$ smallest spacings among $S_i(n)$, $1\leqq i\leqq n+1$. By convention, the 0th smallest spacing is 0, and in the case $k=1$, any term involving K_n'' has to be ignored. Applying Lemmas 2.4 and 2.5 we have, using some notation from Lemma 2.4,

$$\begin{aligned}
P(A_n^cA_{n+1})&\leqq P\left(\bigcup_{j=-1}^{n}\left[|X_j-X_{n+1}|\leqq\frac{u_{n+1}}{(n+1)^2}\right], K_n'\leqq\frac{u_{n+1}}{(n+1)^2}\right)\\
&\quad+P(S(X_{n+1},n)\leqq 2u_{n+1}/(n+1)^2; K_n''\leqq u_{n+1}/(n+1)^2)\\
&\leqq 2\frac{u_{n+1}}{(n+1)^2}(n+1)P\left(K_n'\leqq\frac{u_{n+1}}{(n+1)^2}\right)+9\frac{u_{n+1}^2}{n+1}P\left(K_n''\leqq\frac{u_{n+1}}{(n+1)^2}\right)\\
&\quad+P\left(\sum_{i=1}^{n+1}S_i(n)\,I_{[S_i(n)\leqq 2u_{n+1}/(n+1)^2]}\geqq 9u_{n+1}^2/(n+1)\right)\\
&=B_{n1}+B_{n2}+B_{n3}.
\end{aligned}$$

Clearly,

$$B_{n1} \leqq 2(n+1)\frac{u_n}{n^2} P\left(K'_n \leqq \frac{u_n}{n^2}\right) \sim 2\frac{u_n}{n}\frac{u_n^{k-1}}{(k-1)!},$$

and, for $k \geqq 2$,

$$B_{n2} \leqq 9\frac{u_n^2}{n^4}(n+1)^3 P\left(K''_n \leqq \frac{u_n}{n^2}\right) \sim 9\frac{u_n^2}{n}\frac{u_n^{k-2}}{(k-2)!},$$

so that $B_{nl}+B_{n2}$ is summable whenever $\sum u_n^k/n < \infty$. (The term B_{n2} is to be ignored when $k=1$.)
Also,

$$\begin{aligned} B_{n3} &\leqq P\left(\sum_{i=1}^{n+1} I_{[S_i(n) \leqq 2u_{n+1}/(n+1)^2]} \frac{2u_{n+1}}{(n+1)^2} \geqq \frac{9u_{n+1}^2}{n+1}\right) \\ &= P\left(\sum_{i=1}^{n+1} I_{[S_i(n) \leqq 2u_{n+1}/(n+1)^2]} \geqq \tfrac{9}{2}(n+1)\, u_{n+1}\right) \\ &\leqq P\left(\sum_{i=1}^{n+1} E_i \geqq \tfrac{3}{2}(n+1)\right) + P\left(\sum_{i=1}^{n+1} I_{[E_i \leqq 3u_{n+1}/(n+1)]} \geqq \tfrac{9}{2}(n+1)\, u_{n+1}\right) \\ &\leqq \exp(-n/16) + P(Z \geqq \tfrac{9}{2}(n+1)\, u_{n+1}) \end{aligned}$$

where we used Lemma 2.2 and Lemma 2.1, $E_1, \ldots, E_{n+1}$ are independent exponentially distributed random variables, and Z is a binomial $(n+1, p)$ random variable with $p = 1 - \exp(-3u_{n+1}/(n+1))$. The summability of B_{n3} follows from the summability of $P(Z \geqq \frac{9}{2}(n+1)\, u_{n+1})$. Since $p \leqq p' = 3u_{n+1}/(n+1) \to 0$, we may replace Z by a binomial $(n+1, p')$ random variable Z'. By inequality (2.4), we obtain the upper bound

$$\left(\frac{p'e}{\varepsilon}\right)^{n\varepsilon}, \text{ valid for } p' \leqq \varepsilon < 1-p',$$

where

$$\varepsilon = (\tfrac{9}{2}u_{n+1}(n+1) - (n+1)p')/(n+1) = u_{n+1}\left(\frac{9}{2} - \frac{3}{n+1}\right) \geqq 3u_{n+1}.$$

For n large, the upper bound is valid and it is further bounded from above by

$$\left(\frac{e}{n+1}\right)^{n\varepsilon} \leqq \left(\frac{e}{n+1}\right)^{3nu_{n+1}},$$

and these terms are summable in n because $nu_n \to \infty$.

Now we prove that $P(n^2 K_n \leqq u_n \text{ i.o.}) = 1$ under the assumptions that $u_n/n^2 \downarrow$, $\sum u_n^k/n = \infty$ and $\log(1/u_n) = o(n)$ (Remark 3.2). The proof is based upon the following implication:

$$[n^2 K_n \leqq u_n \text{ i.o.}] \supseteq [T_j \leqq u_{n_{j+1}}/n_{j+1}^2 \text{ i.o.}]$$

where T_j is the k-th smallest spacing defined by $X_{n_j+1}, \ldots, X_{n_{j+1}}$ on $[0, 1]$, and n_j is a carefully chosen strictly increasing subsequence of the integers. Since the events $A_j=[T_j \leqq u_{n_{j+1}}/n_{j+1}^2]$ are independent, we need only show that $\sum P(A_j) = \infty$. We distinguish between two cases: 1) There exists a subsequence n_j such that along this subsequence $\liminf u_{n_j} \geqq a > 0$; and 2) $\lim u_n = 0$. In the first case, we can assume without loss of generality that $n_{j+1} \geqq 2n_j$, all j. If F is the gamma (k) distribution function, then by Theorem 2.1,

$$P(A_j)=P(K_{n_{j+1}-n_j} \leqq u_{n_{j+1}}/n_{j+1}^2) \geqq P\left(K_{n_{j+1}/2}\left(\frac{n_{j+1}}{2}\right)^2 \leqq \frac{u_{n_{j+1}}}{4}\right)$$
$$\geqq (1+o(1))\, F\left(\frac{a}{4}\right),$$

and thus $\sum P(A_j)=\infty$. We may therefore assume that $u_n \to 0$. Let $n_j=2^j$, and note that by Theorem 2.2, since $\log(1/u_{2n})=o(n)$,

$$P(A_j)=P(K_{2^j} \leqq u_{2^{j+1}}/2^{2(j+1)}) = P((2^j)^2 K_{2^j} \leqq u_{2^{j+1}}/4) \sim (u_{2^{j+1}}/4)^k/k!\,.$$

Since $u_n/n^2 \downarrow$, we have

$$\sum_{j=1}^{\infty} u_{2^j}^k \geqq \sum_{j=1}^{\infty} (1/2^j) \sum_{i=2^j}^{2^{j+1}-1} (u_i^k/4) \geqq \sum_{j=1}^{\infty} \frac{1}{4} \sum_{i=2^j}^{2^{j+1}-1} \frac{u_i^k}{2i} = \sum_{i=2}^{\infty} \frac{u_i}{8i} = \infty.$$

Thus, $\sum P(A_j)=\infty$. This concludes the proof of Theorem 3.1.

4. Upper Class Sequences

Theorem 4.1. *If u_n is a sequence of positive numbers such that $u_n \to \infty$, $u_n/n^2 \downarrow$, and*

$$\sum_n \frac{u_n^k}{n} e^{-u_n} < \infty, \tag{4.1}$$

then

$$P(n^2 K_n \geqq u_n \text{ i.o.}) = 0. \tag{4.2}$$

Corollary 4.1. *If u_n is a sequence of positive numbers such that eventually $u_n \uparrow$ and $u_n/n^2 \downarrow$, then* (4.1) *implies* (4.2). (*Note that $u_n \uparrow$ and* (4.1) *imply that $u_n \to \infty$.*)

Proof of Theorem 4.1. We may without loss of generality assume that $u_n = o(n^{1/3})$. For otherwise, u_n can be replaced by $u_n' = \min(u_n, n^{1/4})$. Clearly, $u_n' = o(n^{1/3})$. Also, if $u_n \to \infty$, so does u_n', and if $u_n/n^2 \downarrow$, so is u_n'. Finally, when u_n satisfies (4.1), then

$$\sum_n (u_n'^k/n)\, e^{-u_n'} \leqq \sum_n (n^{k/4}/n)\, e^{-n^{1/4}} + \sum_n (u_n^k/n)\, e^{-u_n} < \infty.$$

If $P(n^2 K_n \geqq u_n' \text{ i.o.})=0$, then clearly, $P(n^2 K_n \geqq u_n \text{ i.o.})=0$.

We will use the fact that if A_n is a sequence of events satisfying $P(A_n) \to 0$ and $\sum P(A_n A_{n+1}^c) < \infty$, then $P(A_n \text{ i.o.})=0$ (see for example, Devroye, 1981). Let $A_n=[n^2 K_n \geqq u_n]$. By Theorem 2.2,

$$P(A_n) \sim e^{-u_n} u_n^{k-1}/(k-1)! \to 0.$$

Also, by Lemma 2.4,

$$P(A_n A_{n+1}^c) \leqq P\left(\bigcup_{j=-1}^{n}\left[|X_j - X_{n+1}| \leqq \frac{u_{n+1}}{(n+1)^2}\right], A_n\right)$$
$$\leqq 2\frac{u_{n+1}}{(n+1)^2}(n+1)\,P(A_n) \leqq 2\frac{u_n}{n^2}(n+1)\,P(A_n) \sim \frac{2}{(k-1)!}\frac{u_n^k}{n}e^{-u_n},$$

which is summable in n.

Theorem 4.2. *If u_n is a sequence of positive numbers satisfying $u_n\uparrow$, $u_n/n^2\downarrow$ and*

$$\sum_n \frac{u_n^k}{n} e^{-u_n} = \infty, \tag{4.3}$$

then

$$P(n^2 K_n \geqq u_n \text{ i.o.}) = 1. \tag{4.4}$$

Theorem 4.3. *If u_n is a sequence of positive numbers such that eventually $u_n\uparrow$ and $u_n/n^2\downarrow$, then*

$$P(n^2 K_n \geqq u_n \text{ i.o.}) = \begin{matrix}0\\1\end{matrix} \quad \text{when} \quad \sum_n \frac{u_n^k}{n} e^{-u_n} \begin{matrix}<\infty\\=\infty\end{matrix}.$$

When $p \geqq 4$ is integer, then

$$P(n^2 K_n \geqq \log_2 n + (k+1)\log_3 n + \ldots + (1+\delta)\log_p n \text{ i.o.}) = \begin{matrix}0\\1\end{matrix} \quad \textit{when} \quad \begin{matrix}\delta>0\\ \delta\leqq 0\end{matrix}.$$

Theorem 4.3 follows from Theorem 4.2 and corollary 4.1. Theorem 4.2 is proved along the lines of the proof of Theorem 2 in Robbins and Siegmund (1972). In this proof, the following sequence of integers is very important:

$$n_j = [\exp(aj/\log j)], \; a > 0.$$

Here $[\,.\,]$ denotes the integral part operator. We first offer the following lemma.

Lemma 4.1. *If u_n is a sequence of positive numbers satisfying $u_n/n^2\downarrow$ and $u_n\uparrow$, then* (4.3) *is equivalent to*

$$\sum_j u_{n_j}^{k-1} \exp(-u_{n_j}) = \infty, \qquad \textit{all } a > 0. \tag{4.5}$$

Proof of Lemma 4.1.

Step 1. We first show that for all $a > 0$, (4.3) is equivalent to

$$\sum_j u_{n_j}^{k-1} \exp(-u_{n_j}) \cdot u_{n_j}/\log_2 n_j = \infty. \tag{4.6}$$

We note first that the monotonicity conditions on u_n imply that $u_n^k e^{-u_n}/n = (u_n/n^2)^{1/2}\, u_n^{k-1/2} e^{-u_n}$ is $\downarrow$ for $u_n \geqq k - \frac{1}{2}$. Thus, for $u_n \uparrow U < \infty$, the equivalence of (4.3) and (4.6) is trivial. If $u_n \uparrow \infty$, we observe that $\log_2 n_j \sim \log j$, and that

$$(1 - n_j/n_{j+1}) \sim 1 - \exp(-a/\log j) \sim a/\log j \sim (n_{j+1}/n_j - 1).$$

Thus, for some large integer J, we have

$$\sum_n \frac{u_n^k}{n} e^{-u_n} \leqq J + \sum_{j \geqq J} (u_{n_j}^k/n_j)\, e^{-u_{n_j}}(n_j - n_{j-1})$$
$$\leqq 2 \sum_{j \geqq J} u_{n_j}^{k-1} e^{-u_{n_j}}(u_{n_j}/\log_2 n_j) + J,$$

and

$$\sum_n \frac{u_n^k}{n} e^{-u_n} \geqq \sum_{j \geqq J} (u_{n_j}^k/n_j)\, e^{-u_{n_j}}(n_{j+1} - n_j)$$
$$\geqq \tfrac{1}{2} \sum_{j \geqq J} u_{n_j}^{k-1} e^{-u_{n_j}}(u_{n_j}/\log_2 n_j).$$

Step 2. We may assume, without loss of generality, that $u_n \leqq 2\log_2 n$. If u_n does not satisfy $u_n \leqq 2\log_2 n$, then define $u'_n = \min(u_n, 2\log_2 n)$, and note that $u'_n \uparrow$, $u'_n/n^2 \downarrow$ when $u_n \uparrow$, $u_n/n^2 \downarrow$ respectively. It is also easy to check that (4.3) diverges for u_n if and only if it diverges for u'_n, and that (4.5) holds for u_n if and only if it holds for u'_n. For example, for (4.3) this follows from

$$\sum_n \left| \frac{u_n^k}{n} e^{-u_n} - \frac{u_n'^k}{n} e^{-u'_n} \right| \leqq \sum_{u_n > 2\log_2 n} \frac{u_n^k}{n} e^{-u_n} + \frac{(2\log_2 n)^k}{n(\log n)^2}$$
$$\leqq \sum_{n:\, 2\log_2 n \leqq k} 1 + \sum_n (2\log_2 n)^k/(n \log^2 n) < \infty.$$

Here we used the fact that the function $u^k e^{-u}$ is unimodal with peak at $u = k$. Since $u'_{n_j} \leqq 2\log_2 n_j$, (4.6) (which is equivalent to (4.3)) implies (4.5) for u'_n and thus for u_n.

Step 3. We need only show that (4.5) implies (4.3) (or (4.6)). We may assume without loss of generality that $u_n \geqq \log_2 n$, as we will now show. Set $u''_n = \max(u_n, \log_2 n)$, and note that $u''_n/n^2 \downarrow$ and $u''_n \uparrow$. Also, (4.3) diverges for u_n if and only if it diverges for u''_n, and (4.5) diverges for u_n if and only if it diverges for u''_n. Since $u''_n \geqq u_n \to \infty$, we have for all n large enough, $u_n^k \exp(-u_n)/n \geqq u_n''^k \exp(-u''_n)/n$, and $u_n^{k-1} \exp(-u_n) \geqq u_n''^{k-1} \exp(-u''_n)$, so that the divergence of (4.3) or (4.5) for u''_n implies the corresponding divergence for u_n. Thus, we need only establish the other implications. Assume first that (4.3) holds for u_n. Then either

$$\sum_{n:\, u_n < \log_2 n} u_n^k e^{-u_n}/n = \infty,$$

or

$$\sum_{n:\, u_n \geqq \log_2 n} u_n^k e^{-u_n}/n = \infty.$$

In the latter case, it is clear that (4.3) holds for u''_n. In the former case, there exists a subsequence n' of the integers such that $u_{n'} < \log_2 n'$. By the monotonicity of u''_n and the unimodality of $u^k e^{-u}$, we have for some constant n_0 and all $n' > n_0$,

$$\sum_{n_0}^{n'} u_j''^k e^{-u_j''}/j \geqq u_{n'}''^k e^{-u_{n'}''} \sum_{n_0}^{n'} \frac{1}{j} = u_{n'}''^k e^{-u_{n'}''} \log n'(1+o(1))$$
$$\geqq (\log_2 n')^k (1+o(1)$$

along this subsequence n'. Thus, obviously, (4.3) holds for u_n'' too. Assume now that (4.5) holds for u_n. We can argue as for (4.3), and show that (4.5) holds for u_n'' as well.

Because $u_n'' \geqq \log_2 n$, we see that (4.5) implies (4.6) and thus (4.3), which was to be shown.

Proof of Theorem 4.2. By Lemma 4.1, we have that $u_{n_j}^{k-1} \exp(-n_j)$ is not summable in j for any $a>0$. Also, the assumptions

$$\liminf u_n/\log_2 n \geqq 1$$

and (4.7)

$$u_n \leqq 2 \log_2 n$$

do not affect (4.3) or (4.5). Furthermore, if we can show that $P(n^2 K_n \geqq \max(u_n, \log_2 n) \text{ i.o.}) = 1$, then we will certainly be able to conclude that $P(n^2 K_n \geqq u_n \text{ i.o.}) = 1$. Since $P(n^2 K_n \geqq 2 \log_2 n \text{ i.o.}) = 0$ (Theorem 4.1), we may also replace any u_n given to us by $\min(u_n, 2\log_2 n)$. From now on, (4.7) is assumed to be valid.

Let $A_j = [n_j K_{n_j} \geqq u_{n_j}]$. Since $[A_j \text{ i.o.}]$ is invariant under permutations of $X_1, \ldots, X_m$ for any finite m, we conclude by the Hewitt-Savage zero one law that $P(A_j \text{ i.o.})$ is either 0 or 1 (Hewitt- and Savage (1955); see also Breiman (1968, p. 63)). It suffices to show that for all j_0, there exists a $j_1 > j_0$ such that

$$P\left(\bigcup_{j=j_0}^{j_1} A_j\right) \geqq \tfrac{1}{8}.$$

If we define the events $B_j = A_j A_{j+1}^c \ldots A_{j_1}^c$, $j_0 \leqq j \leqq j_1$, then clearly

$$P\left(\bigcup_{j=j_0}^{j_1} A_j\right) = \sum_{j=j_0}^{j_1} P(B_j)$$

and

$$P(B_j) \geqq P(A_j)\left(1 - \sum_{i=j+1}^{j_1} P(A_i | A_j)\right).$$

Since $u_n \to \infty$, $u_n = o(n^{1/3})$, we see that $P(A_j) \sim u_{n_j}^{k-1} \exp(-u_{n_j})/(k-1)! \to 0$ (Theorem 2.2), and this that $\sum P(A_j) = \infty$. Thus, we can find a constant K_0 such that for all $j_0 \geqq K_0$, there exists a $j_1 > j_0$ such that

$$\frac{1}{4} \leqq \sum_{j=j_0}^{j_1} P(A_j) \leqq \frac{1}{3}. \tag{4.8}$$

We are done then if we can show that

$$\sup_{j_0 \leqq j \leqq j_1} \sum_{i=j+1}^{j_1} P(A_i | A_j) \leqq \tfrac{1}{2}. \tag{4.9}$$

We are still free to choose the constant $a>0$.

We introduce a second constant $c\in(0,1)$. The positive integers $K_0, K_1, K_2, \ldots$ depend upon a and c only. We also need the following facts: for all $i>j\geqq K_1$,

$$-\frac{a(i-j)}{\log i}\leqq\log\left(\frac{n_j}{n_i}\right)\leqq-\frac{a(i-j)}{\log i}\left(1-\frac{1}{\log j}\right)\leqq-\frac{ac(i-j)}{\log i} \tag{4.10}$$

(see for example Lemma 2 of Robbins and Siegmund (1972)). Define the integers $\rho(j)$ and $\sigma(j)$ as follows: $\rho(j)$ is the largest integer i such that $i-j<(\log i)^{2/3}$; $\sigma(j)$ is the smallest positive integer i such that $i-j\geqq(\log i)^2$. It is not hard to check that $\rho(j)\sim\sigma(j)\sim j$, and that $\sigma(j)-\rho(j)\leqq 2+(j+\log^2(\sigma(j)-1))-(j-(\log(1+\rho(j)))^{2/3})<2\log^2 j$ for all $j\geqq K_2$. Also, if

$$ac>1, \tag{4.11}$$

then $i>\sigma(j)$, $j\geqq K_1$ imply that

$$\log\left(\frac{n_j}{n_i}\right)\leqq-\frac{ac(i-j)}{\log i}\leqq-ac\log i\leqq-\log i. \tag{4.12}$$

Let $\delta=2e^k$, and $\alpha=\dfrac{ac}{8}$. Let $a>0$ and $c\in(0,1)$ be chosen such that

$$ac\geqq\max(2k, 16, \sqrt{18}) \tag{4.13}$$

and

$$\delta\left(e^{-\alpha}\alpha^{k-1}+\frac{(k-1)!}{\alpha}\right)\leqq\frac{1}{12}. \tag{4.14}$$

We will prove (4.9) in four steps.

Step 1. $i>j\geqq K_3$, $i\leqq\rho(j)$ implies that

$$P(A_i|A_j)\leqq\delta[\alpha(i-j)]^{k-1}e^{-\alpha(i-j)}. \tag{4.15}$$

Step 2. $i>j\geqq K_4$, $\rho(j)<i\leqq\sigma(j)$ implies that

$$P(A_i|A_j)\leqq\delta(\log i)^{(k-1)/3}\,e^{-(\log i)^{1/3}}. \tag{4.16}$$

Step 3. $i>j\geqq K_5$, $i>\sigma(j)$ implies that

$$P(A_i|A_j)\leqq\tfrac{9}{8}P(A_i). \tag{4.17}$$

Step 4. Let $j_0\geqq K_7=\max(K_0, K_1, K_2, K_3, K_4, K_5, K_6)$ where K_6 is such that $j\geqq K_6$ implies $\log\sigma(j)\leqq 2\log j$. Let j_1 be as in (4.8) and let $j_1\geqq j\geqq j_0\geqq K_7$. Then

$$\begin{aligned}\sum_{i=j+1}^{j_1}P(A_i|A_j)&=\sum_{j+1}^{\rho(j)}+\sum_{\rho(j)+1}^{\sigma(j)}+\sum_{\sigma(j)+1}^{j_1}\\&\leqq\sum_{i=1}^{\infty}\delta e^{-\alpha i}(\alpha i)^{k-1}+(\sigma(j)-\rho(j))\,\delta\,(\log\sigma(j))^{\frac{k-1}{3}}\,e^{-(\log j)^{1/3}}+\frac{9}{8}\sum_{i=j_0}^{j_1}P(A_i)\\&\leqq\delta e^{-\alpha}\alpha^{k-1}+\int_0^{\infty}\delta e^{-\alpha u}(\alpha u)^{k-1}\,du+2\delta\log^2 j(2\log j)^{\frac{k-1}{3}}\,e^{-(\log j)^{1/3}}+\tfrac{3}{8}.\end{aligned} \tag{4.19}$$

Here we used (4.15), (4.16), (4.17), (4.8) and the definitions of K_2 and K_6. We also exploited the unimodality of the function $u^{k-1}e^{-u}$ and the fact that $\alpha > k-1$. Now, the first two terms on the right-hand-side of (4.19) are not greater than the expression in (4.14), and the third term in (4.19) is smaller than 1/24 for $j \geqq j_0 \geqq K_6$ by our choice of K_6. Thus, for $j_0 \geqq K_7$, the expression (4.19) is not greater than $1/12 + 1/24 + 3/8 = 1/2$, which was to be shown. We are done if we can show (4.15), (4.16) and (4.17).

Proof of (4.15). By Lemma 2.5, $P(A_i | A_j) \leqq P(X < k)$ where X is a binomial $(n_i - n_j, p)$ random variable and $p = (n_j + 1 - k) u_{n_i}/n_i^2$ (we assume that K_3 is so large that $j \geqq K_3$ implies $n_j \geqq 4(k-1)$). Let K_3 also be so large that $a/(\log j)^{1/3} \leqq \frac{1}{4}$, and $u_{n_i} \geqq \frac{3}{4}\log_2 n_i \geqq \frac{1}{2}\log i$ for all $i > j \geqq K_3$. Furthermore, $K_3 \geqq K_1$. For $i > j \geqq K_3$, $i \leqq \rho(j)$ we have

$$\begin{aligned}
\left(1 - \frac{n_j}{n_i}\right) &\geqq 1 - \exp\left(-ac\frac{i-j}{\log i}\right) \quad \text{(by (4.10))} \\
&\geqq ac\frac{i-j}{\log i}\left(1 - ac\frac{i-j}{2\log i}\right) \\
&\geqq ac\frac{i-j}{\log i}\left(1 - \frac{ac}{2(\log i)^{1/3}}\right) \quad (\text{since } i-j < (\log i)^{2/3}) \\
&\geqq \frac{7}{8}ac\frac{i-j}{\log i},
\end{aligned}$$

$$\begin{aligned}
\frac{n_j + 1 - 2k}{n_i} &\geqq \frac{3}{4}\frac{n_j}{n_i} \quad \text{(by choice of } K_3) \\
&\geqq \frac{3}{4}\exp\left(-a\frac{i-j}{\log i}\right) \quad \text{(by (4.10))} \\
&\geqq \frac{3}{4}\left(1 - a\frac{i-j}{\log i}\right) \\
&\geqq \frac{3}{4}\left(1 - \frac{a}{(\log i)^{1/3}}\right) \\
&\geqq \tfrac{1}{2} \quad \text{(by choice of } K_3),
\end{aligned}$$

and

$$(n_i - n_j)\, p \geqq \frac{1}{4} ac \frac{i-j}{\log i} u_{ni} \geqq \frac{1}{8} ac(i-j) \geqq \alpha \geqq \max(k, 2)$$

where we used (4.13) and the definition of K_3. By Lemma 2.3, we have

$$\begin{aligned}
P(X < k) &\leqq e^{k-1}\left(\frac{1 + (k-1)!}{(k-1)!}\right)[(n_i - n_j)\, p]^{k-1} e^{-(n_i - n_j)p} \\
&\leqq 2e^k [\alpha(i-j)]^{k-1} e^{-\alpha(i-j)},
\end{aligned}$$

which was to be shown.

Proof of (4.16). By Lemma 2.5 and (2.8),

$$P(A_i|A_j) \leqq P(n_i^2 K_{n_i-n_j} \geqq u_{n_i}) \leqq P(X<k)+P(T \leqq (n_i-n_j)/2)$$

where X is a binomial (n_i-n_j, p) random variable, T is a gamma (n_i-n_j) random variable and $p=1-\exp(-u_{n_i}(n_i-n_j)/(2n_i^2))$. Let K_4 be so large that for $i>j\geqq K_4$, $K_4\geqq K_1$, $u_{n_i}\geqq \frac{1}{2}\log_2 n_i \geqq \frac{1}{4}\log i$, $ac \leqq (\log i)^{1/3}$, $ac(\log i)^{2/3}\leqq n_i$, $acn_i \geqq 2i(\log i)^{1/3}$ and $(\log i)^{1/3}\geqq \max(k-1,2)$. Then

$$\begin{aligned}
\left(1-\frac{n_j}{n_i}\right) &\geqq 1-\exp\left(-ac\frac{i-j}{\log i}\right) \quad \text{(by (4.10))}\\
&\geqq 1-\exp(-ac/(\log i)^{1/3}) \quad (\text{since } i-j\geqq(\log i)^{1/3} \text{ for } i>\rho(j))\\
&\geqq \frac{ac}{(\log i)^{1/3}}\left(1-\frac{ac}{2(\log i)^{1/3}}\right)\\
&\geqq \frac{1}{2}\frac{ac}{(\log i)^{1/3}},
\end{aligned}$$

$$n_i-n_j \geqq \frac{acn_i}{2(\log i)^{1/3}} \geqq i \quad (\text{by choice of } K_4),$$

$$\frac{u_{n_i}(n_i-n_j)}{2n_i^2} \geqq \frac{1}{8}\frac{\log i}{n_i}\left(1-\frac{n_j}{n_i}\right) \geqq \frac{1}{8}\frac{ac(\log i)^{2/3}}{n_i},$$

$$\begin{aligned}
p &\geqq 1-\exp\left(-\frac{ac(\log i)^{2/3}}{8n_i}\right) \geqq \frac{ac(\log i)^{2/3}}{8n_i}\left(1-\frac{ac(\log i)^{2/3}}{16n_i}\right)\\
&\geqq \frac{ac(\log i)^{2/3}}{9n_i} \quad (\text{by choice of } K_4),
\end{aligned}$$

and

$$\begin{aligned}
(n_i-n_j)\,p &\geqq \left(1-\frac{n_j}{n_i}\right)\frac{ac(\log i)^{2/3}}{9} \geqq \frac{(ac)^2(\log i)^{1/3}}{18}\\
&\geqq (\log i)^{1/3} \geqq \max(k-1,2).
\end{aligned}$$

By Lemma 2.3 and Lemma 2.1, we have

$$\begin{aligned}
P(A_i|A_j) &\leqq P(X<k)+\exp(-(n_i-n_j)/8)\\
&\leqq 2e^{k-1}(\log i)^{\frac{k-1}{3}}\exp(-(\log i)^{1/3})+\exp(-i)\\
&\leqq 2e^{k}(\log i)^{\frac{k-1}{3}}\exp(-(\log i)^{1/3}),
\end{aligned}$$

which was to be shown.

Proof of (4.17). Let $K_5\geqq\max(K_1,2)$ be such that $i>j\geqq K_5$ implies $2\log_2 n_i\geqq k-1$, $n_i\geqq 2i^4$, $u_{n_i}\leqq(2n_i/i)$, $u_{n_i}(1-4/i)\geqq k-1$, (4.20) and (4.21). We recall that for $i>\sigma(j)$, $i>j\geqq K_1$, $(1-n_j/n_i)\geqq 1-1/i$ (see (4.12)). Thus,

$$(n_i-n_j)=n_i\left(1-\frac{n_j}{n_i}\right)\geqq n_i\left(1-\frac{1}{i}\right)\geqq\frac{n_i}{2}\geqq i^4. \tag{4.20}$$

By (2.8) and Lemma 2.5,

$$P(A_i|A_j) \leqq P(X<k) + P(T \leqq (n_i - n_j)(1-r)) \tag{4.21}$$

where X is a binomial $(n_i - n_j, p)$ random variable, T is a gamma $(n_i - n_j)$ random variable, $r = (n_i - n_j)^{-1/4}$ and $p = 1 - \exp(-u_{n_i}(n_i - n_j)(1-r)/n_i^2)$. By Lemma 2.1, (4.20), (4.7) and our choice of K_5,

$$\begin{aligned} &P(T \leqq (n_i - n_j)(1-r))/P(A_i) \leqq \exp(-\sqrt{n_i - n_j}/2)/P(A_i) \\ &\leqq \exp(-i^2/2)/P(A_i) \leqq \tfrac{1}{16}. \end{aligned} \tag{4.22}$$

Here we used the fact that

$$\begin{aligned} P(A_i) &\sim u_{n_i}^{k-1} \exp(-u_{n_i})/(k-1)! \geqq (2\log_2 n_i)^{k-1} (\log n_i)^{-2}/(k-1)! \\ &\sim (2\log i)^{k+1} \frac{1}{(ai)^2 (k-1)!}. \end{aligned}$$

Next we note that

$$p \leqq \frac{u_{n_i}}{n_i^2}(n_i - n_j)(1-r) \leqq \frac{u_{n_i}}{n_i},$$

$$\begin{aligned} p &\geqq \frac{u_{n_i}}{n_i^2}(n_i - n_j)(1-r)\left(1 - \frac{u_{n_i}}{2n_i}\right) \\ &\geqq \frac{u_{n_i}}{n_i}\left(1 - \frac{3}{i}\right) \end{aligned}$$

where we used (4.12), (4.20) and our choice of K_5. Also, $(n_i - n_j)p \geqq (1 - \frac{1}{i}) n_i p \geqq \left(1 - \frac{4}{i}\right) u_{n_i}$. By Lemma 2.3,

$$\begin{aligned} P(X<k) &\leqq \exp\left(\frac{u_{n_i}}{n_i}(k-1)\right)\left(1 + \frac{(k-1)!}{u_{n_i}\left(1-\frac{4}{i}\right) - 1}\right) \frac{u_{n_i}^{k-1} \exp\left(-u_{n_i}\left(1 - \frac{4}{i}\right)\right)}{(k-1)!} \\ &\leqq (1 + \tfrac{1}{16}) P(A_i). \end{aligned} \tag{4.23}$$

Combining (4.21), (4.22) and (4.23) gives the desired result.

Remark 4.1. Theorem 4.3 implies that $P(n^2 K_n \geqq \log_2 n \text{ i.o.}) = 1$ for any k, and $P(n^2 K_n \geqq (1+\delta)\log_2 n \text{ i.o.}) = 0$ for any k and any $\delta > 0$. The proof of the former fact is considerably shorter than the proof of Theorem 4.2. We offer one proof below, that uses arguments similar to those found in Kiefer (1972) and Devroye (1981).

Proof of Remark 4.1. We define the subsequence $n_j = [\exp(aj \log j)]$ where $a > 1/(2k)$ is a constant. The proof is based upon the following implication:

$$[n^2 K_n \geqq \log_2 n \text{ i.o.}] \supseteq [n_j^2 K_{n_j} \geqq \log_2 n_j \text{ i.o.}]$$
$$\supseteq [K_{n_j} < \log_2 n_{j+1}/n_{j+1}^2 \text{ f.o.}] \cap [A_j \text{ i.o.}]$$

where

$$A_j = \bigcap_{i=n_j+1}^{n_{j+1}} \bigcap_{l=-1}^{i-1} [|X_i - X_l| \geqq \log_2 n_{j+1}/n_{j+1}^2].$$

By Theorem 3.1, we have $P(K_{n_j} \leqq \log_2 n_{j+1}/n_{j+1}^2 \text{ f.o.}) = 1$ when for all j large enough,

$$\frac{\log_2 n_{j+1}}{n_{j+1}^2} < \frac{1}{n_j^2 (\log n_j \log_2^2 n_j)^{1/k}}. \tag{4.24}$$

Also, let $\mathscr{F}_j$ be the σ-algebra generated by $A_1, \ldots, A_j$, and let $b = \log_2 n_{j+1}/n_{j+1}^2$. By the conditional form of the Borel-Cantelli lemma (Levy, 1965), we have $P(A_j \text{ i.o.}) = 1$ on $\sum P(A_j|\mathscr{F}_{j-1}) = \infty$. In fact, by a result of Dubins and Freedman (1965, Theorem 1) (see also Freedman, 1973, (40)), $\sum_{i=1}^{j} I_{A_i} \Big/ \sum_{i=1}^{j} P(A_i|\mathscr{F}_{i-1}) \to 1$ a.s. on $\sum P(A_j|\mathscr{F}_{j-1}) = \infty$. Because

$$P(A_j|\mathscr{F}_{j-1}) \geqq \prod_{i=n_j+1}^{n_{j+1}} (1 - 2b\,i),$$

it suffices to show that

$$\sum_{j=1}^{\infty} \prod_{i=n_j+1}^{n_{j+1}} (1 - 2b\,i) = \infty. \tag{4.25}$$

Using the facts that $b\,n_{j+1} = o(1)$, $b^3 n_{j+1}^3 = o(1)$, $\log(1-u) \geqq -u - u^2/(2(1-u))$, $0 < u < 1$, and $\sum_{n+1}^{m} i \leqq (m^2 - n^2)/2 + (m-n)$, we have for all j large enough,

$$\begin{aligned}
\prod_{i=n_j+1}^{n_{j+1}} (1 - 2b\,i) &\geqq \exp\left(- \sum_{i=n_j+1}^{n_{j+1}} (-2b\,i - b^2 i^2/(1-2b\,i)\right) \\
&\geqq \exp(-b(n_{j+1}^2 - n_j^2) - 2b(n_{j+1} - n_j) - O(b^2 n_{j+1}^3)) \\
&= \exp(-b(n_{j+1}^2 - n_j^2) + o(1)).
\end{aligned}$$

It is easy to show that $(n_{j+1}/n_j)^2 \sim (ej)^{2a}$, so that $b(n_{j+1}^2 - n_j^2) = \log_2 n_{j+1} + O(\log_2 n_{j+1}/j^{2a}) = \log(aj \log j) + o(1)$. Thus, the j-th term in (4.25) is at least equal to

$$\frac{1 + o(1)}{aj \log j},$$

which is not summable in j.

To prove (4.24), we note that $(n_{j+1}/n_j)^2 \sim (ej)^{2a}$, and that $\log_2 n_{j+1} \cdot (\log n_j \log_2^2 n_j)^{1/k} \sim (aj)^{1/k} (\log j)^{1+3/k}$. This explains our choice of $a > 1/(2k)$.

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