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AN AUTOMATIC METHOD FOR GENERATING RANDOM VARIATES WITH A GIVEN CHARACTERISTIC FUNCTION*

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Abstract. An automatic method is developed for the computer generation of random variables with a characteristic function satisfying certain regularity conditions. The method is based upon a generalization of the rejection method and exploits the duality between densities and their Fourier transforms. It takes finite time almost surely, does not use approximations or inversions, and does not require explicit knowledge of the characteristic function (only its computability is assumed—hence the adjective "automatic"). As a by-product, we show how the sum of n independent random variables with common density f can be generated in time essentially independent of n, at least when its characteristic function satisfies the above mentioned regularity conditions.

Key words. random number generation, characteristic functions, rejection method, expected time complexity, probability inequalities, stable distribution, simulation, convexity, Polya's criterion, Fourier analysis

AMS(MOS) subject classification. Primary 65C10

Computing Reviews categories. CR 8.1, 5.5, 5.25

1. Introduction. We are given a characteristic function $\phi(t)$, $t \in R$, for a real-valued random variable X, and are asked to generate a random variate X on a computer, assuming that this computer can (i) produce an i.i.d. (independent identically distributed) sequence of uniform $[0, 1]$ random variables $U_1, U_2, \cdots$; and (ii) and manipulate real numbers. Both (i) and (ii) are unrealistic assumptions that we will not try to defend. The methods outlined below are exact if we are willing to accept (i) and (ii).

To get a feeling for the problem, we consider the simplest approach first: if F is the distribution function of X, then the solution X of $F(X) = U_1$ has distribution function F. F itself can be obtained from ϕ by standard inversion techniques (see e.g. Kawata (1972, pp. 128–131)). If we assume that all computer operations ($+$, $-$, $\times$, $/$, move, compare, truncate, log, exp, sin, cos, power) take a constant time regardless of the value(s) of the operand(s), then integrals and infinite sums take infinite time, so that the inversion of ϕ can be done with infinite precision only in infinite time. But even if we could compute F from ϕ in a finite amount of time, the numerical solution of $F(X) = U_1$ would require an infinite number of iterations, and thus eliminate this method from contention.

An *automatic method* assumes only that $\phi(t)$ can be computed for all $t \in R$ in one unit of time or less, and that ϕ belongs to Φ, a suitably general class of characteristic functions. We should think of ϕ not as a function that is directly given to us, but rather as a subprogram of sorts with unknown contents.

We should report here some attempts at obtaining algorithms for special classes of characteristic functions. Devroye (1981) considered the class of all characteristic

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functions with two derivatives and finite values for $\int|\phi|$ and $\int|\phi''|$. His method is semi-automatic in that it takes a finite time if the inversion

$$f(x)=(2\pi)^{-1}\int e^{-itx}\phi(t)\,dt$$

can be done in one unit of time. For the class of Polya characteristic functions, the author (1984) noted that the density is a mixture of densities of the form $(\sin x/x)^2$, where the mixing distribution requires explicit knowledge of ϕ (and this violates our definition of an automatic method). Yet, this approach is in a sense practical because it leads to fast exact algorithms for symmetric stable distributions and Linnik–Lukacs distributions (see § 3).

The principles outlined below for an automatic method are universal, and can be used for many a class of characteristic functions Φ. For readability, we will restrict the discussion to a few simple but interesting classes. We will in particular study

$$M=\left\{\phi\colon \phi \text{ is real, convex on } [0,\infty), \text{ and } \int|\phi|<\infty\right\}.$$

Note that ϕ is necessarily monotone on $[0,\infty)$, and that to each ϕ in M corresponds a bounded continuous density on R (Feller (1966, p. 482)). M comprises quite a few important distributions (such as the symmetric stable distribution; see also § 3), and has great didactical value due to its simplicity.

Let us look at parametrized sub-classes of M, and let us call the parameter(s) θ and the subclasses M_θ. Then $M=\bigcup_\theta M_\theta$, by assumption. The sub-classes are such that it is easy to find a uniform dominating curve g_θ, i.e., a function satisfying

$$\sup_{f\in M_\theta} f(x)\leqq g_\theta(x), \qquad x\in R.$$

Here "f" stands for a density corresponding to a characteristic function in M_θ, so that the notation $f\in M_\theta$ is unambiguous. If we can manage to find an integrable g_θ and to generate random variates from the density $g_\theta/\int g_\theta$, then the following *rejection algorithm* would be valid:

A1. REPEAT Generate X with density $g_\theta/\int g_\theta$.
  Generate an independent random variate U uniformly on $[0,1]$.
  Compute $f(X)=(2\pi)^{-1}\int_{-\infty}^{\infty}\cos(tX)\phi(t)\,dt$.
 UNTIL $Ug_\theta(X)\leqq f(X)$.
 EXIT with X.

For a general introduction to random variate generation and in particular to the inversion and rejection methods, see e.g. Bratley, Fox and Schrage (1983) or Schmeiser (1977). The rejection method is usually attributed to von Neumann (1951). For example, the expected number of iterations in A1 is $\int g_\theta$: it is the same for all $f\in M_\theta$.

Unfortunately, the inversion cannot be done in a finite amount of time unless we can find a way to avoid it. The principle that we will apply here is based upon the observation that the decision $Ug_\theta(X)\leqq f(X)$ does not require the exact computation of $f(X)$ (see §§ 4, 5, 6 and 8). In fact, the algorithm of § 6 never requires the computation of f at any point! Because automatic methods for characteristic functions go to the heart of the matter, i.e. the duality between characteristic functions and densities, we need some tools to quantify this duality. This is done via inequalities in § 2. Thus, we have inequalities relating the tail of ϕ to the smoothness of f, and relating the behavior of ϕ near 0 to the tails of f. Unlike asymptotic theory in mathematical statistics, it does not suffice to have inequalities with $O(\cdot)$ and $o(\cdot)$ terms hidden away. The inequalities of § 2 will later be used in the constructive phase of our work.

To find dominating curves g_θ is relatively easy (see § 4). It is another matter to verify whether the acceptance condition is satisfied. The trick is to rewrite f as an integral with a positive integrand. This is basically why ϕ is required to be convex. Once f is represented by such an integral (§ 5), it is relatively straightforward to generalize the rejection algorithm so that f does not have to be computed exactly. Instead, the positive integrand must be computed in every iteration of the algorithm (§§ 3 and 6). The remainder of the paper deals with analysis and applications. Various statements about the moments of the complexity are given in §§ 7 and 8. In § 9, we show how for some classes of distributions, the sum of i.i.d. random variables can be simulated in time essentially independent of n. This is the case when the sequence of characteristic functions ϕ, ϕ^2, ϕ^3, $\phi^4, \cdots$ of successive partial sums are all in the class under consideration. This result is only possible because we have attraction to some symmetric stable law.

We will require some knowledge of modern rejection algorithms and squeeze steps. For careful explanations or insightful applications of the rejection and squeeze principles, see e.g. Marsaglia (1977), Ahrens and Dieter (1974), Schmeiser and Lal (1980), Best (1978) and Cheng and Feast (1980).

2. Inequalities for characteristic functions. The inequalities given here will help us to construct good dominating curves g_θ, and to replace the inversion integral by a finite series with random finite length. The main reference for the proofs not given here is Kawata (1972). *We will assume throughout this section that ϕ is real,* although most inequalities have versions that are generally applicable.

LEMMA 2.1 (the Raikov-Wiener inequality, Raikov (1940), Wiener (1925)).

$$|\phi(t+h)-\phi(t)| \leqq \sqrt{2(1-\phi(h))}, \qquad t, h \in R.$$

LEMMA 2.2 (the four-point inequality).

$$|\phi(t)-\phi(t+h)-\phi(t+u)+\phi(t+h+u)| \leqq \sqrt{2(1-\phi(h))}\sqrt{2(1-\phi(u))}, \qquad t, h, u \in R.$$

Proof of Lemma 2.2. Use the fact that by Bochner's theorem (Bochner (1959), see Kawata (1972, pp. 376-377)), ϕ is nonnegative definite.

LEMMA 2.3. *When $\phi \in M$, then for t, u, $h \geqq 0$, $0 \leqq \Delta \leqq \phi(t)$, where Δ is defined by* $\phi(t)+\phi(t+u+h)-\phi(t+u)-\phi(t+h)$.

LEMMA 2.4 (Kawata (1972, pp. 438-439)). *If $\int |t|^p|\phi(t)|\,dt<\infty$ for $p \geqq 0$ integer, then*

$$f^{(p)}(x)=\frac{(-i)^p}{2\pi}\int e^{-itx}t^p\phi(t)\,dt.$$

Thus,

$$\sup_x |f^{(p)}(x)| \leqq (2\pi)^{-1}\int |t|^p|\phi(t)|\,dt.$$

LEMMA 2.5 (Van Bahr-Esseen type inequalities).

$$|1-\phi(t)| \leqq C_r|t|^r E(|X|^2)$$

where

(i) $C_r \leqq \dfrac{3.38}{2.6^r}$, $\quad 1 \leqq r \leqq 2$ (*Van Bahr and Esseen* (1965)).

(ii) $C_r \leqq 2^{1-r}, \qquad 0 < r \leqq 2.$

(iii) $C_r \leqq \dfrac{\pi}{2\Gamma(r)\sin(\pi r/2)}, \qquad 0 < r \leqq 2 \quad (when\ f \in M).$

Proof of Lemma 2.5, *parts* (ii) *and* (iii). By definition of ϕ,

$$1-\phi(t) = 2\int_0^\infty (1-\cos(xt))f(x)\,dx = 4\int_0^\infty \sin^2\left(\frac{xt}{2}\right)f(x)\,dx$$

$$\leqq E(|X|^r)\sup_{x>0} 2\sin^2\left(\frac{xt}{2}\right)x^{-r}$$

$$= E(|X|^r)|t|^r \sup_{u>0} 2u^{-r}\sin^2\left(\frac{u}{2}\right).$$

But $\sin^2(u/2) \leqq \min(1, u^2/4)$, so that the supremum on the right-hand side of the chain of inequalities is not greater than

$$\sup_{u>0} \min(2u^{-r}, u^{2-r}/2) = 2^{1-r}$$

since the supremum of the last expression must necessarily occur at $u=2$. This concludes the proof of (ii).

To show (iii), we start from the following expression:

$$E(|X|^r) = \frac{2\Gamma(r+1)}{\pi}\sin\left(\frac{\pi r}{2}\right)\int_0^\infty \frac{1-\phi(t)}{t^{1+r}}\,dt$$

(Chung (1974, pp. 158–159)). The integral in this expression is greater than $(1-\phi(t))/(r|t|^r)$ for all $t \neq 0$ (by monotonicity of ϕ). Now, (iii) follows when we note that $\Gamma(r+1) = r\Gamma(r)$.

Remark 2.1. (*About the constant* C_r.) Kawata (1972, p. 430) offers a seemingly different expression for $E(|X|^r)$ than the one used in the proof. It leads to the value

$$C_r = \frac{\sin((1-r)\pi/2)\Gamma(2-r)}{1-r}$$

in (iii). The equivalence between this C_r and the one given in (iii) follows from the well-known identity (see e.g. Whittaker and Watson (1927))

$$\Gamma(1-r)\Gamma(r) = \frac{\pi}{\sin(\pi r)}, \qquad r \in (0,1).$$

Which of the three inequalities in Lemma 2.6 is actually smallest depends upon r. The value of C_r in (iii) approaches 1 as $r \downarrow 0$, equals $\pi/2$ at $r=1$, and increases to ∞ as $r \uparrow 2$.

LEMMA 2.6. *Let s be a nonnegative integer, and let $E(|X|^{2s}) < \infty$ (this is equivalent to ϕ being $2s$ times differentiable at the origin, and in fact $E(X^{2s}) = (-1)^s\phi^{(2s)}(0)$, see e.g. Kawata* (1972, *pp.* 410–411). *Then*

$$\sup_x x^{2s}f(x) \leqq \frac{1}{2\pi}\int |\phi^{(2s)}(t)|\,dt.$$

(*Note.* The right-hand side of this inequality is not necessarily finite.)

Proof of Lemma 2.6. With $b = E(X^{2s})$, $x^{2s}f(x)/b$ is a density with characteristic function $(-1)^s\phi^{(2s)}(t)/b$, and the result follows from Lemma 2.4.

LEMMA 2.7. *Let ϕ be a nonnegative real-valued characteristic function with density f, and let $\int t^2\phi(t)\,dt<\infty$. Then f'' exists everywhere and*

$$\sup_t t^2\phi(t)\leqq\int |f''|.$$

Proof of Lemma 2.7. Apply Lemma 2.6 to the "density" $\phi/\int\phi$ (considered for $t\in R$), which has "characteristic function" $2\pi f/\int\phi$ (where x is the running variable now).

As an immediate corollary of Lemma 2.6, we have the following global bound for f:

LEMMA 2.8. *Let s be a positive integer, and assume that $E(|X|^{2s})<\infty$. Then, if f is the density of an absolutely integrable characteristic function ϕ,*

$$f(x)\leqq(2\pi)^{-1}\min\left(\int|\phi|,\int|\phi^{(2s)}|/x^{2s}\right).$$

We note that the bound in Lemma 2.8 is integrable. An easy computation shows that its integral is equal to

$$\frac{1}{\pi}\frac{2s}{2s-1}\left(\int|\phi|\right)^{1-1/2s}\left(\int|\phi^{(2s)}|\right)^{1/2s}.$$

Thus, this integral would be the expected number of iterations in the rejection Algorithm A1 when it is applied to densities f bounded as shown in Lemma 2.8. In § 8, we will show how random variates with density $g/\int g$ (where g is the upper bound for f) can be generated. Algorithm A1 would of course also be applicable to densities with smaller values for $\int|\phi|$ and $\int|\phi^{(2s)}|$. It is interesting to note that the efficiency of the algorithm is good when f is small ($\int|\phi|$ measures a uniform bound for f), and has small tails ($\int|\phi^{(2s)}|$ measures the smallness of the tails of f). As pointed out in Devroye (1981), the integral shown above is relatively close to 1, and thus acceptable, in many important cases. For example, for the Cauchy distribution, it is equal to $4/\pi$.

3. A class of densities. The parametrization of the class M is done as follows: let M_θ be the class of all densities f (with corresponding characteristic function ϕ) satisfying:

(i) ϕ is real, convex on $[0,\infty)$ and absolutely integrable. (Note that this implies that ϕ is symmetric, nonnegative, and nonincreasing on $[0,\infty)$. Also, $E(|X|)=\infty$.)

(ii)

$$\sup_t |t|^{1+\alpha}\phi(t)\leqq A,$$

where $A>0$ and $\alpha\in(0,1]$ are constants.

(iii)

$$\sup_t\frac{1-\phi(t)}{|t|^\beta}\leqq B,$$

where $B>0$ and $\beta\in(0,1]$ are constants.

(iv)

$$\frac{1}{\pi}\int_0^\infty\phi(t)\,dt=C<\infty.$$

The fact that (i) implies $E(|X|)=\infty$ can be shown as follows. By Taylor's series expansion, we know that there exists a function $z(x)$ with $0\leqq z(x)\leqq x$, such that

$$\begin{aligned}1-\phi(t)&=2\int_0^\infty tz(x)\sin(tz(x))f(x)\,dx\\&\leqq 2t^2\int_0^{\delta/t} z^2(x)f(x)\,dx+2\int_{\delta/t}^\infty txf(x)\,dx\\&\leqq 2\delta t\int_0^{\delta/t} xf(x)\,dx+2t\int_{\delta/t}^\infty xf(x)\,dx.\end{aligned}$$

If $\int |x|f(x)\,dx<\infty$, then, as $t\downarrow 0$, the last term is $o(t)$, and the first term $\sim \delta t\int |x|f(x)\,dx$. Thus,

$$\int |x|f(x)\,dx\geqq\frac{1}{\delta}\limsup_{t\downarrow 0}\frac{1-\phi(t)}{t}.$$

Since the given limit supremum is positive (by convexity), and δ is arbitrary, we conclude that $\int |x|f(x)\,dx$ cannot be finite.

Remark 3.1. We will refer to A, B, C, α and β as the parameters of the class M_θ, even though they are not parameters in the classical sense of the word, i.e. they do not uniquely characterize one member in the class. All parameters must be known before we can apply the algorithm to be given below: we must know to which M_θ our f belongs. What is particularly bothersome is that C must be known exactly. For A and B we can get away with upper bounds. The fact that we must know to which M_θ, f belongs is considered by us as the major drawback of the algorithm that will follow. The determination of the parameters should of course always be deferred to a preprocessing step.

Condition (ii) is smoothness condition on f. For example, for $\alpha=1$, we obtain from Lemma 2.7:

$$\sup_t t^2\phi(t)\leqq\int |f''|.$$

Thus, we can replace A by $\int |f''|$ in the few instances where the latter quantity is known.

The supremum of (iii) need not be computed if the βth moment of $|X|$ is known. By the generalization of the Van Bahr–Esseen inequality (Lemma 2.5, (ii)), we see that we can take $B=2^{1-\beta}E(|X|^\beta)$. One should of course realize that something is lost if B is chosen in this way.

The finiteness of C follows from (ii). In fact, for all real ϕ,

$$\pi C\leqq\frac{\alpha+1}{\alpha}A^{1/(\alpha+1)},$$

where $A=\sup |t|^{1+\alpha}\phi(t)$. To see this, we argue as follows:

$$C=\int_0^\infty\phi(t)\,dt\leqq\int_0^s dt+\int_s^\infty At^{-(1+\alpha)}\,dt=s+\frac{A}{\alpha}s^{-\alpha}=\frac{\alpha+1}{\alpha}A^{1/(\alpha+1)}$$

if we take $s^{\alpha+1}=A$, a choice which minimizes the upper bound.

After these introductory remarks about M_θ, we shall proceed with the description of some characteristic functions that belong to the class. First, it is well known that any real even continuous function taking the value 1 at 0 and convex on $[0,\infty)$ is a characteristic function (this is *Polya's criterion* (Polya (1949)). Fortunately, most characteristic functions satisfying (i) also satisfy (ii) and (iii) for some values of the parameters. More importantly, products of functions satisfying (i)-(iii) for some parameters satisfy (i)-(iii) for some other values of the parameters, an observation that will be crucial in the simulation of sums (§ 9). Thus, we have in our class $M^* = UM_\theta$ (note that M^* is strictly included in M):

$\phi(t) = (1-|t|)_+^\alpha, \quad \alpha \geqq 1;$

$\phi(t) = (1-|t|^\alpha)_+, \quad 0<\alpha \leqq 1;$

$\phi(t) = \exp(-|t|^\alpha), \quad 0<\alpha\leqq 1$ (the *symmetric stable* distribution);

$\phi(t) = (1+|t|^\alpha)^{-1}, \quad 0<\alpha\leqq 1$ (the Linnik-Lukacs distribution).

The symmetric stable distribution. The methods developed below can be applied to the symmetric stable distribution. This is an instance in which the parameters of the distribution are known beforehand, yet f is not known except as an integral or an infinite series. For example, taking $\alpha = 1$ in (ii), we have

$$A = \sup t^2 \phi(t) = \sup_{t>0} t^2 e^{-|t|^a} = \left(\frac{2}{ae}\right)^{2/a},$$

where $a \in (0, 1]$ is the parameter of the stable distribution. Also,

$$\pi C = \int_0^\infty e^{-t^a}\, dt = \Gamma\left(\frac{1}{a}+1\right),$$

and

$$B = \sup_{t>0} (1-e^{-t^a})/t^\beta$$

is finite for all $\beta \leqq a$. In fact, for $\beta = a$, $B = 1$.

For direct exact generators for the symmetric stable distribution, see Chambers, Mallows and Stuck (1976) and Devroye (1984).

Densities with bounded spectrum. We say that a density has a bounded spectrum when its characteristic function ϕ vanishes outside some finite interval $[-T, T]$. In many applications, T is known. Since this imposes a smoothness condition on f, we can expect that A and C can be bounded in terms of T. By convexity, we have $\phi(t) \leqq (1-|t/T|)_+$, and a simple argument shows that on $[0, T]$, $t^{1+\alpha}(1-t/T)$ is maximal for $t = T(1+\alpha)/(2+\alpha)$. Thus,

$$A = \sup |t|^{1+\alpha}\phi(t) \leqq T^{1+\alpha}\frac{(1+\alpha)^{1+\alpha}}{(2+\alpha)^{2+\alpha}}.$$

Also, $C \leqq T/2\pi$.

4. A general rejection principle. We will develop a method here that will allow us to avoid the inversion integral for characteristic functions, by introducing an additional randomization in the t-domain. Assume that f, our density, can be represented as follows:

$$f(x) = \int g(t, x) h(t, x)\, dt$$

where

(i) $g(t, x)$ is a density in t for all x;

(ii) $0 \leqq h(t, x) \leqq H(x)$ for all t, x, where $\int H < \infty$.

Note that f is *not* written here as a standard mixture since both g and h depend upon t and x! From condition (i) we also conclude that $f \leqq H$. The following algorithm produces a random variate X with density f:

A2. REPEAT Generate X with density $H/\int H$.
Generate T with density $g(\cdot, X)$.
Generate a uniform $[0, 1]$ random variate U, independent of (X, T).
UNTIL $UH(X) \leqq h(T, X)$.
EXIT with X.

Proof. To prove our statement, we can argue as follows. Let E be the event "an exit occurs in the first iteration", and let A be an arbitrary Borel set of R. Then

$$P(E|X = x) = \int g(t, x) \frac{h(t, x)}{H(x)} \, dt = \frac{f(x)}{H(x)}.$$

Thus, if X is the random variate obtained when the algorithm stops, then

$$P(X \in A, E) = \int_A P(E|X = x) \frac{H(x)}{\int H} \, dx = \int_A \frac{f}{\int H}.$$

Also, $P(E) = 1/\int H$. By the independence between iterations,

$$P(X \in A) = \sum_{i=0}^{\infty} \left(1 - \frac{1}{\int H}\right)^i \int_A \frac{f}{\int H} = \int_A f.$$

Since A was arbitrary, X must have a density a.e. equal to f.

The number of iterations N in Algorithm A2 is geometrically distributed:

$$P(N = i) = \frac{1}{\int H}\left(1 - \frac{1}{\int H}\right)^{i-1}, \qquad i \geqq 1,$$

and

$$E(N) = \int H.$$

5. Representations for densities. In this section we will construct representations for f in the form needed for Algorithm A2 (§ 4). We will derive a representation which will be valid for small $|x|$ in Lemma 5.2. For large $|x|$, another representation is needed, such as the one provided by Lemma 5.3. We have to determine a threshold value below which Lemma 5.2 is used. This is done by minimizing the integral under the function H will respect to the threshold value: see Lemma 5.4. For future reference, we will need the following fact:

LEMMA 5.1. *For* $\alpha \in (0, 1]$, $x > 0$,

$$\int_0^{\infty} \frac{1 - \cos(tx)}{t^{\alpha+1}} \, dt = C_\alpha x^\alpha,$$

where

$$C_\alpha = \frac{\Gamma(1-\alpha)}{\alpha} \cos\left(\frac{\pi\alpha}{2}\right) = \frac{\pi}{2\Gamma(\alpha+1) \sin(\pi\alpha/2)}.$$

Proof. See Feller (1966, pp. 542–543) or Chung (1974, pp. 158–159). The equivalence of both formulas follows from the identities $\sin(2u) = 2\sin(u)\cos(u)$, and $\Gamma(1-\alpha)\Gamma(\alpha) = \pi/\sin(\pi\alpha)$.

LEMMA 5.2. *Let $f \in M_\theta$ with parameters A, B, C, α and β as defined in* § 3. *Then, for $|x| \leqq (\pi C/AC_\alpha)^{1/\alpha}$, we have $f(x) = \int g(t,x)h(t,x)\,dt$ for the following functions g and h satisfying the conditions of* § 4:

$$g(t,x) = \frac{2\sin^2(tx/2)}{C_\alpha |x|^\alpha t^{\alpha+1}}, \qquad t > 0,$$

$$h(t,x) = C - \frac{1}{\pi} C_\alpha |x|^\alpha t^{\alpha+1} \phi(t) \leqq C = H(x).$$

Proof. To obtain our representation, we start from the inversion formula for characteristic functions (applicable because ϕ is real, even and absolutely integrable):

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_0^\infty \cos(tx)\phi(t)\,dt \\ &= \frac{1}{\pi}\int_0^\infty \phi(t)\,dt - \frac{1}{\pi}\int_0^\infty (1-\cos(tx))\phi(t)\,dt \\ &= C - \frac{1}{\pi}\int_0^\infty 2\sin^2\left(\frac{tx}{2}\right)\phi(t)\,dt \\ &= C\int_0^\infty g(t,x)\,dt - \int_0^\infty g(t,x)\frac{1}{\pi}C_\alpha |x|^\alpha t^{\alpha+1}\phi(t)\,dt \\ &= \int_0^\infty g(t,x)h(t,x)\,dt. \end{aligned}$$

This concludes the proof of Lemma 5.2.

We have not used the convexity condition (i) and the tail condition (iii) in the definition of M_θ. These will however become crucial in the following representation:

LEMMA 5.3. *Let $f \in M_\theta$ with parameters A, B, C, α and β as defined in* § 3. *Then, for all $x \neq 0$, we have $f(x) = \int g(t,x)h(t,x)\,dt$ for the following functions g and h satisfying the conditions of* § 4:

$$g(t,x) = |x|\cos(tx), \qquad 0 \leqq t \leqq \frac{\pi}{2|x|},$$

$$h(t,x) = \frac{1}{\pi|x|}\sum_{j=0}^\infty \psi_{j,t}(x) \leqq B\pi^{\beta-1}(2+2^{\beta-1})/|x|^{1+\beta} = H(x).$$

Here

$$\psi_{j,t}(x) = \phi\left(t + \frac{2\pi j}{|x|}\right) - \phi\left(t + \frac{2\pi j + \pi}{|x|}\right) - \phi\left(\frac{\pi}{|x|} - t + \frac{2\pi j}{|x|}\right) + \phi\left(\frac{\pi}{|x|} - t + \frac{2\pi j + \pi}{|x|}\right).$$

Proof of Lemma 5.3. Let us first verify that g and h are valid functions: g is indeed a density in t for all $x \neq 0$. Furthermore, $h \geqq 0$ because each $\psi_{j,t}$ is nonnegative (by the convexity of ϕ on $[0,\infty)$; note that $t \in [0, \pi/(2|x|)]$ when verifying this). This leaves us with two tasks: first, we should prove that $h(t,x) \leqq H(x)$, and then, we must make sure that f does indeed have the said representation.

By eliminating the periodicity of $\cos(tx)$, we have

$$f(x) = \frac{1}{\pi}\int_0^\infty \cos(tx)\phi(t)\,dt = \frac{1}{\pi}\int_0^{\pi/(2|x|)} \cos(tx)\sum_{j=0}^\infty \psi_{j,t}(x)\,dt,$$

from which the representation follows. Let ϕ_l' and ϕ_r' be the left- and right-hand derivatives of ϕ. By convexity, we have for $0<t<s<\infty$,

$$|\phi_r'(0)|\geqq|\phi_l'(t)|\geqq|\phi_r'(t)|\geqq|\phi_l'(s)|\geqq|\phi_r'(s)|.$$

Thus, for $x>0$,

$$\begin{aligned}\psi_{j,t}(x)&\leqq\left(\frac{\pi}{x}-2t\right)\left(\left|\phi_l'\left(t+\frac{2\pi j}{x}\right)\right|-\left|\phi_r'\left(\frac{\pi}{x}-t+\frac{2\pi j+\pi}{x}\right)\right|\right)\\&\leqq\frac{\pi}{x}\left(\left|\phi_l'\left(\frac{2\pi j}{x}\right)\right|-\left|\phi_r'\left(\frac{2\pi j+\pi}{x}\right)\right|\right).\end{aligned}$$

Hence, using the convexity, we have for positive integers J,

$$\sum_{j=J}^{\infty}\psi_{j,t}(x)\leqq\frac{\pi}{x}\left|\phi_l'\left(\frac{2\pi J}{x}\right)\right|\leqq\frac{\pi}{x}\frac{1-\phi(2\pi J/x)}{(2\pi J/x)}.$$

Now, by the four-point inequality (Lemma 2.2),

$$\begin{aligned}h(t,x)=\frac{1}{\pi x}\sum_{j=0}^{\infty}\psi_{j,t}(x)&\leqq\frac{1}{\pi x}\left(\psi_{0,t}(x)+\frac{1-\phi(2\pi/x)}{2}\right)\\&\leqq\frac{1}{\pi x}\left(2\sqrt{(1-\phi(\pi/x))(1-\phi(\pi/x-2t))}+\frac{1}{2}\left(1-\phi\left(\frac{2\pi}{x}\right)\right)\right)\\&\leqq\frac{1}{\pi x}\left(2(1-\phi(\pi/x))+\frac{1}{2}(1-\phi(2\pi/x))\right).\end{aligned}$$

But we can use assumption (iii): $1-\phi(t)\leqq Bt^\beta$, $t>0$, to bound the last expression from above. The bound then formally reads

$$B\frac{1}{\pi x}\left(2\left(\frac{\pi}{x}\right)^\beta+\frac{1}{2}\left(\frac{2\pi}{x}\right)^\beta\right)=H(x).$$

This concludes the proof of Lemma 5.3.

We are now in a position to give a global representation for f, valid for all $x\neq 0$, by merging Lemmas 5.2 and 5.3 together. Note that the representation of Lemma 5.3 is also valid for all $x\neq 0$, but that the dominating function H is not integrable. It is precisely the integrability of H that is required in Algorithm A2. In fact, we should merge Lemmas 5.2 and 5.3 in such a manner that the integral under the new function H is minimal. All of this is captured in the following lemma, which follows directly from Lemmas 5.2 and 5.3.

LEMMA 5.4. *Let $f\in M_\theta$ with parameters A, B, C, α and β as defined in* § 3. *Define*

$$x_0'=\left(\frac{\pi C}{C_\alpha A}\right)^{1/\alpha},\quad x_0''=\left(\frac{DB}{C}\right)^{1/(\beta+1)},\quad x_0=\min(x_0',x_0''),$$

where $D=\pi^{\beta-1}(2^{\beta-1}+2)$. Then $f(x)=\int g(t,x)h(t,x)\,dt$ for all $x\neq 0$, where g and h are as in Lemma 5.2 when $|x|\leqq x_0$, and as in Lemma 5.3 otherwise. In particular, the dominating curve H is

$$H(x)=\begin{cases}C, & |x|\leqq x_0,\\ DB/|x|^{1+\beta}, & |x|>x_0,\end{cases}$$

and

$$\int H=2\left(Cx_0+\frac{DB}{\beta x_0^\beta}\right).$$

The cut-off point x_0 between the bounds C and $BD/|x|^{1+\beta}$ given in Lemma 5.4 minimizes $\int H$ over all possible choices of such cut-off points. This can be seen by setting the derivative of $\int H$ with respect to x_0 equal to 0 and noting that its solution is $x_0^{\beta+1}=(DB/C)$. If x_0'' is greater than x_0', the representation of Lemma 5.2 is not valid, and we must take the minimum of x_0'' and x_0' as our cut-off point.

If x_0 in Lemma 5.4 is replaced by x_0', the Algorithm A2 that corresponds to the representation of Lemma 5.4 remains valid, but $\int H$ has increased or remained the same.

6. The algorithm. The algorithm can now be spelled out in more detail. The evaluation of h and the generators for $g(t, x)$ will be the subject of § 8. For the density $H/\int H$, we can use the inversion method. Indeed, the symmetrized density $2H(x)/\int H$, $x>0$, has distribution function

$$2Cx\Big/\int H, \qquad 0<x\leqq x_0, \qquad 2\left(Cx_0+\frac{DB}{\beta}(x_0^{-\beta}-x^{-\beta})\right)\Big/\int H, \qquad x>x_0.$$

Inversion of this distribution function gives us a random variate with density $2H(x)/\int H$, $x>0$. We summarize:

A3. [*Prepocessing.*] Choose α and β from $(0, 1]$.
Compute A, B, C. Compute $D=\pi^{\beta-1}(2^{\beta-1}+2)$.
Set $C_\alpha=\pi\left(2\Gamma(\alpha+1)\sin\left(\frac{\pi\alpha}{2}\right)\right)^{-1}$, $x_0'=\left(\frac{\pi C}{C_\alpha A}\right)^{1/\alpha}$, $x_0''=(DB/C)^{1/(\beta+1)}$,

$x_0=\min(x_0', x_0'')$, $I=2(Cx_0+DB\beta^{-1}x_0^{-\beta})=\int H$, $p=2Cx_0/I$.

REPEAT
Generate X with density $H/\int H$, i.e.
Generate a uniform $[-1, 1]$ random variate V.
IF $|V|\leqq p$ THEN $X\leftarrow\left(\frac{I}{C}\right)V$
ELSE $X\leftarrow\operatorname{sign} V\Big/\left(x_0^{-\beta}-(I|V|-Cx_0)\left(\frac{\beta}{D\beta}\right)\right)^{1/\beta}$.
Generate a uniform $[0, 1]$ random variate U, independent of X.
IF $|X|\leqq x_0$ (i.e. $|V|\leqq p$)
THEN Generate T with density $g(t, X)=2\sin^2(tX/2)/(C_\alpha|X|^\alpha t^{\alpha+1})$, $t>0$.
ELSE Generate T with density $g(t, X)=|X|\cos(tX)$, $0\leqq t\leqq\pi/(2|X|)$.
UNTIL $UH(X)\leqq h(T, X)$,

where for $|V|<p$, $h(T, X)=C-\frac{1}{\pi}C_\alpha|X|^\alpha T^{\alpha+1}\phi(T)$, $H(X)=C$.

and for $|V|\geqq p$, $h(T, X)=(\pi|X|)^{-1}\sum_{j=0}^{\infty}\psi_{j,T}(X)$, $H(x)=DB/|X|^{1+\beta}$.

7. Analysis of the performance of the algorithm. If N is the number of iterations needed before A3 terminates, then N is geometrically distributed, and $E(N)=\int H$ (see Lemma 5.4 for $\int H$). Thus, $\int H$ is an appropriate measure of the expected time taken by the generators for X and U and by the overhead of the REPEAT/UNTIL loop. There are two other components that are *not*, at least not as far as we know until now, measured by $\int H$, i.e. the total time taken by the generators of T, and the total time taken by the evaluators of $h(T, X)$. Both times will be analyzed in § 8, just after the generators and evaluator are presented.

It is helpful to verify just how close $\int H$ is to 1 in some important examples. Example 1 below has computations that will be used in § 9 on the simulation of sums.

Example 7.1. (*The symmetric stable distribution.*) For the characteristic function $\phi(t)=\exp(-|t|^a)$, $0<a\leqq 1$, we have (see § 3) for $\alpha=1$, $\beta=a$,

$$A=\left(\frac{2}{ae}\right)^{2/a},\quad B=1,\quad C=\frac{1}{\pi}\Gamma\left(\frac{1}{a}+1\right),$$

and thus $C_\alpha=C_1=\pi/2$, $x_0'=(2/\pi)\Gamma(1/a+1)(ae/2)^{2/a}$. Resubstitution in the formula of $\int H$, with x_0' as cut-off point, gives

$$\int H|_{a=1}=\frac{e^2}{\pi^2}+\frac{12\pi}{e^2}=5.850\cdots,$$

$$\int H|_{a=1/2}=\frac{e^4}{16\pi^2}+\frac{32(1+\sqrt{8})}{e^2\sqrt{2}}=12.069\cdots.$$

Using Stirling's approximation ($\Gamma(1/a+1)\sim(2\pi/a)^{1/2}(1/ae)^{1/a}$ as $a\downarrow 0$), it is quickly verified that as $a\downarrow 0$, $\int H$ increases as $20\,(\pi e a^2)^{-1}$.

It is important to notice that the expected number of iterations is not uniformly bounded in $a\in(0,1]$ for the symmetric stable family. This suggests that the design of the algorithm could be improved upon. Nevertheless, for values of a near 1, $E(N)$ is quite acceptable when one considers that the algorithm is not specifically designed for the symmetric stable family. Also, no attempt was made to optimize α and β.

8. Practical details and more analysis. It is important to know how much time is spent *globally*, i.e., in the basic Algorithm A3 (§ 6) with the generation of random variates with density g. We offer the following crucial lemma:

LEMMA 8.1. *Let N be the number of iterations required to generate a random variate T with density $g(t,X)$ where X has density $H/\int H$. Let N_{total} be the total number of iterations spent on this generator before Algorithm* A3 *halts. Then*

$$E(N_{\text{total}})=E(N)\int H.$$

Proof. N_{total} is distributed as $N_{r1}+\cdots+N_{rK}+N_a$ where K is the number of rejections in the Algorithm A3, $N_{r1},\cdots,N_{rK}$ are i.i.d. random variables distributed as N, conditioned on rejection (i.e. $UH(X)>h(T,X)$), and N_a is independent of the N_{ri}'s, and is distributed as N, conditioned on acceptance. Taking expectations, we have

$$E(N_{\text{total}})=E(K)E(N_{r1})+E(N_a).$$

We also have $E(K)+1=1/p=\int H$, and $E(N)=pE(N_a)+(1-p)E(N_{r1})$. Combining this shows that $E(N_{\text{total}})=E(N)/p$.

We will discuss random variate generation for the densities $g(t,x)$ of Lemmas 5.2 and 5.3. Since these generators are fixed once and for all, they should be implemented with some care.

LEMMA 8.2. *The following algorithms can be used to generate a random variate T with density $g(t,x)=|x|\cos(tx)$, $0<t<\pi/(2|x|)$.*

A4. Set $T\leftarrow|1/x|\arcsin(U)$ where U is a uniform $[0,1]$ random variate.

A5. REPEAT Generate (U,T) uniformly on $[0,1]\times[0,\pi/2]$.

IF $U<1-T^2/2$ THEN EXIT with $T/|x|$
ELSE IF $U<1-T^2/2+T^4/24$
THEN IF $U<\cos(T)$ THEN EXIT with $T/|x|$
UNTIL False (This is an infinite loop.)

The number of iterations, N, in A5 *is a geometrically distributed random variable with $E(N)=\pi/2$.*

Proof. A4 is an inversion algorithm because $|x|\cos(tx)$ has distribution function $\sin(t|x|)$, $0<t<\pi/(2|x|)$. A5 is a rejection algorithm with bounding function $|x|$. It uses "squeeze" steps based on the inequalities $1-t^2/2\leqq\cos(t)\leqslant 1-t^2/2+t^4/24$, designed to avoid the $\cos(\cdot)$ evaluation most of the time.

Lemma 8.1 shows that where expected time is concerned, we can study the expected number of iterations in Algorithm A3, and the expected number of iterations for the g-generator separately.

As we have seen in Lemma 8.2, if $|X|>x_0$, $E(N|X)=\pi/2$. We will see in Lemma 8.4 that for $|X|\leqq x_0$, there exists a g-generator for which $E(N|X)\leqq 2.0662$. Thus, by Lemma 8.1, $E(N_{\text{total}})\leqq 2.0662\int H$. In fact, better bounds are obtainable by considering the probability $P(|X|>x_0)$ and taking a convex combination of the individual bounds. Before we give the second g-generator, we will show an algorithm that does not require the exact evaluation of cos.

LEMMA 8.3. *The following algorithm is equivalent to Algorithm* A5 (*i.e. it produces a random variate T with the same distribution.*)

A6. REPEAT Generate (U, T) uniformly on $[0,1]\times[0,\pi/2]$.
Set $k\leftarrow 0$, $S\leftarrow 1$, $P\leftarrow 1$.
REPEAT Set $k\leftarrow k+2$, $P\leftarrow PT^2/(k(k-1))$, $S\leftarrow S-P$
IF $U\leqq S$ THEN EXIT with $T\leftarrow T/|x|$.
ELSE Set $k\leftarrow k+2$, $P\leftarrow T^2P/(k(k-1))$, $S\leftarrow S+P$
UNTIL $U>S$
UNTIL False.

Furthermore, the expected number of inner loop iterations does not exceed

$$\sum_{j=0}^{\infty}\left(\frac{\pi}{2}\right)^{4j+1}\frac{1}{(4j+1)!}\leqq 1.08\frac{\pi}{2}.$$

Proof. Lemma 8.3 is based upon the fact that $\cos(t)$ is sandwiched between successive partial sums in the series

$$\cos(t)=1-\frac{1}{2!}t^2+\frac{1}{4!}t^4-\cdots.$$

Since for each t this sum converges, algorithm A6 halts with probability one. Let $N_{U,T}$ be the number of iterations in the inner loop for a given pair (U, T). (Thus, $N_{U,T}$ is a deterministic function of U and T.) Now,

$$\begin{aligned}P(N_{U,T}>j)&=P\left(\sum_{k=0}^{2j-1}(-1)^{2k}\frac{T^{2k}}{(2k)!}<U<\sum_{k=0}^{2j}(-1)^{2k}\frac{T^{2k}}{(2k)!}\right)\\&\leqq E(\min(1, T^{4j}/(4j)!))\leqq E(T^{4j}/(4j)!)=\int_0^{\pi/2}\frac{2}{\pi}\frac{t^{4j}}{(4j)!}\,dt\\&=\left(\frac{\pi}{2}\right)^{4j+1}\frac{1}{(4j+1)!}\frac{2}{\pi}.\end{aligned}$$

Thus,

$$E(N_{U,T}) = \sum_{j=0}^{\infty} P(N_{U,T} > j) \leqq \frac{2}{\pi} \sum_{j=0}^{\infty} \left(\frac{\pi}{2}\right)^{4j+1} \frac{1}{(4j+1)!} < 1.08.$$

Let N be the number of inner loop iterations taking into account that many outer loops could be executed before A6 halts. As in Lemma 8.1, it is easy to prove that the total expected number of inner loop iterations is the product of the number of outer loop iterations ($\pi/2$, by Lemma 8.2) and $E(N_{U,T})$. This concludes the proof of Lemma 8.3.

LEMMA 8.4. *The following algorithm generates* T *with density* $g(t, x) = 2\sin^2(tx/2)/(C_\alpha |x|^\alpha t^{\alpha+1})$, $t > 0$.

A7. REPEAT Generate (U, V) uniformly on $[0, 1]^2$.

IF $U < \alpha/2$ THEN $T \leftarrow \left(\frac{8U}{\alpha 2^\alpha}\right)^{1/(2-\alpha)}$

ELSE $T \leftarrow 2\left(1 - \alpha\left(\frac{2U}{\alpha(2-\alpha)} - \frac{1}{2-\alpha}\right)\right)^{-1/\alpha}$

$V \leftarrow V \min(1, T^2/4)$.

IF $V \leqq \left(\frac{T}{2} - T^3/48\right)^2$ THEN EXIT with $T \leftarrow T/|x|$

ELSE IF $V \leqq \sin^2(T/2)$ THEN EXIT with $T \leftarrow T/|x|$.

UNTIL False.

The number of iterations, N, is a geometrically distributed random variable with

$$E(N) = \frac{4(1-\alpha)}{\Gamma(3-\alpha)\cos(\pi\alpha/2)2^\alpha}.$$

We have $\lim_{\alpha\downarrow 0} E(N) = 2$, $\lim_{\alpha\uparrow 1} E(N) = 4/\pi$, *and*

$$\sup_{0<\alpha\leqq 1} E(N) \leqq 2\exp\left(\frac{(1.5-\gamma-\log 2)^2}{(2\pi^2/3-5)}\right) < 2.0662,$$

where $\gamma = 0.577215664\cdots$ *is Euler's constant.*

Proof. The random variate $T|x|$ has density $2\sin^2(t/2)/(C_\alpha t^{\alpha+1})$, $t > 0$, if T has density $g(t, x)$. For this density, a rejection algorithm is easily constructed, because

$$2\sin^2\left(\frac{t}{2}\right) \Big/ (C_\alpha t^{\alpha+1}) \leqq \frac{2}{C_\alpha} \min\left(1, \frac{t^2}{4}\right) \frac{1}{t^{\alpha+1}}.$$

Let us call the dominating curve $g^*(t)$, $t > 0$. If rejection is used from g^*, then the expected number of iterations is

$$E(N) = \int g^* = \frac{2}{C_\alpha}\left(\frac{1}{2^\alpha} \frac{2}{\alpha(2-\alpha)}\right) = \frac{4(1-\alpha)}{\Gamma(3-\alpha)\cos(\pi\alpha/2)2^\alpha}.$$

The proofs of the statements about $E(N)$ will be deferred for the time being. The density $g^*/\int g^*$ can be written as

$$2^\alpha \frac{\alpha(2-\alpha)}{2} \min\left(\frac{1}{t^{\alpha+1}}, \frac{t^{1-\alpha}}{4}\right).$$

The corresponding distribution function is

$$2^\alpha \frac{\alpha}{8} t^{2-\alpha}, \quad 0 \leqq t \leqq 2, \quad \frac{\alpha}{2} + \frac{1}{\alpha}\left(\frac{1}{2^\alpha} - \frac{1}{t^\alpha}\right) \cdot 2^\alpha \frac{\alpha(2-\alpha)}{2}, \quad t > 2.$$

The random variate T in Algorithm A7 is obtained from this by straightforward inversion. For the rejection/acceptance step, we need to verify whether $V \min(1, T^2/4) \leqq \sin^2(T/2)$ where V is a uniform $[0, 1]$ random variate. The Algorithm A7 does exactly that. We have added a squeeze step, based upon the inequality $\sin^2(t) \geqq (t - t^3/6)^2$.

Let us now turn to $E(N)$. We will use the inequality $2/\pi \leqq u/\sin(\pi u/2) \leqq 1$, $0 \leqq u \leqq 1$, several times. The values for $\alpha \downarrow 0$ and $\alpha = 1$ are obtained without any calculations. For the uniform upper bound, we will use some properties of the gamma function and the psi-function $\Psi(u) = (\log(\Gamma(u)))' = \Gamma'(u)/\Gamma(u)$, $u > 0$ (see e.g. Whittaker and Watson (1927) for the details). Obviously,

$$\frac{1}{E(N)} \geqq \inf_{0<u<1} \left(\frac{1}{2} \frac{\sin(\pi u/2)}{u} \frac{\Gamma(2+u)}{2^u}\right) \geqq 2 \inf_{0<u<1} \frac{\Gamma(2+u)}{2^{u+2}} = 2 \inf_{2 \leqq u \leqq 3} \frac{\Gamma(u)}{2^u}.$$

We claim that the infimum of the last function $(\Gamma(u)/2^u)$ is reached for some $u \in (2, 3)$. The logarithm of this function is convex in u. Also, its derivative is $\Psi(u) - \log 2$. Since $\Psi(2) = -\gamma + 1$, the derivative is negative at $u = 2$. And because $\Psi(3) = -\gamma + 1 + \frac{1}{2}$, it is positive at $u = 3$. By Taylor's series with remainder and the fact that $\Psi'(u) = \sum_{n=0}^{\infty} (u+n)^{-2}$ is a decreasing function of u, we note that

$$\log(\Gamma(u)) - u \log 2 \geqq -2 \log 2 + (u-3)(1.5 - \gamma - \log 2) + (u-3)^2 \Psi'(3), \qquad 2 \leqq u \leqq 3.$$

The second order (in u) lower bound takes as minimal value

$$-2 \log 2 - \frac{(1.5 - \gamma - \log 2)^2}{4\Psi'(3)}.$$

Now,

$$4\Psi'(3) = 4 \sum_{n=3}^{\infty} \frac{1}{n^2} = 4\left(\frac{\pi^2}{6} - 1 - \frac{1}{4}\right) = \frac{2}{3}\pi^2 - 5.$$

Thus,

$$(E(N))^{-1} \geqq \frac{2}{4} \exp\left(-\frac{(1.5 - \gamma - \log 2)^2}{(2\pi^2/3 - 5)}\right),$$

which was to be shown.

We conclude this section with the problem of the evaluation of the infinite sum $h(t, x) = (\pi|x|)^{-1} \sum_{j=0}^{\infty} \psi_{j,t}(x)$ in finite time, or rather of how to avoid the infinite series. We propose the following algorithm for deciding whether $h(T, X) \geqq UH(X)$:

A8. IF $|X| \leqq x_0$ THEN Compute $h(T, X)$, and set Decision
        $\leftarrow [h(T, X) \geqq UH(X)]$.
      ELSE $Y \leftarrow UH(X)\pi|X|$, $J \leftarrow 0$, $S \leftarrow 0$.
          REPEAT $S \leftarrow S + \psi_{J,T}(X)$, $J \leftarrow J + 1$
          UNTIL $(S > Y)$ OR $(S < Y - (2J)^{-1}(1 - \phi(2\pi J/|X|)))$.
          IF $S > Y$ THEN Decision $\leftarrow$ False
              ELSE Decision $\leftarrow$ True.

In Algorithm A8, "Decision" is a logical variable equal to $[h(T, X) \geqq UH(X)]$. Regardless of its exactness, *the algorithm halts in finite time almost surely*: this follows from the monotonicity of the series (each $\psi_{j,t}$ is nonnegative, a consequence of the convexity of ϕ) and the decreasing nature of the error bound

$$\sum_{j=J}^{\infty} \psi_{j,t}(x) \leqq (2J)^{-1}\left(1-\phi\left(\frac{2\pi J}{|x|}\right)\right)$$

(see proof of Lemma 5.3 for its derivative). But because of this error bound, we also note that *the correct decision is reached almost surely.*

The analysis of the time taken by Algorithm A8 in the global context of algorithm A3 requires some additonal work. To simplify things, we will once again concentrate on the number of iterations. For Algorithm A8, the number of iterations is defined as one when $|X| \leqq x_0$, and as the number of times the REPEAT/UNTIL structure is executed when $|X| > x_0$. The total number of iterations in Algorithm A3 is thus

$$N_{\text{total}} = \sum_{i=1}^{N_0-1} N_{ri} + N_a$$

where N_0 is the number of times the outer loop in Algorithm A3 is executed, and $N_a, N_{r1}, \cdots, N_{rk}, \cdots$ are independent random variables: N_a is distributed as the number of iterations in A8 conditional on the decision that $h(T, X) \geqq U(HX)$ is true, and N_{rk}, $k = 1, 2, \cdots$ are similarly defined, but now conditional on the decision being false. We can argue again as in Lemma 8.1, and conclude that

$$E(N_{\text{total}}) = E(N) \int H$$

where N is the number of iterations taken by algorithm A8 without any conditioning (i.e., X has density $H/\int H$, T has density $g(t, X)$, and U is uniformly distributed on $[0, 1]$). Unfortunately, as we will see below, $E(N) = \infty$ in most cases except the most trivial ones, and Lemma 8.1 is of little direct use to us. However, we can offer the following generalization of Lemma 8.1:

LEMMA 8.5. *For $r \in (0, 1)$, $E(N_{\text{total}}^r) \leqq E(N^r) \int H$.*

Proof. The proof of Lemma 8.5 can be mimicked after one notes that

$$N_{\text{total}}^r \leqq \sum_{i=1}^{N_0-1} N_{ri}^r + N_a^r.$$

LEMMA 8.6 (Performance of the h-evaluator). *Let N_{total}, N be as defined above, and let $f \in M_\theta$.*

(1) *For $n \geqq 1$,*

$$P(N > n) \leqq \frac{1}{\pi(n+1)\int H}\left(\frac{1}{\beta} + \log_+(\Delta(n+1))\right)$$

where

$$\Delta = \frac{2\pi B^{1/\beta}}{x_0} \leqq \frac{\pi\beta \int H}{D}.$$

(2) *For $0 < r < 1$,*

$$E(N^r) \leqq 1 + \frac{1}{\pi \int H} \frac{r}{1-r}\left(\frac{1}{\beta} + \Delta^{1-r}\right),$$

$$E(N_{\text{total}}^r) \leqq \int H + \frac{1}{\pi} \frac{r}{1-r}\left(\frac{1}{\beta} + \Delta^{1-r}\right).$$

(3) *For* $n \geqq e$,

$$P(N_{\text{total}} \geqq n) \leqq \frac{e \int H}{n} + \frac{e \log(n)}{\pi n}\left(\frac{1}{\beta} + \Delta^{1/(\log n)}\right)$$
$$\leqq \frac{e}{n}\left(\int H + \frac{1}{\pi}(\Delta - 1)_+\right) + \frac{e(\beta+1)\log(n)}{\pi\beta n}$$
$$\leqq \frac{e\int H}{n}\left(1 + \frac{\beta}{D}\right) + \frac{e(\beta+1)\log(n)}{\pi\beta n}$$
$$\leqq \frac{e\int H}{n}\left(1 + \frac{2\pi}{5e\log(\pi)}\right) + \frac{e(\beta+1)\log(n)}{\pi\beta n}$$
$$\leqq \frac{e(\beta+1)}{\pi\beta}\,\frac{\log(n)}{n}(1+o(1)).$$

Proof. For $n \geqq 1$, we have $P(N>n) = P(|X|>x_0, N>n)$. Now, by our stopping rule,

$$P(N>n, |X|>x_0 \mid T, X)$$
$$\leqq I_{[|X|>x_0]} P\left(Y - \frac{1}{2(n+1)}\left(1 - \phi\left(\frac{2\pi(n+1)}{|X|}\right)\right) \leqq \sum_{j=0}^{n} \psi_{j,T}(X) \leqq Y \,\Big|\, T, X\right)$$
$$\leqq I_{[|X|>x_0]} \frac{1}{2(n+1)}\left(1 - \phi\left(\frac{2\pi(n+1)}{|X|}\right)\right)\frac{1}{\pi|X|H(X)}$$
$$\leqq I_{[|X|>x_0]} \frac{1}{2(n+1)}\,\frac{1}{\pi|X|H(X)}\min\left(1, B\left(\frac{2\pi(n+1)}{|X|}\right)^{\beta}\right)$$

where I is the indicator function of an event. In the chain of inequalities, we used the fact that U is uniformly distributed, and obtained the last inequality from the definition of M_θ. The upper bound does not depend upon T. Thus, if X has density $H/\int H$, and we take expectations on left and right, we obtain

$$P(N>n) \leqq \frac{1}{2(n+1)}2\int_{x_0}^{\infty} \frac{H(y)}{\int H}\,\frac{1}{\pi y H(y)} B\left(\frac{2\pi(n+1)}{y}\right)^{\beta} I_{[y \geqq 2\pi(n+1)B^{1/\beta}]}\,dy$$
$$+ \frac{1}{2(n+1)}2\int_{x_0}^{\infty} \frac{1}{\pi y H(y)} I_{[y < 2\pi(n+1)B^{1/\beta}]}\,dy$$
$$= \frac{2}{(2\pi(n+1))^{1-\beta}\int H}\int_{x_0}^{\infty} B y^{-(1+\beta)} I_{[y \geqq 2\pi(n+1)B^{1/\beta}]}\,dy$$
$$+ \frac{2}{2\pi(n+1)\int H}\log_+(\Delta(n+1)).$$

There are two cases: If $\Delta(n+1) \geqq 1$, then, the first term becomes

$$\frac{2}{2\pi(n+1)\beta\int H}.$$

If $\Delta(n+1) < 1$, the last term in the upper bound is zero, and the first term becomes

$$\frac{2B}{(2\pi(n+1))^{1-\beta}\int H\beta x_0^{\beta}} = (\Delta(n+1))\frac{2}{2\pi(n+1)\beta\int H} \leqq \frac{2}{2\pi(n+1)\beta\int H}.$$

To prove the bound on Δ, use $\int H = 2(Cx_0 + D\beta/\beta x_0^{\beta})$.

We will use the notation $\underline{u}$ to denote the largest integer not exceeding u. Let $r \in (0, 1)$ be a constant. Then,

$$
\begin{aligned}
E(N^r) &= \int_0^\infty P(N > t^{1/r})\,dt = \int_0^\infty P(N > \underline{t^{1/r}})\,dt \leqq 1 + \int_1^\infty P(N > \underline{t^{1/r}})\,dt \\
&\leqq 1 + \frac{1}{\pi \int H} \int_1^\infty \frac{1}{(\underline{t^{1/r}}+1)} \left(\frac{1}{\beta} + \log_+ (\Delta(\underline{t^{1/r}}+1))\right) dt \quad \text{(by part (1))} \\
&\leqq 1 + \frac{1}{\pi \int H} \int_1^\infty t^{-1/r} \left(\frac{1}{\beta} + \log_+ (\Delta t^{1/r})\right) dt \quad (\text{since } \underline{t^{1/r}} + 1 \geqq t^{1/r}) \\
&= 1 + \frac{1}{\pi \int H} \int_0^\infty e^{-y} \left(\frac{1}{\beta} + \log_+ (\Delta\, e^y)\right) r\, e^{ry}\,dy \quad (\text{set } t = e^{ry}) \\
&= 1 + \frac{1}{\pi \int H} \left(\frac{r}{\beta(1-r)} + \frac{r}{1-r} \Delta^{1-r}\right),
\end{aligned}
$$

which proves (2).

For part (3), we start with Markov's inequality:

$$
P(N_{\text{total}} \geqq n) \leqq \frac{E(N_{\text{total}}^r)}{n^r} \leqq \frac{\int H}{n^r} + \frac{r}{\pi(1-r)n^r} \left(\frac{1}{\beta} + \Delta^{1-r}\right).
$$

Clearly, we can still choose r. Expressions of the form $1/((1-r)n^r)$ are minimized by setting $r = 1 - (\log n)^{-1}$. The corresponding r is in the range (0, 1) if $n \geqq e$. Also, $n^r = n/e$. Resubstitution of the given value of r gives the bound

$$
P(N_{\text{total}} \geqq n) \leqq \frac{e \int H}{n} + \frac{re \log(n)}{\pi n} \left(\frac{1}{\beta} + \Delta^{1/\log(n)}\right).
$$

The chain of inequalities that follows in part (3) is shown as follows: first, $\Delta^{1/\log(n)} \leqq 1 + (1/\log(n))(\Delta - 1)_+$. For the last inequality, it suffices to establish that

$$
\frac{\beta}{D} \leqq \frac{2\pi}{5e \log(\pi)}.
$$

But $\beta/D = 2\pi\beta/(\pi^\beta(2^\beta + 4))$. The factor $2^\beta + 4$ is at least equal to 5. The function β/π^β is maximal when $\beta = 1/\log(\pi)$. Resubstitution gives the maximal value, $1/(e \log(\pi))$. This concludes the proof of the chain of inequalities. The last part of (3) is trivially true.

Remark 8.1. The bottom line of Lemma 8.6 is the collection of inequalities for $P(N_{\text{total}} \geqq n)$. The expectations $E(N_{\text{total}}^r)$, $0 < r < 1$, were only used as a handy tool to obtain bounds for these probabilities in what could be termed a Tauberian argument. The first bound for $P(N_{\text{total}} \geqq n)$ depends positively on $\int H$ and Δ, and negatively on n and β. From this, one could conclude that it is advantageous to pick β as large as possible (preferably equal to 1 if this is feasible). It is more important to realize that the main term in the last upper bound depends upon β and n only: for $\beta = 1$, it is equal to $(2e/\pi)(\log n/n)$. The second term in the upper bound decreases faster, i.e. as $1/n$, but has a coefficient that is proportional to $\int H$.

Remark 8.2. We alluded earlier to the possibility of having $E(N) = \infty$ for the h-evaluator. We will give a brief outline of how this can be proved. It suffices of course to show that for almost all t, x, $E(N|T = t, X = x) = \infty$, where "almost all" refers to the distribution of (T, X). To include this, it suffices in turn to obtain for almost all t, x:

$$
P(N > n | T = t, X = x) \geqq \frac{c(t, x) + o(1)}{n+1},
$$

where $c(t, x)$ is a positive number depending upon t and x only, and $o(1)$ tends to 0 with n, but is allowed to depend upon t and x. When $H(x) > h(t, x)$, it is easily seen that we can take $c(t, x) = (2\pi|x|H(x))^{-1}$.

Remark 8.3. (*The factor* Δ.) In Lemma 8.6(1), we have obtained a simple upper bound for Δ in terms of $\int H$ and β. In fact, for all practical purposes, Δ can be considered as a constant, since it is bounded from below by a positive-valued function of β only:

$$2\pi\left(\frac{1}{\pi^{\beta}(2^{\beta-1}+2)}\frac{\beta}{\beta+1}\right)^{1/(\beta+1)}.$$

This can be shown as follows: let us set $D = \pi^{\beta-1}(2^{\beta-1}+2)$, and note that $x_0 \leqq (DB/C)^{1/(\beta+1)}$. Also, by a geometrical argument, for $\phi \in M_\theta$,

$$C = \frac{1}{\pi}\int_0^\infty \phi \geqq \frac{1}{\pi}\int_0^{B^{-1/\beta}} (1 - Bt^\beta)\,dt = \frac{1}{\pi B^{1/\beta}}\frac{\beta}{\beta+1}.$$

Thus, $(B^{1/\beta}/x_0)^{\beta+1} \geqq \beta/(\pi D(\beta+1))$, which was to be shown.

9. Simulation of sums. Characteristic functions are mainly used in the study of sums of independent random variables. We have seen that *we can generate the sum* $S_n = X_1 + \cdots + X_n$ of n i.i.d. random variables X_i with common characteristic function $\phi \in M_\theta$ *without actually generating the individual* X_i*'s*: just apply the algorithm of § 6 to the characteristic function ϕ^n of S_n, and note that ϕ^n belongs to some M_θ when ϕ belongs to some M_θ (with possibly different θ's).

The real issue here is the following: if the time taken by the algorithm grows linearly with n or faster, it seems indicated to generate S_n by generating the individual X_i's and then taking the sum. Thus, we should verify how the time varies with n, in order to be sure that our algorithm is efficient for simulating sums directly. What we will show here is truly exciting: the time taken by Algorithm A3 remains essentially uniformly bounded by a random variable independent of n. This statement needs some qualification: we will take some $\phi \in M_\theta$ (with parameters A, B, C, α, β), and consider the class $\Phi(\phi) = \{\phi, \phi^2, \phi^3, \cdots, \phi^n, \cdots\}$. Keep α and β fixed throughout, but define

$$A_n = \sup_{t>0} t^{1+\alpha}\phi^n(t), \quad B_n = \sup_{t>0}\frac{1-\phi^n(t)}{t^\beta}, \quad C_n = \frac{1}{\pi}\int_0^\infty \phi^n(t)\,dt.$$

We will also write x_{n0}, H_n, Δ_n, et cetera to make the dependence upon n explicit. For the time being, we assume that A_n, B_n and C_n are all exactly known. This restriction makes the algorithm only efficient when many random variates are needed for fixed n. It should be stressed that the algorithm only requires upper bounds for A_n and B_n, but for the sake of a smooth analysis, it is simpler to work with the exact values throughout.

As is well known from the central limit theorem, the behavior of $\phi(t)$ near the origin determines to a large extent how S_n behaves. Thus, to analyze our algorithm, we will assume that $1-\phi(t) \sim at^b$ as $t \downarrow 0$, for some $a, b > 0$. (Clearly, $b \leqq 1$ for our class of distributions.) The parameters a and b need not be known to the user. They are only needed here in the analysis of the performance.

All our performance measures which were developed for computing the time taken for constant overhead per iteration, for g-generators and for h-evaluators, depend upon H_n and Δ_n only. If we want uniform performances over the class $\Phi(\phi)$, i.e. uniform upper bounds for $E(N)$, (EN^r_{total}), $0 < r < 1$, and $P(N_{\text{total}} \geqq n)$ in the notation of previous sections, we need only insure that

$$\sup_n \int H_n < \infty.$$

(Note that this implies that $\sup_n \Delta_n < \infty$ (Lemma 8.6)). Thus, we should verify how A_n, B_n and C_n vary with n. This is done in Lemma 9.1. In Remark 9.1 we will see that when ϕ is in the domain of attraction of a stable law, then the performance of the algorithm approaches that of the algorithm when applied to the limiting stable law.

LEMMA 9.1 (asymptotic behavior of A_n, B_n, C_n). *Let $\phi \in M_\theta$, and let*

$$1-\phi(t) \sim at^b \quad \text{as } t \downarrow 0$$

for some positive numbers a, b. Then

$$A_n \sim \left(\frac{1+\alpha}{nabe}\right)^{(1+\alpha)/b},$$

$$B_n \sim B^*(na)^{\beta/b}, \quad \text{where } B^* = \sup_{y>0} \frac{1-e^{-y}}{y^{\beta/b}} \quad (=1 \text{ if } \beta = b),$$

$$C_n \sim \frac{1}{\pi}\Gamma\left(1+\frac{1}{b}\right)(na)^{1/b}.$$

Proof. For all $\varepsilon \in (0, 1)$, there exists $t_0 > 0$ such that

$$\int_0^{t_0} (1-a(1+\varepsilon)t^b)^n \, dt \leqq \pi C_n = \int_0^\infty \phi^n \leqq \int_0^{t_0} (1-a(1-\varepsilon)t^b)^n \, dt + \int_{t_0}^\infty \phi^n.$$

But $\int_{t_0}^\infty \phi^n \leqq \phi^{n-1}(t_0) \int_{t_0}^\infty \phi \leqq \pi C_1 \phi^{n-1}(t_0)$, which decreases exponentially fast with n. (We used the fact that ϕ is monotone.)

If we set $at^b n = y$, then

$$\begin{aligned}\int_0^{t_0} (1-at^b)^n \, dt &= (an)^{-1/b} \frac{1}{b} \int_0^{at_0^b n} \left(1-\frac{y}{n}\right) y^{1/b-1} \, dy \\ &\sim (an)^{-1/b} \frac{1}{b} \int_0^\infty e^{-y} y^{1/b-1} \, dy \\ &= (an)^{-1/b} \frac{1}{b} \Gamma\left(\frac{1}{b}\right) = (an)^{-1/b} \Gamma\left(1+\frac{1}{b}\right).\end{aligned}$$

Here we used $(1-u) \leqq \exp(-u)$, $u \geqq 0$, the dominated convergence theorem, and the definition of the gamma integral. We have proved our claim about C_n by the continuity of the right-hand side of the last expression in a, and the arbitrariness of ε.

Consider now A_n, and find for each $\varepsilon \in (0, 1)$ a constant $t_0 > 0$ such that $1-\phi(t)$ is between $a(1-\varepsilon)t^b$ and $a(1+\varepsilon)t^b$, $|t| \leqq t_0$. Clearly, since $\phi \in M_\theta$,

$$\sup_{t>t_0} t^{1+\alpha} \phi^n(t) \leqq A_1 \phi^{n-1}(t_0),$$

which decreases at an exponential rate in n, and is thus asymptotically negligible in comparison with polynomially decreasing terms. Again basing ourselves on a continuity argument (in a), it suffices to show that

$$\sup_{t \leqq t_0} t^{1+\alpha}(1-at^b)^n \sim \left(\frac{1+\alpha}{nabe}\right)^{(1+\alpha)/b}.$$

The supremum is reached on R for $t^*=((1+\alpha)/(nab+a(1+\alpha)))^{1/b}$, as a quick computation shows. Since $t^*<t_0$ for n large enough, we can substitute the value t^* in $t^{1+\alpha}(1-at^b)^n$, and obtain

$$\left(\frac{1+\alpha}{nab+a(1+\alpha)}\right)^{(1+\alpha)/b}\left(1+\frac{1+\alpha}{bn}\right)^{-n}\sim\left(\frac{1+\alpha}{nab}\right)^{(1+\alpha)/b}e^{-(1+\alpha)/b}=\left(\frac{1+\alpha}{nabe}\right)^{(1+\alpha)/b}.$$

For B_n, we employ a similar proof: ε and t_0 are as before. First,

$$\sup_{t>t_0}\frac{1-\phi^n(t)}{t^\beta}\leqq\frac{1-(1-Bt^\beta)^n}{t^\beta}\leqq t_0^{-\beta}(1-(1-Bt_0^\beta)^n)\uparrow t_0^{-\beta}.$$

Thus, by continuity in a of the limit, we need only show that

$$\sup_{0<t\leqq t_0}\frac{1-(1-at^b)^n}{t^\beta}\sim B^*(na)^{\beta/b}.$$

Reparametrize by defining $y=ant^b$. Then, the supremum can be rewritten as

$$\sup_{0<y\leqq ant_0^b}\left(\frac{1-(1-y/n)^n}{y^{\beta/b}}\right)(na)^{\beta/b}\sim B^*(na)^{\beta/b}.$$

This concludes the proof of Lemma 9.1.

LEMMA 9.2. (Uniformly bounded time for $\Phi(\phi)$.) *Let $\phi\in M_\theta$, and assume that $1-\phi(t)\sim at^b$ as $t\downarrow 0$ for some positive numbers a, b. Assume further that $\beta\leqq b$. If N is the number of iterations in Algorithm* A3, *N_g is the number of iterations executed in generating random variates from density g, and N_h is the number of iterations in the h-evaluator, then*

$$\sup_n\int H_n<\infty,\qquad \sup_n\Delta_n<\infty,$$

$$\sup_n E(N)<\infty,\quad \sup_n E(N_g)<\infty,\quad \sup_n E(N_h^r)<\infty,\quad \textit{all } r\in(0,1),$$

and

$$\sup_n P(N_h\geqq i)\leqq(1\text{ to }(1))\frac{e(\beta+1)}{\pi\beta}\frac{\log i}{i}\quad \textit{as } i\to\infty.$$

Proof. The condition $\beta\leqq b$ is needed to insure that B^*, defined in Lemma 9.1, is finite. From Lemma 9.1, we can conclude that

$$x'_{n0}=\left(\frac{\pi}{C_\alpha}\frac{C_n}{A_n}\right)^{1/\alpha}\sim c'(na)^{1/b},$$

$$x''_{n0}=(DB_n/C_n)^{1/(\beta+1)}\sim c''(na)^{1/b},$$

and thus

$$x_{n0}=\max(x'_{n0},x''_{n0})\sim\max(c',c'')(na)^{1/b}$$

as $n\to\infty$, where c', c'' are positive constants. Hence C_nx_{n0} has a positive finite limit, c_1, and $DB_n/(\beta x_{n0}^\beta)$ has another finite limit, c_2. Thus, recalling the definitions of $\int H_n$ and Δ_n, we notice that both tend to a constant as $n\to\infty$, and are therefore uniformly bounded in n. The remainder of the statements of Lemma 9.2 follow directly from this and various lemmas and remarks scattered throughout the paper.

Remark 9.1. (*Attraction to the stable law.*) The condition put on ϕ in Lemmas 9.1 and 9.2 puts it in the domain of attraction of the stable distribution with parameter

b: $\phi(t) = \exp(-|t|^b)$ (see e.g. Feller (1966)). For such laws, if we take $\alpha = 1$, $\beta = b$, we have seen in Example 7.1 that $A = (2/eb)^{2/b}$, $B = 1$, $C = (1/\pi)\Gamma(1+1/b)$. But observe that for ϕ as in Lemmas 9.1, 9.2,

$$A_n(na)^{2/b} \to \left(\frac{2}{eb}\right)^{2/b}, \quad B_n(na)^{-1} \to 1, \quad C_n(na)^{1/b} \to \frac{1}{\pi}\Gamma\left(1+\frac{1}{b}\right).$$

Since all the performance measures encountered until now are continuous functions of A_n, B_n and C_n, it is a straightforward exercise to prove that their asymptotic limits (which we know exist, see Lemma 9.2) are precisely equal to corresponding values for the stable (b) distribution!

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