Journal of Theoretical Probability, Vol. 3, No. 4, 1990

Random Walks on Highly Symmetric Graphs

Luc Devroye[1] **and Amine Sbihi**[1]

Received January 27, 1989; revised July 7, 1989

We consider uniform random walks on finite graphs with n nodes. When the hitting times are symmetric, the expected covering time is at least $\frac{1}{2}n \log n - O(n \log \log n)$ uniformly over all such graphs. We also obtain bounds for the covering times in terms of the eigenvalues of the transition matrix of the Markov chain. For distance-regular graphs, a general lower bound of $(n-1)\log n$ is obtained. For hypercubes and binomial coefficient graphs, the limit law of the covering time is obtained as well.

KEY WORDS: Random walks; covering times; graphs; vertex-transitive graphs; distance-regular graphs.

1. INTRODUCTION

For a finite Markov chain, let the covering time T be the time taken to visit all the states. Aldous[2] introduced an important approach to obtain results on the mean covering time in the context of rapidly mixing random walks on finite groups. He showed that $\mathbf{E}(T)$ is approximately $Rn \log n$, where n is the cardinality of the group and R is the mean number of visits to the initial state in a short time.

Matthews[10] obtained bounds applicable to mean covering times for finite Markov chains: if the state space $S = \{0, 1, \ldots, n\}$ then, for the chain starting at 0

$$\mu_- \cdot H_n \leqslant \mathbf{E}_0(T) \leqslant \mu_+ \cdot H_n \tag{1.1}$$

where H_n is the nth harmonic number,

$$\mu_- = \min_{1 \leqslant j \leqslant n} \min_{0 \leqslant i \leqslant n, i \neq j} \mathbf{E}_i(T_j), \qquad \mu_+ = \max_{1 \leqslant j \leqslant n} \max_{0 \leqslant i \leqslant n, i \neq j} \mathbf{E}_i(T_j)$$

[1] Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Street West, Montréal, Canada H3A 2K6.

0894-9840/90/1000-0497$06.00/0

and $\mathbf{E}_i(T_j)$ is the mean hitting time of j for the chain starting at i. Similar bounds for the moment generating functions of the covering times have also been obtained by Matthews.[11]

Let G be a finite connected graph with n vertices and consider the nearest neighbor random walk on G: for a vertex i, if v_i is the number of edges connected to i, the probability that a particle moves from i to a neighboring vertex j is $1/v_i$. For the nearest neighbor random walk on graphs several results have been obtained recently (see Aldous, 1989), foremost among which is a general lower bound $\mathbf{E}_\pi(T)=\Omega(n\log n)$ for the walk started with the stationary distribution π.[4]

In this note we consider graphs for which the nearest neighbor random walk has symmetrical mean hitting times, that is, $\mathbf{E}_i(T_j)=\mathbf{E}_j(T_i)$ for all distinct vertices i, j. We provide bounds for $\mathbf{E}(T)$ and study in more detail two classes of graphs, namely, vertex-transitive graphs and distance-regular graphs. For the properties of these graphs we refer to Biggs.[8]

2. SYMMETRIC HITTING TIMES

Let G be a finite connected graph with n vertices and e edges for which $\mathbf{E}_i(T_j)=\mathbf{E}_j(T_i)$ for all distinct vertices i, j. We call these graphs symmetric graphs; some families of graphs that are symmetric are considered in the next sections. It turns out that the symmetry hypothesis is powerful enough to obtain the following proposition.

Proposition 1. Let the nodes be ordered according to increasing degrees $v_1\leqslant v_1\leqslant \cdots \leqslant v_n$ and let the starting node be s. Then, for any integer $k\geqslant 1$, and any symmetric graphs, we have

1. $\mathbf{E}_s(T)\geqslant \dfrac{e}{v_k} H_{k-1}$

2. $\mathbf{E}_s(T)\geqslant \dfrac{n-k}{2} H_{k-1}$

3. $\mathbf{E}_s(T)\geqslant \frac{1}{2}n\log n - O(n\log\log n)$ uniformly over all symmetric graphs with n vertices.

Proof. For $1\leqslant i\leqslant n$, let $\pi_i=v_i/2e$ be the stationary distribution. Since $\mathbf{E}_i(T_i)=1/\pi_i$ we have $\sum_{i=1}^n 1/\mathbf{E}_i(T_i)=1$. Clearly then, since $\mathbf{E}_1(T_1)\geqslant \cdots \geqslant \mathbf{E}_n(T_n)$

$$\begin{aligned} 1=\sum_{i=1}^n 1/\mathbf{E}_i(T_i) &\geqslant [k/\mathbf{E}_1(T_1)]+[(n-k)/\mathbf{E}_k(T_k)] \\ &\geqslant (n-k)/\mathbf{E}_k(T_k) \end{aligned}$$

But, for $1 \leqslant i \neq j \leqslant k$, we have by the triangle inequality,

$$n - k \leqslant \mathbf{E}_k(T_k) \leqslant \mathbf{E}_i(T_i) \leqslant \mathbf{E}_i(T_j) + \mathbf{E}_j(T_i) = 2\mathbf{E}_i(T_j)$$

The covering time T is at least equal to $T_{\{1,2,\dots,k\}}$ where $T_{\{1,2,\dots,k\}}$ is the first time to cover the set $\{1, 2,\dots, k\}$. By Matthews's lower bound (1.1) we conclude that

$$\mathbf{E}_s(T) \geqslant \frac{\mathbf{E}_k(T_k)}{2} H_{k-1} \geqslant \frac{n-k}{2} H_{k-1}$$

Part 3 of the proposition follows after taking $k = k(n)$ such that $k \sim n/\log n$. Thus, uniformly over all symmetric graphs,

$$\mathbf{E}_s(T) \geqslant \tfrac{1}{2} n \log n - O(n \log \log n) \qquad \square$$

For symmetric regular graphs all $\mathbf{E}_i(T_i)$ are equal to n. Hence, for all $i \neq j$, $\mathbf{E}_i(T_j) \geqslant n/2$ and thus $\mathbf{E}_s(T) \geqslant (n/2) \cdot H_{n-1}$.

For the upper bound we use the eigenstructure of the transition matrix P. A general upper bound for $\mathbf{E}(T)$ has been obtained by Aleliunas *et al.*[6] in the from $\mathbf{E}(T) \leqslant 2e(n-1)$ where e is the number of edges in the graphs.

The transition matrix of the nearest neighbor random walk is given by $P = DA$ where A is the adjacency matrix and D is the diagonal matrix $D = (D_{ii} = 1/v_i)$. From the fact that P and the symmetric matrix $Q = D^{1/2} A D^{1/2}$ are similar it can be shown (see, for example, Mazo[13]) that the mean hitting times $\mathbf{E}_i(T_j)$ for $i \neq j$ can be given by the following spectral representation:

$$\mathbf{E}_i(T_j) = \sum_{r=2}^{n} \left[\frac{u_r^2(j)}{\pi_j} - \frac{u_r(i)\, u_r(j)}{\sqrt{\pi_i}\sqrt{\pi_j}} \right] \frac{1}{1-\lambda_r} \tag{2.1}$$

where $(\pi_i,\ 1 \leqslant i \leqslant n)$ is the stationary distribution, $u_r = (u_r(i),\ 1 \leqslant i \leqslant n)'$ is, for $1 \leqslant r \leqslant n$, a complete set of orthonormal eigenvectors of Q, associated with the eigenvalues λ_r.

Combining the spectral representation with the symmetry $\mathbf{E}_i(T_j) = \mathbf{E}_j(T_i)$ for $i \neq j$, produces

$$\mathbf{E}_i(T_j) = \frac{1}{2} \sum_{r=2}^{n} \left[\frac{u_r(i)}{\pi_i} - \frac{u_r(j)}{\pi_j} \right]^2 \frac{1}{1-\lambda_r} \tag{2.2}$$

Further, the assumed symmetry is equivalent to

$$\sum_{r=2}^{n} \frac{u_r^2(i)}{\pi_i} \frac{1}{1-\lambda_r} = \sum_{r=2}^{n} \frac{u_r^2(j)}{\pi_j} \frac{1}{1-\lambda_r} \tag{2.3}$$

Observe that the mean hitting times, starting from the stationary distribution π are

$$\mathbf{E}_\pi(T_j) = \sum_i \pi_i \mathbf{E}_i(T_j) = \sum_{r=2}^{n} \frac{u_r^2(j)}{\pi_j} \frac{1}{1-\lambda_r}$$

by using the spectral representation with $u_1(j) = \sqrt{\pi_j}$, $1 \leqslant j \leqslant n$. Hence the symmetry hypothesis is equivalent to $\mathbf{E}_\pi(T_i) = \mathbf{E}_\pi(T_j)$ for all i, j. Denote by C_n this common value,

$$C_n = \sum_j \pi_j \mathbf{E}_\pi(T_j) = \sum_j \pi_j \sum_{r=2}^{n} \frac{u_r^2(j)}{\pi_j} \frac{1}{1-\lambda_r}$$

$$= \sum_{r=2}^{n} \frac{1}{1-\lambda_r} \qquad \text{by the orthonormality of } u_r$$

From (2.2) and (2.3),

$$\mathbf{E}_i(T_j) = C_n - \sum_{r=2}^{n} \frac{u_r(i)\, u_r(j)}{\sqrt{\pi_i}\sqrt{\pi_j}} \frac{1}{1-\lambda_r}$$

and by the Cauchy–Schwarz inequality,

$$\sum_{r=2}^{n} \frac{|u_r(i)\, u_r(j)|}{\sqrt{\pi_i}\sqrt{\pi_j}} \frac{1}{1-\lambda_r} \leqslant C_n$$

Thus,

$$\mathbf{E}_i(T_j) \leqslant 2C_n = 2 \sum_{r=2}^{n} \frac{1}{1-\lambda_r} = 2\left(n-1+\sum_{r=2}^{n} \frac{\lambda_r}{1-\lambda_r}\right)$$

which combined with (1.1) gives

$$\mathbf{E}(T) \leqslant 2\left(n-1+\sum_{r=2}^{n} \frac{\lambda_r}{1-\lambda_r}\right) H_{n-1}$$

Hence we have proved the following proposition.

Proposition 2. For a symmetric graph with n vertices, let $1 = \lambda_1 > \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant -1$ be the eigenvalues of the transition matrix P. The covering time T, starting from any vertex, is bounded by

$$\mathbf{E}(T) \leqslant 2\left(n-1+\sum_{r=2}^{n} \frac{\lambda_r}{1-\lambda_r}\right) H_{n-1}$$

Remark. If we assume in addition that the graph is regular we can obtain from the proof of Proposition 2 that

$$n\frac{1}{1-\lambda_n}\leqslant \mathbf{E}_i(T_j)\leqslant n\frac{1}{1-\lambda_2} \qquad \text{for} \quad i\neq j$$

and then, by (1.1), the covering time T starting from any vertex is bounded by

$$n\frac{1}{1-\lambda_n}\cdot H_{n-1}\leqslant \mathbf{E}(T)\leqslant n\frac{1}{1-\lambda_2}\cdot H_{n-1}$$

3. VERTEX-TRANSITIVE GRAPHS

As an example of graphs to which the previous remark applies, let us consider the class of vertex-transitive graphs. A vertex-transitive graph $G=(V,E)$ is a graph for which the group of automorphisms $H(G)$ acts transitively on V, that is for each node i, we can find for each $j\in V-\{i\}$ some $g\in H(G)$ such that $gi=j$.

Vertex-transitive graphs are k-valent, for some k, since the transition probabilities are $H(G)$-invariant:

$$\mathbf{P}(i,j)=\mathbf{P}(gi,gj)$$

for all $i,j\in V$, $g\in H(G)$. For the hitting times $T_i=\min\{n\geqslant 0,\ X_n=i\}$ we observe the following:

Lemma 1. For any vertex-transitive graph G with n vertices, for any vertices i, j, and for $|u|\leqslant 1$, we have

1. $\mathbf{E}_i(u^{T_j})=\mathbf{E}_j(u^{T_i})$
2. $\mathbf{E}_\pi(u^{T_j})=\dfrac{1}{1+(1-u)\sum_{r=2}^{n}1/(1-\lambda_r u)}$

where $1=\lambda_1>\lambda_2\geqslant\cdots\geqslant\lambda_n\geqslant -1$ are the eigenvalues of the transition matrix P.

Proof. For this, consider any permutation h of the vertex set V with permutation matrix $M=(h_{ij}=1$ if $i=hj$, 0 otherwise). $h\in H(G)$ if and only if $MA=AM$ where A is the adjacency matrix of G; then, for any integers $n\geqslant 0$, $A^n=M^tA^nM$.

Using the vertex-transitivity it follows that $(A^n)_{ii}=(A^n)_{jj}$, for all $i,j\in V$, $n\geqslant 0$. In particular, since the transition matrix associated with the nearest neighbor random walks (X_n) on G is $P=(1/k)A$, where k is the common valency of the vertices, we have $\mathbf{P}_i(X_n=i)=(1/k^n)(A^n)_{ii}$ is independent of the vertex i. As $P(i,j)=P(j,i)$ for all vertices i, j, it follows, by Chapman–Kolmogorov that

$$\mathbf{P}_i(X_n=j)=\mathbf{P}_j(X_n=i) \qquad \text{for all} \quad i, j \in V, \quad n \geqslant 0$$

Conclude by using the well-known relation

$$\mathbf{E}_i(u^{T_j})=\frac{\sum_{n\geqslant 0} u^n \mathbf{P}_i(X_n=j)}{\sum_{n\geqslant 0} u^n \mathbf{P}_j(X_n=j)} \tag{3.1}$$

For the second part consider the spectral representation of the symmetric matrix P, from which we have, for $|u|<1$,

$$\sum_{m\geqslant 0} u^m \mathbf{P}_i(X_m=j)=\sum_{r=1}^{n} \frac{1}{1-\lambda_r u} x_r(i)\, x_r(j)$$

where $x_1,..., x_n$ are orthogonal eigenvectors associated to the eigenvalues $1=\lambda_1>\lambda_2\geqslant \cdots \geqslant \lambda_n \geqslant -1$ of P.

Hence, from (3.1) and the symmetry $\mathbf{E}_i(u^{T_j})=\mathbf{E}_j(u^{T_i})$ it follows immediately that

$$\mathbf{E}_i(u^{T_j})=\frac{1+(1-u)n \sum_{r=2}^{n} \frac{1}{1-\lambda_r u} x_r(i)\, x_r(j)}{1+(1-u) \sum_{r=2}^{n} \frac{1}{1-\lambda_r u}}$$

and then

$$\mathbf{E}_\pi(u^{T_j})=\frac{1}{1+(1-u) \sum_{r=2}^{n} \frac{1}{1-\lambda_r u}} \qquad \square$$

Note that sharp conditions for the asymptotic exponentiality of the hitting times have been obtained recently by Aldous.[5]

A connection with particular random walks on groups can be made by introducing the Cayley graphs. Let F be a abstract finite group with identity 1 and set of generators Ω with the properties $\Omega=\Omega^{-1}$ and $1 \notin \Omega$. The Cayley graph $C(F, \Omega)$ of F is the graph with vertex set F and edge set $\{(x, y) \mid x^{-1}y \in \Omega\}$.

The Cayley graph $C(F, \Omega)$ is vertex-transitive and the nearest neighbor random walk on C can be viewed as the random walk on the group F, defined by the probability measure μ on F, which support Ω and

$P(x, y) = \mu(yx^{-1})$ such that μ is constant on the conjugacy classes: $\mu(xyx^{-1}) = \mu(y)$ for all $x, y \in F$, and μ symmetric: $\mu(x) = \mu(x^{-1})$ for all $x \in F$.

4. DISTANCE-REGULAR GRAPHS

In this section we consider graphs having the following regularity in their paths: given a graph G of diameter d, we assume that, for any vertices u, v at distance i, the number $s_{hj}(u, v) = \#\{w\colon d(u, w) = h,\ d(v, w) = j\}$ for $0 \leqslant h,\ j \leqslant d$, is independent of the choice of the vertices u, v. Denote $s_{hj}(u, v) = s_{hji}$. In particular the number of vertices at distance i of a given vertex is s_{ii0} and hence the graph is regular. We define $k_i = s_{ii0}$.

The purpose of this analysis is to obtain, in special cases, the limit distribution of the covering time T. Thus, in the two examples considered at the end of this section, we show that for all fixed x,

$$\mathbf{P}\left(\frac{T - n\log n}{n} < x\right) \to e^{-e^{-x}} \qquad \text{as} \quad n \to \infty$$

where n is the number of vertices. Modulo a normalization it seems that a similar result can be obtain for other distance-regular graphs that are rapidly mixing (for more on rapidly mixing Markov chains see Aldous[1] and Diaconis[9]).

A distance-regular graph G has the property (Ref. 8, p. 136) that the zero-one matrices $A_0, A_1, \dots, A_d$ where $A_0 = I_d$, $A_1 = A$ adjacency matrix, $A_r = [(A_r)_{ij} = 1$ if $d(i, j) = r$, 0 otherwise$]$ form a basis for the algebra $A(G)$ of polynomials in A. From this it follows that the transition matrix P of the nearest neighbor random walk $(X_n,\ n \geqslant 0)$ on a distance-regular graph G is such that

$$P^n = \frac{1}{k^n} \cdot A^n = \frac{1}{k^n} \cdot \sum_{s=0}^{d} t_{n,s} \cdot A_s$$

for some non-negative integers $t_{n,s}$, and $k = k_1$.

In particular, for any vertices x, y at distance i, $n \geqslant 0$

$$P^n(x, y) = \frac{1}{k^n} \cdot t_{n,i}$$

and for $T_y = \min\{n \geqslant 0,\ X_n = y\}$

$$\mathbf{E}_x(u^{T_y}) = \frac{\sum_{n \geqslant 0} u^n t_{n,i}}{\sum_{n \geqslant 0} u^n t_{n,0}}$$

are both independent of the choice of the vertices x, y at distance i.

Let $\mathbf{E}_x(T_y) = M_r$ for any vertices at distance r. Define $s_{1,r-1,r} = c_r$, $s_{1,r,r} = a_r$ and $s_{1,r+1,r} = b_r$, where the parameters a_r, b_r, c_r have the following meaning: for vertices x, y at distance r, c_r, $1 \leqslant r \leqslant d$, is the number of vertices at distance $r-1$ of y and adjacent to x; b_r, $0 \leqslant r \leqslant d-1$, is the number of vertices at distance $r+1$ of y and adjacent to x. Clearly $a_r + b_r + c_r = k$. We can easily show that, for a distance-regular graph on N vertices:

Lemma 2. For $1 \leqslant r \leqslant d$,

$$M_r = \sum_{i=1}^{r} \frac{k}{k_{i-1} b_{i-1}} \cdot \sum_{j=i}^{d} k_j$$

is an increasing function of r with minimal value $M_1 = \sum_{j=1}^{d} k_j = N-1$.

Hence we have immediately, by using (1.1), the following proposition.

Proposition 3. For distance-regular graphs with N vertices the mean covering time is bounded by

$$\min_s \mathbf{E}_s(T) \geqslant (N-1) H_{N-1} \geqslant (N-1) \log N$$

Note that it is impossible to derive a general upper bound of the order of $N \log N$, since for the cycle graph, $\mathbf{E}_s(T) = N(N-1)/2$.

Proof of Lemma 2. From the Markov property we have, for $|u| < 1$,

$$\mathbf{E}_x(u^{T_y}) = G_{xy}(u) = u \sum_{z \in V} P(x, z) \, \mathbf{E}_z(u^{T_y})$$

$$\mathbf{E}_x(u^{T_x}) = 1$$

for distinct nodes x, y, and arbitrary vertex set V. Then, if $d(x, y) = r$,

$$G_{xy}(u) = G_r(u) = \frac{u}{k} \sum_{z \text{ adjacent to } x} \mathbf{E}_z(u^{T_y})$$

$$= \frac{u}{k} \sum_{l=0}^{d} s_{1lr} G_l(u)$$

since $\#\{w\colon d(x, w) = 1, \; d(y, w) = l\} = s_{1lr}$ and using the fact that the $\mathbf{P}_z$-distribution of T_y depends on the distance between z and y only.

By definition, the parameters s_{1lr} are null if $l \neq r$, $r-1$, $r+1$, hence we have

$$G_0(u) = 1$$

$$G_r(u) = \frac{u}{k} \{c_r G_{r-1}(u) + a_r G_r(u) + b_r G_{r+1}(u)\}, \qquad 1 \leqslant r \leqslant d-1$$

$$G_d(u) = \frac{u}{k} \{c_d G_{d-1}(u) + a_d G_d(u)\}$$

It follows immediately, by differentiating with respect to u and letting $u \to 1$ that the quantities $M_r \triangleq \mathbf{E}_x(T_y) = G_r'(1)$ (for nodes x, y at distance r) satisfy

$$M_0 = 0$$

$$(M_r - M_{r-1})\frac{c_r}{k} = 1 + (M_{r+1} - M_r)\frac{b_r}{k}, \qquad 1 \leqslant r \leqslant d-1$$

$$(M_d - M_{d-1})\frac{c_d}{k} = 1$$

The desired result follows by using the relation $k_r c_r = k_{r-1} b_{r-1}$, $1 \leqslant r \leqslant d$. □

In order to derive for the generating functions $G_r(u)$, $0 \leqslant r \leqslant d$, a more useful relation, we define a process $(Y_n, n \geqslant 0)$ on $\{0, 1,..., d\}$ by

$$Y_n = j \qquad \text{if and only if } X_b \in \Gamma_j, \qquad 0 \leqslant j \leqslant d$$

where, for a fixed vertex, $\Gamma_0, \Gamma_1,..., \Gamma_d$ are the set of nodes at distance $0, 1,..., d$. It is easy to verify that $(Y_n, n \geqslant 0)$ is a Markov chain on $\{0, 1,..., d\}$ with transition probabilities

$$\hat{P}(i, i+1) = \frac{b_i}{k}, \qquad \hat{P}(i, i) = \frac{a_i}{k}, \qquad \hat{P}(i, i-1) = \frac{c_i}{k}$$

with $c_0 = b_d = 0$. Hence the transition matrix $\hat{P} = (1/k)B^t$, where B is a tridiagonal matrix.

We state in the following lemma some known facts about the matrix B (Biggs,[8] pp. 140–143).

Lemma 3. Let G be a distance-regular graph with N vertices, valency k, and diameter d. Then

(i) G has $(d+1)$ distinct eigenvalues $\lambda_0 = k > \cdots > \lambda_d \geqslant -k$ which are the eigenvalues of B.

(ii) Let, for $0 \leqslant r \leqslant d$, $u_r = (u_r(i), 0 \leqslant i \leqslant d)^t$, $v_r = (v_r(i), 0 \leqslant i \leqslant d)^t$ be, respectively, the left and right eigenvectors of B associated to λ_r. Then, there are unique standard $[u_r(0) = v_r(0) = 1, 0 \leqslant r \leqslant d]$ eigenvectors u_r, v_r for which the inner product

$$(u_i, v_l) = 0, \qquad i \neq l, \quad \text{and} \quad (u_r, v_r) = \frac{N}{m(\lambda_r)}, \qquad 0 \leqslant r \leqslant d$$

where $m(\lambda_r)$ is the multiplicity of λ_r as eigenvalue of G.

From the lemma, B has the spectral representation

$$B = \sum_{r=0}^{d} \lambda_r \frac{v_r u_r^t}{(u_r, v_r)}$$

and, since $\hat{P} = (1/k) B^t$, it follows that

$$\hat{P}^n = \frac{1}{N} \sum_{r=0}^{d} \left(\frac{\lambda_r}{k}\right)^n m(\lambda_r)\, u_r v_r^t$$

But obviously

$$\begin{aligned} \hat{P}^n(0, 0) &= \hat{P}^n(x, x) \qquad \text{for any vertex } x \\ &= \frac{1}{N} \sum_{r=0}^{d} \left(\frac{\lambda_r}{k}\right)^n m(\lambda_r) \end{aligned}$$

and, for two vertices x, y at distance i,

$$\begin{aligned} \hat{P}^n(x, y) &= \frac{1}{k_i} \hat{P}^n(0, i) \\ &= \frac{1}{N} \sum_{r=0}^{d} \left(\frac{\lambda_r}{k}\right)^n m(\lambda_r) \frac{v_r(i)}{k_i} \end{aligned}$$

Hence, using the relation (3.1) we obtain the following representation for the hitting time generating functions:

Proposition 4. Under the notation introduced above, for $1 \leqslant i \leqslant d$, $|u| \leqslant 1$,

$$G_i(u) = \frac{1 + (1-u) \sum_{r=1}^{d} m(\lambda_r) \frac{v_r(i)}{k_i} \frac{1}{1 - u\lambda_r/k}}{1 + (1-u) \sum_{r=1}^{d} m(\lambda_r) \frac{1}{1 - u\lambda_r/k}} \tag{4.1}$$

and by differentiation,

$$M_i = \sum_{r=1}^{d} m(\lambda_r) \frac{1}{1 - \lambda_r/k} \left[1 - \frac{v_r(i)}{k_i}\right]$$

Since $Bv_r = \lambda_r v_r$ we have

$$v_r(0) = 1, \qquad v_r(1) = \lambda_r$$

$$b_{i-1} v_r(i-1) + a_i v_r(i) + c_{i+1} v_r(i+1) = \lambda_r v_r(i), \qquad 1 \leqslant i \leqslant d-1$$

From this three-term recurrence relation an interesting property can be derived. Indeed, it can be shown (see, for example, Bannai and Ito,[7] p. 197, Theorem 1.3) that

$$\sum_{r=0}^{d} v_r(i)\, v_r(j)\, m(\lambda_r) = N \cdot k_i \cdot \delta_{ij}$$

and that the $v_r(i)$'s are obtainable from a family of orthogonal polynomials.

A rich subclass of the distance-regular graphs is formed by the distance-transitive graphs (or two-point homogeneous graphs) defined as follows: for all vertices u, v, x, y of a graph G, with $d(u, v) = d(x, y)$, there is some automorphism g of the group of automorphism $H(G)$ such that $gu = x$, $gv = y$. By taking $u = v$, $x = y$ it follows that $H(G)$ acts transitively on the vertex set. Stanton[14] shows that $(H(G), K)$ (where $K = \{g \in H(G), gx = x\}$ is, for a fixed vertex x, the stabilizer subgroup) forms a Gelfand pair, so that in this case the $v_r(i)/k_i$'s are just the spherical functions of the Gelfand pair. In many cases the $v_r(i)$'s have been explicitly determined. Let us consider two examples.

Example 1: Hypercube

The vertices consist of m-tuples $(x_1,\dots, x_m)$, $1 \leqslant x_i \leqslant s$, $s \geqslant 2$, with two vertices are joined if their Hamming distance (number of distinct coordinates) is one. The covering time for the hypercube has been studied by Aldous,[3] Matthews,[12] and others for the case $s = 2$. Matthews has proved that $[T - 2^m(1 + 1/m)\log 2^m]/2^m(1 + 1/m)$ converges in law to $\exp(-e^{-x})$ as m tends to infinity. We will consider the cases when $s \to \infty$ or $m \to \infty$. For this family of graphs $N = s^m$, $d = m$, $m(\lambda_i) = k_i = (s-1)^i \binom{m}{i}$, $0 \leqslant i \leqslant m$, and $a_i = i(s-2)$, $1 \leqslant i \leqslant m$, $b_i = (m-i)(s-1)$, $0 \leqslant i \leqslant m-1$, $c_i = i$, $1 \leqslant i \leqslant m$. The $v_r(i)$'s are given by the Krawtchouk polynomials:

$$v_r(i) = \sum_{j=0}^{i} (-1)^j (s-1)^{i-j} \binom{r}{j} \binom{m-r}{i-j}$$

and in particular, $v_r(1) = \lambda_r = m(s-1) - sr$, $0 \leqslant r \leqslant d$.

Lemma 4. For the hypercube, we have, for $s \geqslant 2$,

$$\begin{aligned} M_m &\leqslant s^m \left(1 + \frac{8}{ms} + 2ms^{-m/2}\right) \\ &\geqslant s^m \left[1 + \frac{1}{(m-1)(s-1)} - \frac{2m}{(m-1)s^m}\right] \end{aligned}$$

Proof. The expression of M_i in Proposition 4 reduces to

$$M_i = \sum_{r=1}^{m} \binom{m}{r} (s-1)^r \cdot \frac{m(s-1)}{rs} \left[1 - \frac{v_r(m)}{k_m}\right]$$

But since

$$\frac{v_r(m)}{k_m} = \left(\frac{1}{1-s}\right)^r$$

it follows that

$$M_m = \frac{m(s-1)}{s} \sum_{r=1}^{m} \frac{s^r}{r}$$

$$\leqslant s^m \left[1 + \sum_{r=1}^{\infty} \frac{m}{s^r} \cdot \frac{1}{(m-r)(m-r+1)}\right]$$

By considering the sum for $r \leqslant m/2$ and $r > m/2$, respectively,

$$M_m \leqslant s^m \left[1 + \frac{m}{(m/2)^2} \sum_{r=1}^{\infty} \frac{1}{s^r} + m \sum_{r > m/2} \frac{1}{s^r}\right]$$

$$\leqslant s^m \left(1 + \frac{8}{ms} + 2ms^{-m/2}\right) \qquad \text{by using } s-1 \geqslant s/2$$

Lower bound follows trivially by using Lemma 2. □

The following proposition can be derived directly from Aldous (Ref. 5, Proposition 8). However, we use an analytic approach for some additional terms needed further on.

Proposition 5. For every $t < 1$, uniformly in i, as $m \to \infty$, $s \geqslant 3$, or as $s \to \infty$, $m \geqslant 2$,

$$G_i(e^{t/N}) \to \frac{1}{1-t}$$

To obtain this result, the existence of $G_i(u)$, for all i, in a ball of radius greater than 1, must be shown. For this, sufficient conditions are given by Matthews,[11] namely, that

$$\begin{aligned}
&1.\quad \frac{1}{|u|} > \frac{\lambda_1}{k} \\
&2.\quad -\frac{1}{|u|} > \frac{\lambda_{m-1}}{k} \\
&3.\quad 1 + (1-|u|) \sum_{i=1}^{m} \frac{m(\lambda_i)}{1-|u|\,\lambda_i/k} > 0 \\
&4.\quad -\frac{1}{|u|} \leqslant \frac{\lambda_m}{k}
\end{aligned} \tag{4.2}$$

Proof of Proposition 5. Let θ, θ' be constants (possibly depending upon s, m, N, t) such that $|\theta|, |\theta'| \leqslant 1$.

Define

$$Q = \sum_{r=1}^{d} \frac{m(\lambda_r)}{1-\lambda_r/k} \qquad \text{and} \qquad V = \frac{N-1}{(1-\lambda_1/k)^2}$$

Then, we have from (4.1) for $|u| \leqslant 1$,

$$G_i(u) = \frac{1-(1-u)M_i + (1-u)Q + (1-u)^2\,\theta V}{1+(1-u)Q+(1-u)^2\theta' V} \tag{4.3}$$

In this case we have

$$\frac{Q}{s^m} = \sum_{r=1}^{m} \binom{m}{r}\left(1-\frac{1}{s}\right)^r \left(\frac{1}{s}\right)^{m-r} \frac{m(s-1)}{sr}$$

As

$$\frac{1}{r/[m(1-1/s)]} = \frac{1}{1-\gamma} = 1+\gamma+\gamma^2+\gamma^3+\frac{\gamma^4}{1-\gamma}$$

using the moments of the binomial $(m, 1-1/s)$ gives

$$\frac{Q}{s^m} = 1 + \frac{1}{m(s-1)} + O\left(\frac{1}{m^2(s-1)}\right) \qquad \text{as} \quad m \to \infty$$

uniformly in $s \geqslant 3$, and $Q/s^m = 1 + O(1/m(s-1))$ as $s \to \infty$ uniformly in $m \geqslant 2$. Note that this is more than we need for now, but the additional terms are needed further on. For $-\infty < t < 0$, the existence of $G_i(e^{t/N})$ is known for all i. Consider $0 < t < 1$. Since $|\lambda_r/k| \leqslant 1$ and $e^{t/N} \to 1$ as $N \to \infty$ the conditions 1, 2, and 4 of (4.2) are clearly satisfied. For the condition 3, consider for N large

$$\begin{aligned}
&1+(1-e^{t/N}) \sum_{r=1}^{m} \frac{m(\lambda_r)}{1-e^{t/N}\lambda_r/k} \\
&= 1 - \left[\frac{t}{N} + O(N^{-2})\right] Q + O\left(\frac{m^2}{N}\right) \qquad \text{as} \quad N \to \infty \\
&= \begin{cases} 1 - t\left[1 + O\left(\dfrac{1}{m(s-1)}\right)\right] & \text{as } s \to \infty \\ 1 - t\left[1 + \dfrac{1}{m(s-1)} + O\left(\dfrac{1}{m^2(s-1)}\right)\right] & \text{as} \quad m \to \infty, \quad s \geqslant 3 \end{cases}
\end{aligned}$$

which, for $t<1$ and $m(s-1)$ large enough, is positive. Thus, condition 3 is satisfied. In (4.3) the common denominator is, for $u=e^{t/N}$, $1-t+O(1/m(s-1))$ as $s\to\infty$ or $1-t-t/m(s-1)+O(1/m^2(s-1))$ as $m\to\infty$, $s\geqslant 3$. The numerator of $G_1(e^{t/N})$ is, by the expansion (4.3), $1+O(1/m(s-1))$ as $s\to\infty$ or $1-t/m(s-1)+O(1/m^2(s-1))$ as $m\to\infty$. Then for $t<1$, N large

$$G_1(e^{t/N})=\begin{cases}\dfrac{1+O\left(\dfrac{1}{m(s-1)}\right)}{1-t+O\left(\dfrac{1}{m(s-1)}\right)} & \text{as} \quad s\to\infty\\[2ex] \dfrac{1-\dfrac{t}{m(s-1)}+O\left(\dfrac{1}{m^2(s-1)}\right)}{1-t-\dfrac{t}{m(s-1)}+O\left(\dfrac{1}{m^2(s-1)}\right)} & \text{as} \quad m\to\infty\end{cases}$$

Recall that all the O terms are uniform in the other parameter. From Lemma 4,

$$M_m=s^m\left[1+O\left(\frac{1}{ms}\right)\right] \qquad \text{as} \quad m\to\infty \quad \text{or} \quad s\to\infty$$

hence the numerator of $G_m(e^{t/N})$, is, for N large, $1+O(1/ms)$ as $m\to\infty$ or $s\to\infty$.

Thus, for $t<1$, N large

$$G_m(e^{t/N})=\begin{cases}\dfrac{1+O\left(\dfrac{1}{m(s-1)}\right)}{1-t+O\left(\dfrac{1}{ms}\right)} & \text{as} \quad s\to\infty\\[2ex] \dfrac{1+O\left(\dfrac{1}{ms}\right)}{1-t-\dfrac{t}{m(s-1)}+O\left(\dfrac{1}{m^2s}\right)} & \text{as} \quad m\to\infty\end{cases}$$

Clearly, as $t>0$, $G_1(e^{t/N})\leqslant G_2(e^{t/N})\leqslant \cdots \leqslant G_m(e^{t/N})$, with the inequalities reversed for $t\leqslant 0$. □

Matthews[11] obtained general bounds for the moment generating function of the covering time $T=T_{\{1,\dots,N\}}$ in the following form:

For any starting point 1

$$\frac{\Gamma(N)\,\Gamma(1/f^{-}(t))}{\Gamma(N-1+1/f^{-}(t))} \leqslant \mathbf{E}_1(e^{tT}) \leqslant \frac{\Gamma(N)\,\Gamma(1/f^{+}(t))}{\Gamma(N-1+1/f^{+}(t))} \tag{4.4}$$

where $f^{+}(t)=\max_{2\leqslant i\leqslant N}\max_{1\leqslant j\leqslant N, j\neq i}\mathbf{E}_j(e^{tT_i})$, $f^{-}(t)$ defined similarly with min instead of max.

For the hypercube case, note that for $t>0$, $f^{+}(t)=G_m(e^t)$ and $f^{-}(t)=G_1(e^t)$, and for $t\leqslant 0$, $f^{-}(t)=G_m(e^t)$, $f^{+}(t)=G_1(e^t)$. Since $G_i(e^{t/N})\to 1/(1-t)$ as $N\to\infty$ the bounds of (4.4) are tight (in the sense that RHS/LHS $\to 1$) whenever $G_m(e^{t/N})-G_1(e^{t/N})=o(1/\log N)$. By the expansion (4.3) this forces $(M_m-M_1)/N=o(1/\log N)$, that is, $ms/\log N\to\infty$, or $s\to\infty$. Hence, for the model $s\to\infty$, a direct use of the Matthews's bounds (4.4) would produce an asymptotic distribution for the covering time. For fixed $s=2$, Matthews[12] improves the lower bound to get a limit law. Similar arguments not produced here could be used for all fixed $s>2$.

Proposition 6. For the covering time T on the hypercube, for every $t<1$, and for any starting point, we have

$$\mathbf{E}(e^{t(T/N-\log N)})\to\Gamma(1-t) \qquad \text{as} \quad s\to\infty$$

That is to say, $(T-N\log N)/N$ converges in distribution, as $s\to\infty$, to the extreme value distribution $e^{-e^{-x}}$.

Proof. Consider for $0<t<1$, the bound

$$\mathbf{E}(e^{tT})\geqslant\frac{\Gamma(N)\cdot\Gamma(1/G_1(e^{t/N}))}{\Gamma(N-1+1/G_1(e^{t/N}))}$$

where, from the proof of Proposition 5,

$$G_1(e^{t/N})=\frac{1+O(1/ms)}{1-t+O(1/ms)} \qquad \text{for } s \text{ large, } t<1$$

On the other hand,

$$\begin{aligned}\frac{\Gamma(N)}{\Gamma(N-1+1/G_1(e^{t/N}))}&=N^{t+O(1/ms)}e^{O(1/N)} \qquad \text{as} \quad s\to\infty\\ &=N^t[1+o(1)]\end{aligned}$$

so the lower bound becomes, as $s\to\infty$

$$N^t\Gamma\left(\frac{1-t+O(1/ms)}{1+O(1/ms)}\right)[1+o(1)]$$

Since the bounds of (4.4) are tight for $s \to \infty$, and since for $t \leqslant 0$ the bounds are reversed, the conclusion follows. $\square$

Example 2: Binomial Coefficient Graphs

The set V of vertices consists of the $\binom{n}{m}$ subsets of size m of a set S of size n. Two vertices are joined if their intersection has cardinality $m-1$. Without loss of generality one can assume, by symmetry, that $m \leqslant n/2$. Formally, we have vertex set $V = \{S_1 \subset S,\ |S_1| = m\}$ and edge set $E = \{(S_1, S_2),\ d(S_1, S_2) = 1\}$ where $d(S_1, S_2) = m - |S_1 \cap S_2|$ is the minimal path distance between two nodes. The parameters of these graphs are, for $0 \leqslant i \leqslant d = m$,

$$a_i = i(n-2i), \qquad b_i = (m-i)(n-m-i), \qquad c_i = i^2$$

$$k_i = \binom{m}{i}\binom{n-m}{i}, \qquad m(\lambda_i) = \binom{n}{i} - \binom{n}{i-1}$$

and $v_r(i)$ are given by the Eberlein polynomials

$$v_r(i) = \sum_{j=0}^{i} (-1)^j \binom{r}{j}\binom{m-r}{i-j}\binom{n-m-r}{i-j}$$

In particular, $v_r(1) = \lambda_r = (m-r)(n-m-r+1) - m$, for $0 \leqslant r \leqslant m$. Similar results to the previous example can be obtained by the same procedure, in particular

Lemma 5. For the Binomial coefficient graph, we have

$$M_m \leqslant N - 1 + \frac{N-1}{(m-1)(n-m-1)}\left(1 + \frac{10}{n-m-2}\right)$$

$$\geqslant (N-1)\left[1 + \frac{1}{(m-1)(n-m-1)}\right]$$

provided that $m \geqslant 2$, $n \geqslant 2m+3$.

Proof. Here it is simpler to proceed by using the formula

$$M_m = \sum_{i=1}^{m} \frac{k}{k_{i-1} \cdot b_{i-1}} \sum_{j=i}^{m} k_j$$

Thus,

$$M_m \leqslant \sum_{i=1}^{m-2} \frac{N-1}{\binom{m-1}{i-1}\binom{n-m-1}{i-1}} + \sum_{i=m-1}^{m} \frac{1}{\binom{m-1}{i-1}\binom{n-m-1}{i-1}} \sum_{j=m-1}^{m} k_j$$

$$\triangleq \mathrm{I} + \mathrm{II}$$

The first term in I is $N-1$, the second term is $(N-1)/(m-1)(n-m-1)$. For the others note that $\binom{m-1}{i-1} \geqslant \binom{m-1}{2}$ provided that $i \leqslant m-2$. Therefore, for $n > m+2$,

$$\mathrm{I} \leqslant N-1+\frac{N-1}{(m-1)(n-m-1)}\left(1+\frac{4}{n-m-2}\right)$$

Finally, for $n \geqslant 2m+3$,

$$\begin{aligned}\mathrm{II}/(N-1) &\leqslant \frac{1}{\binom{n-m-1}{m-1}}+\frac{1}{(m-1)\binom{n-m-1}{m-2}}\\ &\leqslant \frac{6}{(n-m-1)(n-m-2)(n-m-3)}\left(1+\frac{1}{m-1}\right) \qquad \square\end{aligned}$$

Proposition 7. For every $t < 1$, uniformly in i, as $N \to \infty$ with $\limsup(m/n) = c < 1/2$,

$$G_i(e^{t/N}) \to \frac{1}{1-t}$$

Proof (outline). Consider

$$\begin{aligned}Q &= \sum_{r=1}^{m} \frac{m(\lambda_r)}{1-\lambda_r/k} = \sum_{r=1}^{m} \left\{\binom{n}{r}-\binom{n}{r-1}\right\} \frac{m(n-m)}{r(n+1-r)}\\ &= \binom{n}{m}\frac{n-m}{n-m+1} - \frac{m(n-m)}{n}\\ &\quad + \sum_{r=1}^{m-1} \binom{n}{m-r} \frac{m(n-m)[n-2(m-r)]}{(m-r)(m-r+1)(n-m+r+1)(n-m+r)}\\ &\triangleq \mathrm{I}+\mathrm{II}+\mathrm{III}\end{aligned}$$

Using

$$\binom{m}{m-j}\Big/\binom{n}{m} \leqslant \left(\frac{m}{n-m+1}\right)^j$$

we see that

$$\mathrm{III}\Big/\binom{n}{m} \leqslant \sum_{j=1}^{m-1} \left(\frac{m}{n-m+1}\right)^j \cdot \frac{m(n-m)}{(n-m+j)(m-j)^2}$$

Considering the sum for $1 \leqslant j \leqslant [\sqrt{m}]$ and $[\sqrt{m}] < j \leqslant m-1$, respectively, and replacing finite sums by infinite geometric sums, we get

$$\mathrm{III}\Big/\binom{n}{m} \leqslant \left(\frac{m}{m-\sqrt{m}}\right)^2 \frac{1}{n-2m+1} + m\left[\frac{1-c+o(1)}{1-2c}\right]\left[\frac{c}{1-c}+o(1)\right]^{[\sqrt{m}]}$$

Hence, by using the estimates $M_m = (N-1)[1 + O(1/(m(n-m)))]$ and $Q = N[1 + O(1/(n-m))]$, the proof follows in the same way as that of Proposition 5. □

As in the previous example, the following can be shown.

Proposition 8. For $m \geqslant 2$, $m = o(n/\log n)$, the normalized covering time $(T - N\log N)/N$ converges in distribution to the extreme value distribution $e^{-e^{-x}}$.

ACKNOWLEDGMENTS

Research was sponsored by NSERC grant No. A3456 and FCAR grant No. EQ-1679.

REFERENCES

1. Aldous, D. J. (1983). Random walks on finite groups and rapidly mixing Markov chains. In *Séminaire de Probabilités XVII* (Lecture Notes in Mathematics 986), Springer-Verlag, Berlin, pp. 243–297.
2. Aldous, D. J. (1983). On the time taken by random walks on finite groups to visit every state. *Z. Wahrsch. verw. Gebiete* **62**, 361–393.
3. Aldous, D. J. (1983). Minimization algorithms and random walk on the *d*-cube. *Ann. Prob.* **11**, 403–413.
4. Aldous, D. J. (1989). Lower bounds for covering times for reversible Markov chains and random walks on graphs. *J. Theoret. Prob.* **2**, 91–100.
5. Aldous, D. J. (1989). Hitting times for random walks on vertex-transitive graphs. *Math. Proc. Cambridge Phil. Soc.* **106**, 179–191.
6. Aleliunas, R., Karp, R. M., Lipton, R. J., Lovasz, L., and Rackoff, C. (1979). Random walks, universal transversal sequences, and the complexity of maze problems. In Proceedings of the 20th Annual Symposium on the Foundations of Computer Science, pp. 218–223.
7. Bannai, E., and Ito, T. (1984). *Algebraic Combinatorics I.* Benjamin, New York.
8. Biggs, N. (1974). *Algebraic Graph Theory.* Cambridge University Press, Cambridge.
9. Diaconis, P. (1988). *Group Representation in Probability and Statistics.* IMS-Lecture notes—Monograph series. Vol. 11, IMS, Hayward, CA.
10. Matthews, P. (1988). Covering problems for Brownian motion on spheres. *Ann. Prob.* **16**, 189–199.
11. Matthews, P. (1988). Covering problems for Markov chains. *Ann. Prob.* **16**, 1215–1228.
12. Matthews, P. (1989). Some sample path properties of random walk on the cube. *J. Theor. Prob.* **2**, 129–146.
13. Mazo, J. E. (1982). Some extremal Markov chains. *Bell Syst. Tech. J.* **61**, 2065–2080.
14. Stanton, D. (1984). Orthogonal polynomials and Chevalley groups. In R. A. Askey *et al.*, *Special Functions: Group Theoretical Aspects and Applications*, pp. 87–92, D. Reidel, Dordrecht, Boston.