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# THE $L_1$ CONVERGENCE OF KERNEL DENSITY ESTIMATES[1]

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Let $X_1, \cdots, X_n$ be a sequence of independent random vectors taking values in $\mathbb{R}^d$ with a common probability density $f$. If $f_n(x) = (1/h) h_n^{-d}\Sigma_{i=1}^n K((x - X_i)/h_n)$ is the kernel estimate of $f$ from $X_1, \cdots, X_n$ then conditions on $K$ and $\{h_n\}$ are given which insure that $\int |f_n(x) - f(x)|\, dx \to_n 0$ in probability or with probability one. No continuity conditions are imposed on $f$.

Let $X_1, \cdots, X_n$ be a sequence of independent random vectors taking values in $\mathbb{R}^d$ with a common probability density $f$. The kernel estimate of $f$ from $X_1, \cdots, X_n$ is given by

$$f_n(x) = (1/n)h_n^{-d}\Sigma_{i=1}^n K((x - X_i)/h_n)$$

where the kernel $K$ is a bounded probability density on $\mathbb{R}^d$ and $\{h_n\}$ is a sequence of positive numbers. We are concerned here with the conditions of $f$, $K$ and $\{h_n\}$ which insure the $L_1$ convergence of $f_n$ to $f$, namely,

(1) $$\int_{\mathbb{R}^d} |f_n(x) - f(x)|\, dx \to_n 0 \qquad \text{in probability (or w.p. 1).}$$

This concern is motivated by the observation (Scheffé (1947)) that

$$2 \sup_{B \in \mathfrak{B}} |\mu_n(B) - \mu(B)| = \int_{\mathbb{R}^d} |f_n(x) - f(x)|\, dx$$

where $\mathfrak{B}$ is the class of Borel sets in $\mathbb{R}^d$ and $\mu_n$ and $\mu$ are the measures on $\mathfrak{B}$ corresponding to $f_n$ and $f$ respectively. Consequently, whenever (1) holds

(2) $$\sup_{B \in \mathfrak{B}} |\mu_n(B) - \mu(B)| \to_n 0 \qquad \text{in probability (or w.p. 1).}$$

Of course, (2) reminds one of the Glivenko-Cantelli theorem and its extensions (Winter (1973), Glick (1974)), namely,

$$\sup_{B \in \mathcal{C}} |\nu_n(B) - \nu(B)| \to_n 0 \qquad \text{in probability (or w.p. 1),}$$

where $X_1, \cdots, X_n$ are independent, identically distributed with an arbitrary probability measure $\nu$ on $\mathfrak{B}$, $\nu_n$ is the empirical measure for $X_1, \cdots, X_n$ and $\mathcal{C}$ is a strict subclass of Borel sets (Rao (1962), Vapnik and Chervonenkis (1971)). For our case it is easy to see that $\mu$ must be absolutely continuous if (2) is to hold for $\mu_n$ which correspond to kernel estimates.

Glick (1974) has shown that whenever $f_n$ is a probability density on $\mathbb{R}^d$ which is a measurable function of $x$ and $X_1, \cdots, X_n$, then (1) follows from

(3) $$f_n(x) \to_n f(x) \qquad \text{in probability (or w.p. 1) almost everywhere in } x.$$

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Whenever $f$ is almost everywhere continuous on $\mathbb{R}^d$, (3) then follows immediately from the known pointwise consistency conditions for kernel estimates. (See, for example, Rosenblatt (1957), Parzen (1962), Cacoullos (1965), Nadaraya (1965), Van Ryzin (1969), Deheuvels (1974).) This argument fails for those densities on $\mathbb{R}^d$ which do not have an almost everywhere continuous version. The main result of this note is that (1) follows without any continuity requirements on $f$ and, consequently, (2) holds for all absolutely continuous probability measures.

For comparison, we note that the nearest neighbor density estimate of $f$ (Loftsgaarden and Quesenberry (1965), Moore and Yackel (1977)) will never satisfy (1) or (2) since its integral over $\mathbb{R}^d$ is always infinite. Abou-Jaoude (1976a, 1976b) has shown, however, that (1) holds for different types of histogram estimates with no assumptions on $f$.

THEOREM. *Let $K$ be a bounded probability density on $\mathbb{R}^d$ with*

$$L(u) = \sup_{\|x\| \geqslant u} K(x)$$

*for $u \geqslant 0$. If $\{h_n\}$ is a sequence of positive numbers then* (1) *follows whenever*

$$h_n \to_n 0 \tag{4}$$

$$nh_n{}^d \to_n \infty \ \left(\textstyle\sum_1^\infty e^{-\alpha n h_n{}^d} < \infty \text{ for all } \alpha > 0\right) \tag{5}$$

*and one of the following conditions holds*:

$$\|x\|^d K(x) \to 0 \text{ as } \|x\| \to \infty \text{ and } f \text{ is almost everywhere continuous}, \tag{6}$$

$$f \text{ is bounded}, \tag{7}$$

$$\textstyle\int_0^\infty u^{d-1} L(u)\, du < \infty. \tag{8}$$

REMARK. The condition in (6) imposed on $K$ is equivalent to

$$u^d L(u) \to 0 \quad \text{as} \quad u \to \infty$$

which is only slightly weaker than (8).

PROOF. Starting, as usual, with

$$|f_n(x) - f(x)| \leqslant |f_n(x) - Ef_n(x)| + |Ef_n(x) - f(x)|$$

we first show that

$$Ef_n(x) \to_n f(x) \qquad \text{almost everywhere in } x. \tag{9}$$

The usual argument shows that (9) is implied by (4) and (6) (e.g., use the $d$-dimensional version of Theorem 1A of Parzen (1962)). Next

$$|Ef_n(x) - f(x)| \leqslant \textstyle\int_{\|y\| < \delta h_n} |f(x-y) - f(x)| h_n^{-d} K(y/h_n)\, dy + \textstyle\int_{\|y\| \geqslant \delta h_n} |f(x-y) - f(x)| h_n^{-d} K(y/h_n)\, dy. \tag{10}$$

If $\lambda(B)$ denotes the Lebesgue measure of the Borel set $B \subseteq \mathbb{R}^d$ and if $S(x, r)$ denotes the closed sphere of radius $r$ centered at $x$ then the first term of the right-hand side of (10) is bounded by

$$\sup_y K(y) \lambda(S(0, \delta)) \textstyle\int_{S(x, \delta h_n)} \{|f(y) - f(x)| / \lambda(S(x, \delta h_n))\}\, dy$$

which tends to 0 for almost every $x$ and every $\delta > 0$ if $h_n \to_n 0$ (see, for example, Zygmund (1959, 1969)). If $f$ is bounded the second term of (10) is bounded by

$$2 \sup_y f(y) \int_{\|y\| > \delta} K(y)\, dy$$

which can be made arbitrarily small for all $n$ by taking $\delta$ large enough. Thus (4) and (7) imply (9). Using a theorem of Stein ((1970), pages 62–63) we see that (4) and (8) imply (9).

Looking at $f_n(x) - Ef_n(x)$ we see that it equals

$$\frac{1}{n}\Sigma_1^n(Y_{ni} - EY_{ni})$$

where

$$Y_{ni} = h_n^{-d} K((x - X_i)/h_n).$$

Letting $\sup_y K(y) = M$, we have

$$0 \leqslant Y_{ni} \leqslant M/h_n^{\,d},$$

and

$$EY_{ni}^{\,2} \leqslant (M/h_n^{\,d}) Ef_n(x),$$

so that, by Bennett's inequality (Bennett (1962)),

$$P\{|f_n(x) - Ef_n(x)| \geqslant \varepsilon\} \leqslant \exp(-2n\varepsilon^2 h_n^{\,d}/(2MEf_n(x) + M\varepsilon)).$$

At each point $x$ for which $Ef_n(x) \to_n f(x)$ the sequence $\{Ef_n(x)\}$ remains bounded so that, almost everywhere in $x$,

$$f_n(x) - Ef_n(x) \to_n 0 \qquad \text{in probability or w.p. 1}$$

depending on whether $nh_n^d \to_n \infty$ or $\Sigma_1^\infty e^{-\alpha n h_n^d} < \infty$ for all $\alpha > 0$. Since (3) follows from the conditions of the theorem, (1) now follows from Glick's result.

REMARK. The proof also yields the strong pointwise consistency of $f_n$ whenever $K$ is a bounded probability density and

(i) $h_n \to_n 0$,
(ii) $\Sigma_1^\infty e^{-\alpha n h_n^d} < \infty$ for $\alpha > 0$, and
(iii) $\|x\|^d K(x) \to 0$ as $\|x\| \to \infty$ or $f$ is bounded.

This result is similar to the one obtained by Deheuvels (1974).

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