ON THE SPANNING RATIO OF GABRIEL GRAPHS AND β-SKELETONS

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Abstract. The spanning ratio of a graph defined on n points in the Euclidean plane is the maximum ratio, over all pairs of data points (u, v), of the minimum graph distance between u and v divided by the Euclidean distance between u and v. A connected graph is said to be a S-spanner if the spanning ratio does not exceed S. For example, for any S, there exists a point set whose minimum spanning tree is not a S-spanner. At the other end of the spectrum, a Delaunay triangulation is guaranteed to be a 2.42-spanner[11]. For proximity graphs *between* these two extremes, such as Gabriel graphs[8], relative neighborhood graphs[16] and β-skeletons[12] with $\beta \in [0, 2]$ some interesting questions arise. We show that the spanning ratio for Gabriel graphs (which are β-skeletons with $\beta = 1$) is $\Theta(\sqrt{n})$ in the worst case. For all β-skeletons with $\beta \in [0, 1]$, we prove that the spanning ratio is at most $O(n^\gamma)$ where $\gamma = (1 - \log_2(1 + \sqrt{1 - \beta^2}))/2$. For all β-skeletons with $\beta \in [1, 2)$, we prove that there exist point sets whose spanning ratio is at least $\left(\frac{1}{2} - o(1)\right)\sqrt{n}$. For relative neighborhood graphs[16] (skeletons with $\beta = 2$), we show that there exist point sets where the spanning ratio is $\Omega(n)$. For points drawn independently from the uniform distribution on the unit square, we show that the spanning ratio of the (random) Gabriel graph and all β-skeletons with $\beta \in [1, 2]$ tends to ∞ in probability as $\sqrt{\log n / \log \log n}$.

Keywords and phrases. Gabriel graph, β-skeletons, spanners, proximity graphs, probabilistic analysis, computational geometry, geometric spanners.

CR Categories: 3.74, 5.25, 5.5.

1. Introduction.

Many problems in geometric network design, pattern recognition and classification, geographic variation analysis, geographic information systems, computational geometry, computational morphology, and computer vision use the underlying *structure* (also referred to as the *skeleton* or *internal shape*) of a set of data points revealed by means of a *proximity graph* (see for example [16, 13, 7, 9]). A proximity graph attempts to exhibit the relation between points in a point set. Two points are joined by an edge if they are deemed *close* by some proximity measure. It is the measure that determines the type of graph that results. Many different measures of proximity have been defined, giving rise to many different types of proximity graphs. An extensive survey on the current research in proximity graphs can be found in Jaromczyk and Toussaint [9].

We are concerned with the spanning ratio of proximity graphs. Consider n points in $\mathbb{R}^2$, and define a graph on these points, such as the Gabriel graph [8], or the relative neighborhood graph [16]. For a pair of data points (u, v), the length of the shortest path between u and v in the graph, where edge length is measured by Euclidean distance, is denoted by $L(u, v)$, while the direct Euclidean distance is $D(u, v)$. The *spanning ratio* of the graph is defined by

$$S \stackrel{\text{def}}{=} \max_{(u,v)} \frac{L(u, v)}{D(u, v)} ,$$

where the maximum is over all $\binom{n}{2}$ pairs of data points. Note that if the graph is

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not connected, the spanning ratio is infinite. In this paper, we will concentrate on connected graphs.

Graphs with small spanning ratios are important in some applications (see [7] for a survey on spanners). The history for the Delaunay triangulation is interesting. First, Chew [2, 3] showed that in the worst case, $S \geq \pi/2$. Subsequently, Dobkin et al.[5] showed that the Delaunay triangulation was a $((1+\sqrt{5})/2)\pi \approx 5.08$ spanner. Finally, Keil and Gutwin [10, 11] improved this to $2\pi/(3\cos(\pi/6)$ which is about 2.42. It is conjectured that the spanning ratio of the Delaunay triangulation is $\pi/2$. The complete graph has $S = 1$, but is less interesting because the number of edges is not linear but quadratic in n. In this paper, we concentrate on the parametrized family of proximity graphs known as β-skeletons [12] with β in the interval $[0, 2]$. The family of β-graphs contains certain well-known proximity graphs such as the Gabriel graph [8] when $\beta = 1$ and the relative neighborhood graph [16] when $\beta = 2$. As graphs become sparser, their spanning ratios increase. For example, it is trivial to show that there are minimal spanning trees with n vertices for which $S \geq n - 1$, whereas the Delaunay triangulation has a constant spanning ratio.

In this note, we probe the expanse between these two extremes. We show that for any n there exists a point set, in the plane, whose Gabriel graph satisfies $S \geq c\sqrt{n}$, where c is a universal constant. We also show that for any Gabriel graph in the plane, $S \leq c'\sqrt{n}$ for another constant c'. For all β-skeletons with $\beta \in [0, 1]$, we prove that the spanning ratio is at most $O(n^{\gamma})$ where $\gamma = (1 - \log_2(1 + \sqrt{1-\beta^2}))/2$. For all β-skeletons with $\beta \in [1, 2)$, we prove that there exist point sets whose spanning ratio is at least $\left(\frac{1}{2} - o(1)\right)\sqrt{n}$. For relative neighborhood graphs, we show that there exist point sets where the spanning ratio is $\Omega(n)$. The second part of the paper deals with point sets drawn independently from the uniform distribution on the unit square. We show that the spanning ratio of the (random) Gabriel graph and all β-skeletons with $\beta \in [1, 2]$ tends to ∞ in probability as $\sqrt{\log n / \log\log n}$.

2. Preliminaries. We begin by defining some of the graph theoretic and geometric terminology used in this paper. For more details see [1] and [15].

A *graph* $G = (V, E)$ consists of a finite non empty set $V(G)$ of *vertices*, and a set $E(G)$ of unordered pairs of vertices known as *edges*. An edge $e \in E(G)$ consisting of vertices u and v is denoted by $e = uv$; u and v are called the *endpoints* of e and are said to be *adjacent* vertices or *neighbors*. A *path* in a graph G is a finite non-null sequence $v_1 v_2 \ldots v_k$ with $v_i \in V(G)$ and $v_i v_{i+1} \in E(G)$ for all i. The vertices v_1 and v_k are known as the *endpoints* of the path. A graph is *connected* if, for each pair of vertices $u, v \in V$, there is a path with endpoints u and v (i.e. a path from u to v).

Intuitively speaking, a *proximity graph* on a finite set $P \subset \mathbb{R}^2$ is obtained by connecting pairs of points of P with line segments if the points are considered to be *close* in some sense. Different definitions of closeness give rise to different proximity graphs. One technique for defining a proximity graph on a set of points is to select a geometric region defined by two points of P—for example the smallest disk containing the two points—and then specifying that a segment is drawn between the two points if and only if this region contains no other points from P. Such a region will be referred to as a *region of influence* of the two points.

Given a set P of points in $\mathbb{R}^2$, the *relative neighborhood graph of* P, denoted by $RNG(P)$, has a segment between points u and v in P if the intersection of the open disks of radius $D(u, v)$ centered at u and v is empty. This region of influence is referred to as the *lune* of u and v. Equivalently, $u, v \in P$ are adjacent if and only if

$$D(u,v) \le \max\{D(u,w), D(v,w)\},\ \text{for all } w \in P, w \ne u, v.$$

The *Gabriel graph of* P, denoted by $GG(P)$, has as its region of influence the closed disk having segment $\overline{uv}$ as diameter. That is, two vertices $u, v \in P$ are adjacent if and only if

$$D^2(u,v) < D^2(u,w) + D^2(v,w),\ \text{for all } w \in P, w \ne u, v.$$

A *Delaunay triangulation* of a set P of points in the plane, denoted by $DT(P)$, is a triangulation of P such that for each interior face, the triangle which bounds that face has the property that the circle circumscribing the triangle contains no other points of the graph in its interior. A set P may admit more than one Delaunay triangulation, but only if P contains four or more co-circular points. A list of properties of the Delaunay triangulation can be found in [15].

We describe another graph, a *minimum spanning tree*, which is not defined in terms of a region of influence. Given a set P of points in the plane, consider a connected straight-line graph G on P, that is, a graph having as its edge set E a collection of line segments connecting pairs of vertices of P. Define the *weight* of G to be the sum of all of the edge lengths of G. Such a graph is called a *minimum spanning tree of* P, denoted by $MST(P)$, if its weight is no greater than the weight of any other connected straight-line graph on P. (It is easy to see that such a graph must be a tree.) In general, a set P may have many minimum spanning trees (for example, if P consists of the vertices of a regular polygon).

The following relationships among the different proximity graphs hold for any finite set P of points in the plane.

Lemma 2.1 ([15]). $MST(P) \subseteq RNG(P) \subseteq GG(P) \subseteq DT(P)$

A *β-skeleton* of a set P of points in the plane is a proximity graph in which the region of influence, $R(u,v,\beta)$, for two points $u, v \in P$ is a function of β:

1. For $\beta = 0$, $R(u,v,\beta)$ is the line segment $\overline{uv}$.
2. For $0 < \beta < 1$, $R(u,v,\beta)$ is the intersection of the two disks of radius $D(u,v)/(2\beta)$ passing through both u and v.
3. For $1 \le \beta < \infty$, $R(u,v,\beta)$ is the intersection of the two disks of radius $\beta D(u,v)/2$ centered at the points $(1-\beta/2)u+(\beta/2)v$ and $(\beta/2)u+(1-\beta/2)v$.
4. For $\beta = \infty$, $R(u,v,\beta)$ is the infinite strip perpendicular to the line segment $\overline{uv}$

The edge uv is in the β-skeleton of P if $R(u,v,\beta) \cap P \setminus \{u,v\} = \emptyset$. Notice that different values of the parameter β give rise to different graphs. Note also that different graphs may result for the same value of β if the regions of influence are constructed with open rather than closed disks, however, these boundary effects do not alter our results. When necessary, we will explicitly state whether the region of influence is open or closed. These graphs will be referred to as open β-skeletons and closed β-skeletons, respectively. The closed 1-skeleton is the Gabriel graph and the open 2-skeleton is the relative neighborhood graph.

As the value of β increases, β-skeletons become sparser since each region of influence expands:

Observation 1. *If $\beta \le \beta'$, then the β'-skeleton is a subset of the β-skeleton of a point set.*

β-skeletons with $\beta > 2$ may be disconnected, so we will concentrate on the interval $\beta \in [0,2]$.

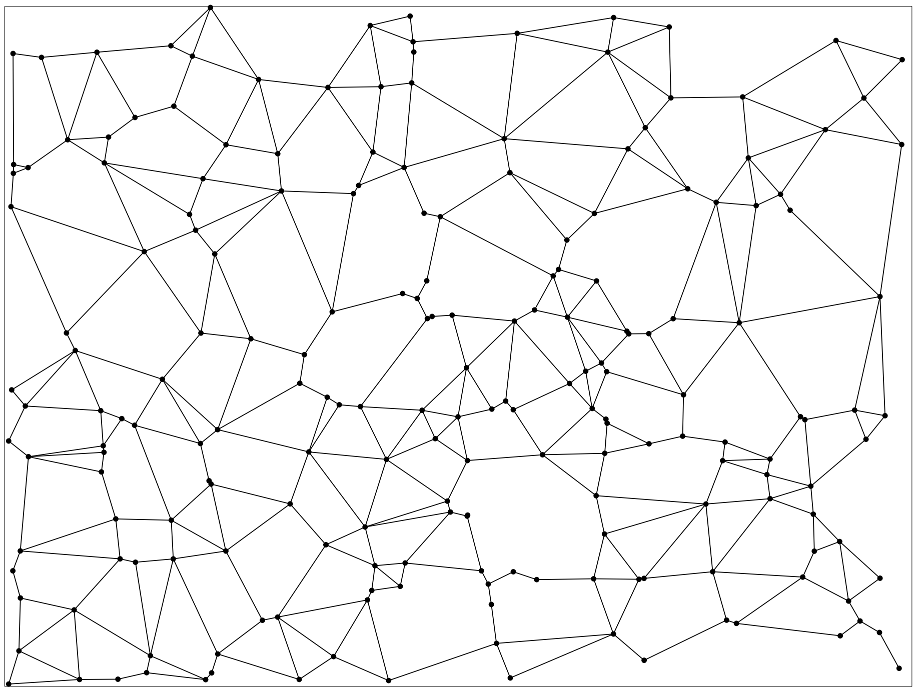

FIG. 2.1. *Gabriel graph for a random point set.*

3. Lower bounds. When $\beta = 0$, the β-skeleton of a point set has spanning ratio 1. When β is in the interval $(0, 1]$, Eppstein[6] presents an elegant fractal construction that proves a non-constant lower bound on the spanning ratio. His result is summarized in the following theorem.

THEOREM 3.1 (Eppstein [6]). *For any $n = 5^k + 1$, there exists a set of n points in the plane whose β-skeleton with $\beta \in (0, 1]$ has spanning ratio $\Omega(n^c)$, where $c = \log_5(5/(3 + 2\cos\theta))$ and $\theta < (2/3)\sin^{-1}\beta$.*

Our lower bounds apply to β-skeletons with $\beta \in [1, 2]$. The tower construction developed here in the proof of Theorem 3.2 is similar to the tower-like configuration we later use in lower bounding the spanning ratio of random Gabriel graphs.

THEOREM 3.2. *For any $n \geq 2$, there exists a set of n points in the plane whose β-skeleton with $\beta \in [1, 2]$ has spanning ratio*

$$S \geq \left(\frac{1}{2} - o(1)\right)\sqrt{n}\ .$$

Note that the closed 1-skeleton is the Gabriel graph and that all β-skeletons with $\beta > 1$ are subgraphs of the Gabriel graph. Therefore, it suffices to prove the theorem for the Gabriel graph. Also, the $1/2 - o(1)$ factor can be improved to $2/3$.

Proof. Let $m = \lfloor n/2 \rfloor$. Place points p_i and q_i at locations $(-r_i, y_i)$ and (r_i, y_i) respectively $(1 \leq i \leq m)$ where

$$r_i = 1 - (i-1)/n$$
$$y_i = (i-1)/\sqrt{n}$$

If n is odd place the remaining point at the same location as p_1.

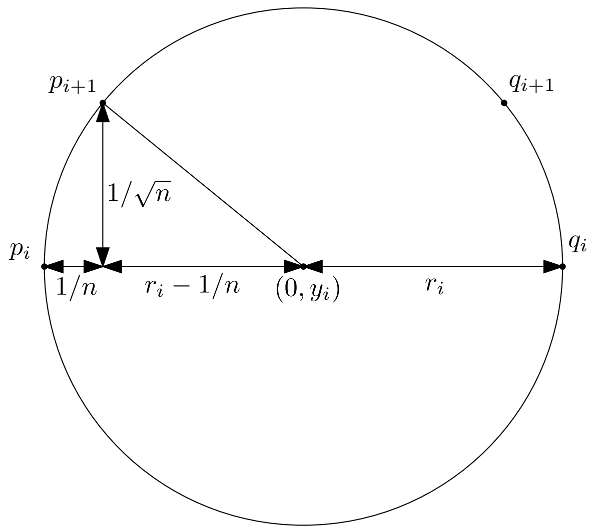

FIG. 3.1. *Illustration of one level in the Gabriel graph tower construction.*

We claim that for each pair p_i, q_i, the circle with diameter p_iq_i contains the points p_{i+1} and q_{i+1} $(1 \le i \le m-1)$. Let d be the distance from the center of the circle with diameter p_iq_i to the point p_{i+1}. For p_{i+1} to lie within this circle, d must be at most r_i. By construction,

$$d = \sqrt{(r_i - 1/n)^2 + 1/n} \ .$$

Thus we require $(r_i - 1/n)^2 + 1/n \le r_i^2$ or, equivalently, $r_i \ge 1/2 + 1/(2n)$, which holds for $1 \le i \le m-1$.

It follows that when $i \le j$, edge p_iq_j does not belong to the Gabriel graph of these points (unless $i = j = m$), since p_{i+1} lies in or on the circle with diameter p_iq_j. Similarly, when $i > j$, edge p_iq_j is precluded by point q_{j+1}.

The Euclidean distance between p_1 and q_1 is two. However, the shortest path from p_1 to q_1 using Gabriel graph edges is at least $2y_m$, which results in a spanning ratio of

$$S = y_m = (\lfloor n/2 \rfloor - 1)/\sqrt{n} = \left(\frac{1}{2} - o(1)\right)\sqrt{n} \ .$$

□

Note that for Gabriel graphs ($\beta = 1$), Eppstein's result (Theorem 3.1) implies a ratio of $\Omega(n^c)$ with $0.138 < c < 0.139$, while Theorem 3.2 provides a much stronger bound of $\Omega(\sqrt{n})$.

For relative neighborhood graphs ($\beta = 2$), the lower bound is $\Omega(n)$.

THEOREM 3.3. *For any $n \ge 2$, there exists a set of n points in the plane whose relative neighborhood graph (open 2-skeleton) has spanning ratio $\Omega(n)$.*

Proof. Refer to Figure 3.2. Let $\theta = 60 - \epsilon$ and $\alpha = 60 + 2\epsilon$. We will fix ϵ later. Since $\alpha + 2\theta = \pi$, the points $a_0, a_1, \ldots, a_n$ are colinear. Similarly, the points $b_0, b_1, \ldots, b_n$ are colinear. The point a_{i+1} blocks the edge a_ib_i. An edge a_ib_j for $i < j$ is blocked by a_{i+1} and an edge a_ib_j for $i > j$ is blocked by b_{i+1}. Thus, the only edges in the relative neighborhood graph of these points are a_ia_{i+1}, b_ib_{i+1} and a_nb_n. Let $A_i = \|a_{i+1} - a_i\|$. Let $B_i = \|b_{i+1} - b_i\|$.

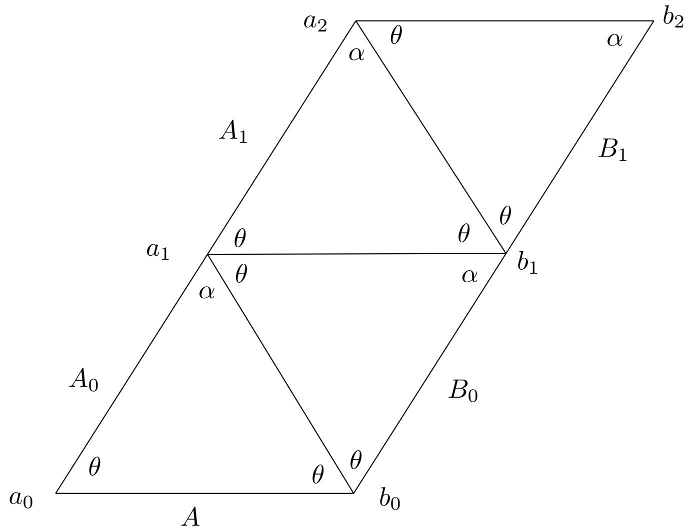

FIG. 3.2. *Relative neighborhood graph tower*

Triangle(a_0, a_1, b_0) and triangle(a_1, b_1, b_0) are similar, therefore, $B_0 = A_0^2/A$. By the same argument, $A_1 = A_0^3/A^2$, and $B_1 = A_0^4/A^3$. In general, $A_i = A_0^{2i+1}/A^{2i}$ and $B_i = A_0^{2i+2}/A^{2i+1}$.

We choose an ϵ so that $A_0/A > (1/2)^{1/(2n)}$. Let L be the length of the path from a_0 to b_0. $L > \sum_{i=0}^{n-1} A_i + B_i = \sum_{i=0}^{2n-1} A_0(A_0/A)^i$. Since $A_0/A > (1/2)^{1/(2n)}$, we have that $\sum_{i=0}^{2n-1} A_0(A_0/A)^i > 1/2 \sum_{i=0}^{2n-1} A_0 = A_0 n$. Therefore, $L > A_0 n$. ∎

4. Upper bounds. We start with a straight-forward upper bound that applies to all β-skeletons for $\beta \in [0, 2]$.

THEOREM 4.1. *For any $\beta \in [0, 2]$, the spanning ratio of the β-skeleton of a set of n points is at most $n - 1$.*

Proof. Let G be the β-skeleton of a set of n points P. Note that the minimum spanning tree $MST(P)$ is contained in G. Every edge in the unique path from u to v in $MST(P)$ has length at most $D(u, v)$, otherwise $MST(P)$ is not minimum. Therefore the shortest path in G from u to v has length at most $(n - 1)D(u, v)$. ∎

The rest of this section establishes an upper bound for β-skeletons when $\beta \in [0, 1]$. The β-skeleton of a point set P for $\beta \in [0, 1]$ is a graph in which points x and y in P are connected by an edge if and only if there is no other point $v \in P$ such that $\angle xvy > \pi - \sin^{-1}\beta$.

To upper bound the spanning ratio of β-skeletons, we show that there exits a special walk $SW_\beta(x, y)$ in the β-skeleton between the endpoints of any Delaunay edge xy. We upper bound the length $|SW_\beta(x, y)|$ of $SW_\beta(x, y)$ as a multiple of $D(x, y)$. We then combine this with an upper bound on the spanning ratio of Delaunay triangulations [10, 11] to obtain our result.

Let $DT(P)$ be the Delaunay triangulation of a points set P. In order to describe the walk between the endpoints of a Delaunay edge, we define the *peak* of a Delaunay edge.

LEMMA 4.2. *Let xy be an edge of $DT(P)$. For $\beta \in [0, 1]$, either xy is an edge of the β-skeleton of P or there exists a unique z (called the peak of xy) such that triangle(xyz) is in $DT(P)$ and z lies in the β-region of xy.*

Proof. Suppose $xy \in DT(P)$ is not an edge in the β-skeleton of P. Then there exists a point $v \in P$ such that $\angle xvy > \pi - \sin^{-1}\beta$. Since xy is an edge of $DT(P)$, there exists a unique z on the same side of xy as v such that disc(xyz) is empty. This implies $\angle xzy \geq \angle xvy$ and thus z lies in the β-region of xy. Since $\beta \leq 1$, disc(xyz) contains that part of the β-region of xy which lies on the other side of xy from z. Since this circle is empty, z is unique. ▯

We now define the walk $SW_\beta(x, y)$ between the endpoints of the Delaunay edge xy. (Note that in a walk edges may be repeated. See Bondy and Murty for details [1].)

$$SW_\beta(x,y) = \begin{cases} xy & \text{if } xy \in \beta\text{-skeleton of } P \\ SW_\beta(x,z) \cup SW_\beta(z,y) & \text{otherwise } (z \text{ is the peak of } xy) \end{cases}$$

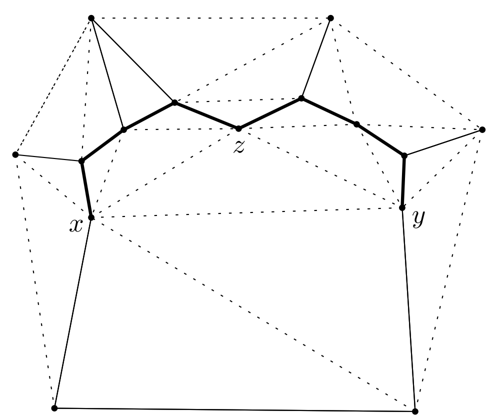

FIG. 4.1. *The solid lines form the Gabriel graph of the point set with $SW_1(x,y)$ in bold. All edges together form the Delaunay triangulation.*

LEMMA 4.3. *Given a set P of n points in the plane. If $xy \in DT(P)$ then the number of edges in $SW_\beta(x,y)$ is at most $6n - 12$, for $\beta \in [0, 1]$.*

Proof. Since a Delaunay edge is adjacent to at most two Delaunay triangles, an edge can occur at most twice in the walk $SW_\beta(x, y)$. Since there are at most $3n - 6$ edges in $DT(P)$ by Euler's formula, $SW_\beta(x, y)$ can consist of at most $6n - 12$ edges. ▯

LEMMA 4.4. *Let P be a set of n points in the plane. For any $\beta \in [0, 1]$, for all $x, y \in P$, if $xy \in DT(P)$ then*

$$|SW_\beta(x,y)| \leq m^\gamma D(x,y)$$

where $\gamma = (1 - \log_2(1 + \sqrt{1-\beta^2}))/2$ and m is the number of edges in $SW_\beta(x,y)$.

Proof. [1] The proof is by induction on the number of edges m in $SW_\beta(x, y)$. When $m = 1$, i.e. $SW_\beta(x, y)$ is simply the line segment from x to y, the lemma clearly holds.

If $m > 1$, then $|SW_\beta(x,y)| = |SW_\beta(x,z)| + |SW_\beta(z,y)|$ for z the peak of xy. Let k be the number of edges in $SW_\beta(x, z)$. Thus, $m - k$ is the number of edges in $SW_\beta(z, y)$. Let $a = D(x, y)$, $b = D(x, z)$, and $c = D(y, z)$. Since xz and zy are

[1] Thanks to Ansgar Grüne and Sébastien Lorenz at the University of Bonn for pointing out a flaw in an earlier proof.

Delaunay edges, by induction, $|SW_\beta(x,z)| \le bk^\gamma$ and $|SW_\beta(z,y)| \le c(m-k)^\gamma$. Thus it suffices to prove that

$$bk^\gamma + c(m-k)^\gamma \le am^\gamma \ .$$

By the law of cosines, $a^2 = b^2 + c^2 - 2bc\cos A$ where A is the angle at the peak z. With this substitution for a, after dividing both sides by c and letting $\delta = b/c$, it remains to show,

$$\delta k^\gamma + (m-k)^\gamma \le m^\gamma \le \sqrt{1+\delta^2 - 2\delta\cos A}$$

where we may assume without loss of generality that $\delta \in [0,1]$. As a function of k the left-hand side of the equation is maximized when $k = m/(1+\delta^{-s})$ where $s = 1/(1-\gamma)$. With this substitution for k, after factoring m^γ, it suffices to show,

$$\frac{\delta + \delta^{-\gamma s}}{(1+\delta^{-s})^\gamma} \le \sqrt{1+\delta^2 - 2\delta\cos A} \qquad \text{when } \delta \in [0,1].$$

We can simplify the left-hand side using the fact that $s = 1/(1-\gamma)$:

$$\frac{\delta + \delta^{-\gamma s}}{(1+\delta^{-s})^\gamma} = \frac{\delta(1+\delta^{-s})}{(1+\delta^{-s})^\gamma} = \delta(1+\delta^{-s})^{1-\gamma} = (\delta^s + 1)^{1-\gamma}.$$

Thus, after squaring both sides of the inequality, it suffices to show,

$$(1+\delta^s)^{2/s} \le 1 + \delta^2 - 2\delta\cos A \qquad \text{when } \delta \in [0,1].$$

The angle A is minimized (thus minimizing the right-hand side of the inequality) when z lies on the boundary of the β-region. For such z, $\cos A = -\sqrt{1-\beta^2}$, and it remains to show,

$$(1+\delta^s)^{2/s} \le 1 + \delta^2 + 2\delta\sqrt{1-\beta^2} \qquad \text{when } \delta \in [0,1].$$

Let $L(\delta)$ be the left-hand side and $R(\delta)$ the right-hand side of this inequality. We want to show that $L(\delta) \le R(\delta)$ when $\delta \in [0,1]$. The maximum of $L(\delta) - R(\delta)$ (for $\delta \in [0,1]$) occurs at $\delta = 0$ or $\delta = 1$ or at some value δ with $L'(\delta) = R'(\delta)$. At $\delta = 0$, $L(0) = R(0) = 1$. At $\delta = 1$,

$$L(1) = 2^{2/s} \qquad \text{and} \qquad R(1) = 2 + 2\sqrt{1-\beta^2}.$$

Since $\gamma = (1 - \log_2(1+\sqrt{1-\beta^2}))/2$, $s = 2/(1+\log_2(1+\sqrt{1-\beta^2}))$, and $L(1) = R(1)$. The derivatives of $L(\delta)$ and $R(\delta)$ are

$$L'(\delta) = 2\delta^{s-1}(1+\delta^s)^{2/s-1} \qquad \text{and} \qquad R'(\delta) = 2\delta + 2\sqrt{1-\beta^2}.$$

For $\beta \in [0,1]$, $L'(0) \le R'(0)$, and for our chosen value of γ, $L'(1) = R'(1)$. For $\beta \in [0,1]$, γ lies in $[0, 1/2]$, which implies $s \in [1,2]$. Thus,

$$L'''(\delta) = 2(1+\delta^s)^{2/s-3}\delta^{s-3}(s-1)(s-2)(1-\delta^s) \le 0$$

and the function $L'(\delta)$ is concave. Since $R'(\delta)$ is linear and $L'(1) = R'(1)$, there is at most one value of $\delta \in (0,1)$ where $L'(\delta) = R'(\delta)$. Since $L'(0) \le R'(0)$, $L(\delta) - R(\delta)$ is a minimum at this value. Thus the maximum of $L(\delta) - R(\delta)$ is 0 for $\gamma = (1 - \log_2(1 + \sqrt{1-\beta^2}))/2$. ∎

THEOREM 4.5. *For $\beta \in [0,1]$, the spanning ratio of the β-skeleton of a set P of n points in the plane is at most*

$$\frac{4\pi(6n-12)^\gamma}{3\sqrt{3}}$$

where $\gamma = (1 - \log_2(1 + \sqrt{1-\beta^2}))/2$.

Proof. Given two arbitrary points x, y in P, let $M = e_1, e_2, \ldots, e_j$ represent the shortest path between x and y in $DT(P)$. Keil and Gutwin [10, 11] have shown that the length of P is at most $2\pi/(3\cos(\pi/6))$ times $D(x,y)$.

For each edge e_i in M, by Lemma 4.3 and Lemma 4.4, we know there exists a path in the β-skeleton whose length is at most $(6n-12)^\gamma$ times the length of e_i. Therefore, the shortest path between x and y in the β-skeleton has length at most $2\pi(6n-12)^\gamma/(3\cos(\pi/6))$ times $D(x,y)$. The theorem follows. ∎

COROLLARY 4.6. *The spanning ratio of the Gabriel graph ($\beta = 1$) of an n-point set is at most*

$$\frac{4\pi}{3}\sqrt{2n-4}\ .$$

When β lies strictly between 0 and 1, there is a gap between the upper bound and lower bound on the spanning ratio of β-skeletons. As noted in Section 3, the spanning ratio is at least $\Omega(n^c)$ where $c = \log_5(5/(3+2\cos\theta))$ and $\theta < (2/3)\sin^{-1}\beta$. We have shown here that the spanning ratio is at most $O(n^\gamma)$ where $\gamma = (1 - \log_2(1 + \sqrt{1-\beta^2}))/2$. Refer to Figure 4.2 for a graph of the exponents of the upper and lower bound. For Gabriel graphs ($\beta = 1$), the lower bound construction given in Section 3, together with the upper bound given here, show that the spanning ratio is indeed $\Theta(\sqrt{n})$.

5. Random Gabriel graphs. If n points are drawn uniformly and at random from the unit square $[0,1]^2$, the spanning ratio of the induced Gabriel graph grows unbounded in probability. In particular, we have the following.

THEOREM 5.1. *If n points are drawn uniformly and at random from the unit square $[0,1]^2$, and S is the spanning ratio of the induced Gabriel graph then*

$$\mathbf{P}\left\{ S < c\sqrt{\frac{a\log n}{\log\log n}} \right\} \le 2e^{-2n^{1-12a-o(1)}}$$

for constants c and $a < 1/12$. Thus, for $a < 1/12$, with probability tending exponentially quickly to one,

$$S \ge c\sqrt{a\log n/\log\log n}\ .$$

Proof. The main idea is to show that a set of n points randomly distributed in the unit square contains many tower-like structures of size $c\log n/\log\log n$, each of which has spanning ratio approximately the square root of its size. We first define what a tower-like structure is and then show that the expected number of such structures is large.

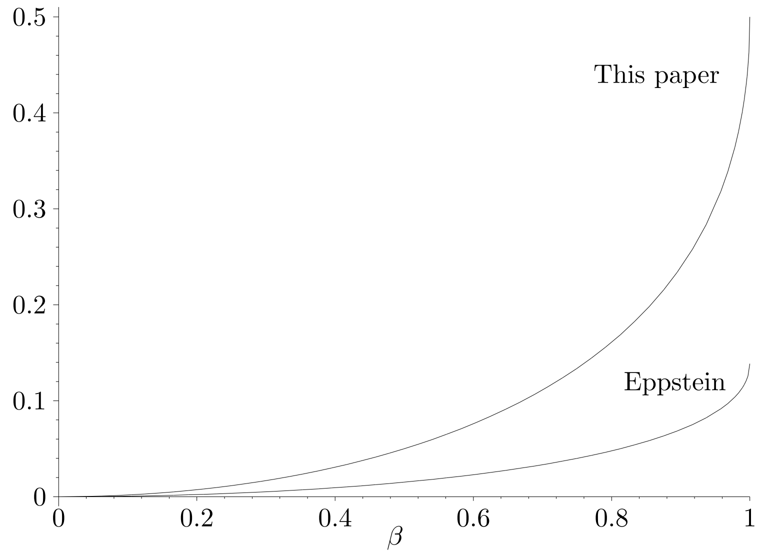

FIG. 4.2. *Exponents of n in the upper and lower bound on the spanning ratio of β-skeletons when $\beta \in [0,1]$.*

A tower-like structure resembles the towers of Section 3 but the points may be slightly perturbed. For $i = 1, ..., k$, let A_i and B_i be discs both of radius d/k (the constant d will be specified later) located at (r_i, y_i) and $(-r_i, y_i)$ respectively, where the sequences r_i and y_i are given below.

$$r_i = 1 - \frac{i-1}{2k}$$
$$y_i = (i-1)\sqrt{\frac{1/2 - (1+\sqrt{2})d}{k}\left(1 - \frac{1/2 - (1+\sqrt{2})d}{k}\right)}.$$

The value of d is chosen so that y_i is positive ($d < 1/(2+2\sqrt{2})$).

Let C be the smallest square enclosing the A_i and B_i within a border of width y_k. Typically, when k is large enough and the tower is taller than it is wide, C extends from $(-3y_k/2 - d/k, -y_k - d/k)$ to $(3y_k/2 + d/k, 2y_k + d/k)$. See Figure 5.2 for an example of such a square, and note that in this figure, the discs A_i and B_i would be smaller than the dots used to represent points.

Assume that each of the A_i and B_i contain exactly one point and C contains no other data point beyond these $2k$ points. We claim that among the points in C, the only edges are those connecting A_1 with A_2, A_2 with A_3, and so forth, up to A_{k-1} and A_k. Then A_k connects with B_k, B_k with B_{k-1} and so forth down to B_1. The proof of this claim is rather technical and is deferred to Appendix A. Note that the A_i's and B_i's are disjoint.

Let u and v be the points in A_1 and B_1 respectively. We have $D(u,v) \leq 2 + 2d/k$. Also, any path from u to v entirely in C must be equal in length to the chain, which is longer than $2y_k$. If the path leaves C, then at least two edges leave C, and those edges have length at least $2y_k$, taken together. Thus, $L(u,v) \geq 2y_k$ and

$$S \geq \frac{L(u,v)}{D(u,v)} \geq \frac{y_k}{1+d/k} \geq c\sqrt{k}$$

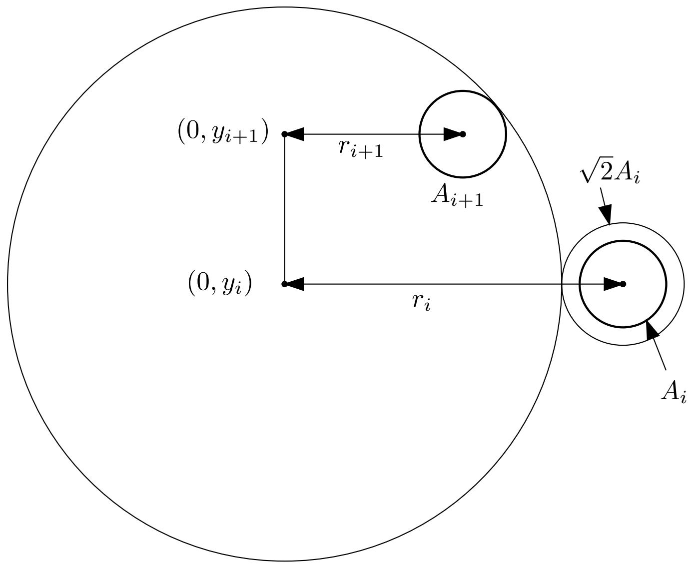

FIG. 5.1. *The construction of A_i and A_{i+1}.*

for sufficiently large k where c is a constant that depends on d.

Let bC denote the scaled down set $\{bx : x \in C\}$.

Divide $[0,1]^2$ into n non-overlapping *tiles* of size $1/\sqrt{n} \times 1/\sqrt{n}$. For $b = 1/(4\sqrt{kn})$, bC fits within one of these tiles. Thus we may place n non-overlapping copies of bC within the unit square. For a given data set, we call a tile *tower-like* if it contains exactly $2k$ data points, one each for bA_i and bB_i, $1 \le i \le k$ within it. Let N be the number of tiles that are tower-like.

Clearly, since the distribution is uniform,

$$\mathbf{E}N = n\mathbf{P}\{\text{a tile is tower-like}\}\ .$$

Pick one tile and partition the n data points over the following disjoint sets: the bA_i's, the bB_i's, $bC - \cup bA_i \cup bB_i$, and $[0,1]^2 - bC$. The cardinalities of these sets, taken together, form a multinomial random vector with probabilities given by the areas of the sets involved. For example, area $(bA_i) = b^2\pi d^2/k^2$. According to the formula for the multinomial distribution,

$$\begin{aligned}
\mathbf{P}\{\text{a tile is tower-like}\} &= \frac{n!}{(n-2k)!}\left(\frac{b^2\pi d^2}{k^2}\right)^{2k}(1-1/n)^{n-2k} \\
&\ge (n-2k+1)^{2k}\left(\frac{\pi d^2}{16nk^3}\right)^{2k}(1-1/n)^n \\
&\ge \frac{1}{4}\left(\frac{(n-2k+1)\pi d^2}{16nk^3}\right)^{2k} \\
&\ge \frac{1}{4}\left(\frac{\pi d^2}{32k^3}\right)^{2k}
\end{aligned}$$

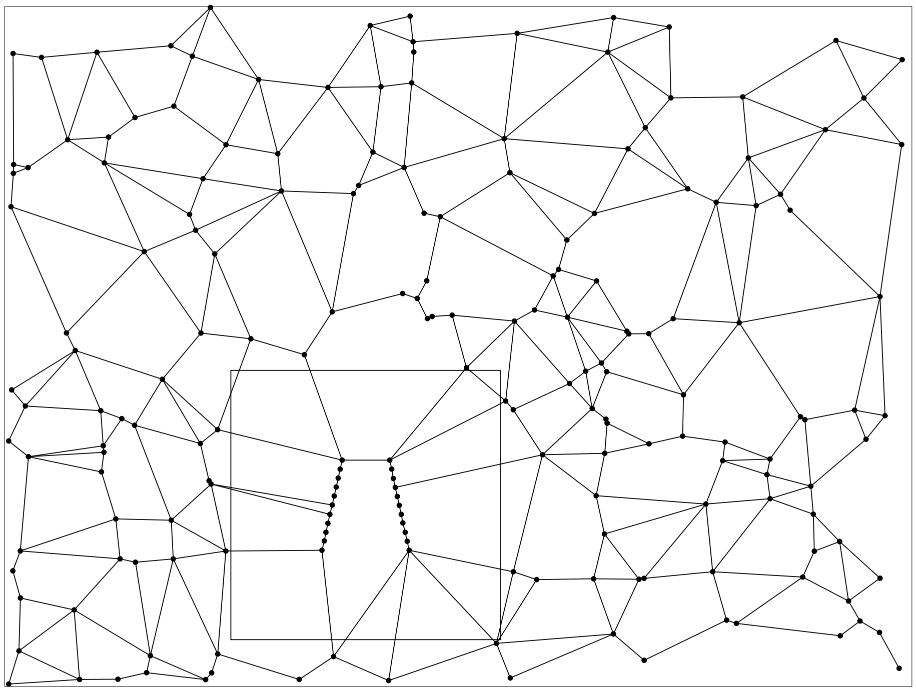

FIG. 5.2. *Gabriel graph with tower-like square.*

provided that n is sufficiently large and $k < (n+2)/4$. We conclude that

$$\mathbf{E}N \geq \frac{n}{4} \left(\frac{\pi d^2}{32k^3} \right)^{2k} .$$

If $k = a \log n / \log\log n$ for a constant $a < 1/6$, then

$$\mathbf{E}N \geq n^{1-6a-o(1)} \to \infty .$$

For each one of these tower-like squares, there is a pair of data points for which the spanning ratio is at least

$$c\sqrt{k} \geq c\sqrt{\frac{a \log n}{\log\log n}} .$$

Change one of the n data points. That will change the number N by at most one. But then, by McDiarmid's inequality [14], we have

$$\mathbf{P}\{|N - \mathbf{E}N| \geq t\} \leq 2e^{-2t^2/n} .$$

In particular, for fixed $\epsilon > 0$,

$$\mathbf{P}\{|N - \mathbf{E}N| \geq \epsilon \mathbf{E}N\} \leq 2e^{-2\epsilon^2 n^{1-12a-o(1)}} \to 0$$

when $a < 1/12$. This shows that $N/\mathbf{E}N \to 1$ in probability for such a choice of a (and thus k), and thus that for every $\epsilon > 0$,

$$\mathbf{P}\{N < (1-\epsilon)\mathbf{E}N\} \to 0 .$$

	$\beta = 0$	$0 < \beta < 1$	$\beta = 1$	$1 < \beta < 2$	$\beta = 2$	$\beta > 2$
Lower Bound	1	$\Omega(n^c)$[6]	$\Omega(\sqrt{n})$	$\Omega(\sqrt{n})$	$\Omega(n)$	∞
Upper Bound	1	$O(n^\gamma)$	$O(\sqrt{n})$	$O(n)$	$O(n)$	∞

$c = \log_5(5/(3 + 2\cos\theta))$ and $\theta < (2/3)\sin^{-1}\beta$.
$\gamma = (1 - \log_2(1 + \sqrt{1 - \beta^2}))/2$.

TABLE 6.1
Summary Table of Results on the Spanning Ratio of β-skeletons

As another application, we have

$$\begin{aligned}\mathbf{P}\{S < c\sqrt{a \log n / \log\log n}\} &\leq \mathbf{P}\{N = 0\} \\ &= \mathbf{P}\{N - \mathbf{E}N \leq -\mathbf{E}N\} \\ &\leq 2e^{-2n^{1-12a-o(1)}} \\ &\to 0 \ .\end{aligned}$$

Note that this probability decreases exponentially quickly with n. □

We have implicitly shown several other properties of random Gabriel graphs. For example, a Gabriel graph partitions the plane into a finite number of polygonal regions. The outside polygon which extends to ∞ is excluded. Let D_n be the maximum number of vertices in these polygons. Then $D_n \to \infty$ in probability, because D_n is larger than the maximum size of any tower that occurs in the point set, and this was shown to diverge in probability. From what transpired above, this is bounded from below in probability by $\Omega(a \log n / \log\log n)$.

6. Conclusion. We studied the spanning ratio of β-skeletons with β ranging from 0 to 2. This class of proximity graphs includes the Gabriel graph and the relative neighborhood graph. Table 6.1 summerizes our results. For $\beta > 2$, β-skeletons lose connectivity; thus, their spanning ratio leaps to infinity. For points drawn independently from the uniform distribution on the unit square, we showed that the spanning ratio of the (random) Gabriel graph (and all β-skeletons with $\beta \in [1, 2]$) tends to ∞ in probability as $\sqrt{\log n / \log\log n}$

Several open problems arise from this investigation. It would be interesting to close the gap between upper and lower bounds for β-skeletons in the ranges $0 < \beta < 1$ and $1 < \beta < 2$. Also, for random point sets, it would be interesting to try to find a matching upper bound for the spanning ratio.

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Appendix A. Tower-like construction.

The purpose of this section is to show that the only Gabriel edges among the points in the tower-like construction described in Section 5 are between A_i and A_{i+1}, B_i and B_{i+1}, and A_k and B_k for $i = 1 \ldots k-1$. Recall that the tower-like construction consists of $2k$ points, one in each of the discs A_i, B_i for $i = 1 \ldots k$, where A_i and B_i are discs of radius d/k centered at (y_i, r_i) and $(y_i, -r_i)$ respectively. The definition of the sequences r_i and y_i is repeated here:

$$r_i = 1 - \frac{i-1}{2k}$$
$$y_i = (i-1)\sqrt{\frac{1/2-(1+\sqrt{2})d}{k}\left(1-\frac{1/2-(1+\sqrt{2})d}{k}\right)}\,.$$

In a Gabriel graph, two points are connected by an edge if and only if the disc whose diameter is the segment joining those points is empty. In our construction, we do not have precise information as to the location of the points. We only know that a point lies within a small disc (whose location we do know). Thus a basic problem is, given two discs, what is the region that, if it contains a point, will forbid an edge between a point in one disc and a point in the other. After we have determined this region, we must show for any two discs in our construction between which we claim no edge exists, that there is a third disc contained within that pair's region.

Let A and B be two discs each of radius s, whose centers are at $p = (r, 0)$ and $q = (-r, 0)$. The region Q we are interested in is the intersection of all discs whose diameter has one endpoint a in A and the other endpoint b in B. See Figure A.1.

We will determine the upper boundary of Q (the points with positive y coordinate). The lower boundary is symmetric. Consider a ray with origin $(0, -r)$ that

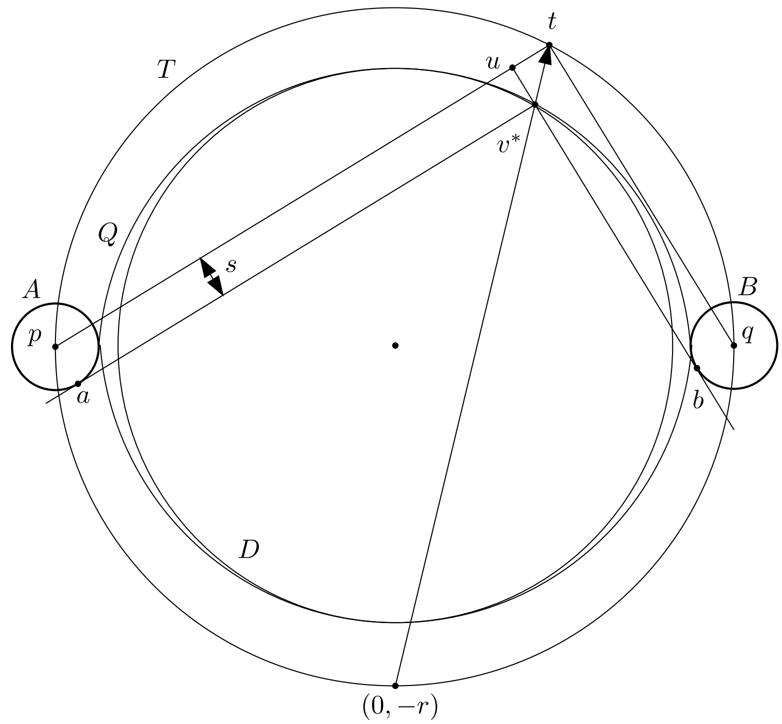

FIG. A.1. *If the region Q contains a point then, for any $a \in A$ and $b \in B$, the edge ab is not in the Gabriel graph of the point set. We insure that there is a point in the disc $D \subset Q$.*

intersects the segment pq. A point v on this ray is inside Q if and only if for all points $a \in A$ and $b \in B$, $\angle avb \geq \pi/2$. For each point v, the points $a \in A$ and $b \in B$ that minimize $\angle avb$ are the tangent points of the lines through v tangent to the A and B respectively. (Strictly speaking, there are two tangent lines from a point to a disc; and a and b are defined by those tangent lines which form the minimum of the resulting four possible angles.) For v with positive y coordinate, this minimum angle is a continuous, decreasing function of the distance between v and $(0, -r)$. Thus the upper boundary of Q intersects the ray at a single point v^* where $\min_{a \in A, b \in B} \angle av^*b = \pi/2$.

Let T be the circle whose diameter is the segment pq, and let t be the point (other than $(0, -r)$) where our chosen ray intersects T. We claim that v^* is the point on the ray that is distance $s\sqrt{2}$ from t. To show this, consider the lines from t to p and q. These lines are parallel to the tangent lines from v^* to a and b respectively, where $a \in A$ and $b \in B$ minimize $\angle av^*b$. In order to establish that $\overline{v^*a}$ is parallel to $\overline{tp}$, drop a line perpendicular to $\overline{tp}$ from v^* to a point u on $\overline{tp}$. Since $\angle p, t, (0, -r) = \pi/4$, the triangle $\triangle tuv^*$ is a right, isoceles triangle. Its hypotenuse has length $s\sqrt{2}$ so its sides have length s. Thus $\overline{v^*a}$ is consistently distance s from $\overline{tp}$. The same argument applies to $\overline{v^*b}$ and $\overline{tq}$. Since $\angle ptq = \pi/2$, the claim is established. It is perhaps surprising that the region Q does not touch A or B.

For our tower-like construction we use a disc to approximate the region Q. The disc D centered at $(0, 0)$ with radius $r - s\sqrt{2}$ is contained within Q. (Note: The point v^* appears to lie on the boundary of D in Figure A.1. This is misleading. The point v^* does not lie on the boundary of D except for v^* with x-coordinate equal to 0.) Thus if a point lies within D, there is no Gabriel edge between any two points $a \in A$

and $b \in B$. The tower-like construction insures that this is the case for any pair of discs $A = A_i$ and $B = B_i$, by placing the discs A_{i+1} and B_{i+1} within the disc D. Also the disc D for any pair $A = A_i$ and $B = B_j$ with $i \neq j$ contains either A_{i+1} if $i < j$ or B_{i+1} if $i > j$. Finally, for $A = A_i$ and $B = A_j$ with $i < j - 1$, the disc D contains A_k where $i < k < j$. (This holds for B_i discs by symmetry.)

It remains to show that the remaining edges in the tower-like construction do exist. A similar argument to the one presented above establishes that the union of the discs with diameter ab with $a \in A$ and $b \in B$ is a region contained in the disc $\hat{D}$ of radius $r + s\sqrt{2}$ centered at the origin. This region is empty for each pair $A = A_i$ and $B = A_{i+1}$ since $r_i \geq 1/2$ while the distance between the centers of A and B is $O(1/\sqrt{k})$.