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THE SERIES METHOD FOR RANDOM VARIATE GENERATION AND ITS APPLICATION TO THE KOLMOGOROV - SMIRNOV DISTRIBUTION

Luc Devroye
McGill University
School of Computer Science
805 Sherbrooke Street West
Montreal, Canada H3A 2K6

SYNOPTIC ABSTRACT

This paper presents a series method for the computer generation of a random variable X with density f when $f = \lim_{n\to\infty} f_n = \lim_{n\to\infty} g_n$ and f_n and g_n are given sequences of functions satisfying $f_n \geq f \geq g_n$; f is never evaluated. This method can be used when f is given as an infinite series. Three complete examples are given, and a computer program is included for the generation of random variates from the Kolmogorov-Smirnov distribution.

1. INTRODUCTION

Consider the problem of computer generation of a random variable X with density f, where f is a (complicated) function which can be approximated from above and below by simpler functions f_n and g_n. In particular, assume that:

(i) there exist sequences of functions f_n and g_n such that

$$f_n \geq f \geq g_n \text{ , for all } n \text{ ,} \qquad (1)$$

where

(ii) $$f_n \to f,\ g_n \to f \text{ as } n \to \infty \text{ ;}$$

and

(iii) there exists an integrable nonnegative function h such that

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$$h \geq f \ . \qquad (2)$$

Here h is proportional to a density which is easy to sample from, and f_n and g_n are sequences of functions which are easy to evaluate. (Note that g_n is not necessarily positive, and that f_n need not be integrable.) The following rejection-type algorithm can be used to generate X.

Algorithm S0 (The Series Method).

0.1 Generate X with density ch (c is a constant), generate an independent uniform (0,1) random variate U, set n = 0 and T = Uh(X).

0.2 n = n+1. If $T \leq f_n(X)$, exit with X.

0.3 If $T > g_n(X)$, go to 0.1.

0.4 Go to 0.2.

If f can be written as an alternating series

$$f(x) = h(x)\ (1-a_1(x)+a_2(x)-a_3(x)+\ldots) \qquad (3)$$

(where a_n is a sequence of functions satisfying

$$a_n(x) \downarrow 0 \ , \text{ for all } x, \qquad (4)$$

and h is a nonnegative integrable function) then X can be generated by:

Algorithm S1 (The Alternating Series Method).

1.1 Generate X with density ch (c is a constant), generate an independent uniform (0,1) random variate U, set n = 0 and T = 0.

1.2 n = n+1 , $T = T+a_n(X)$. If $U \geq T$, exit with X.

1.3 n = n+1 , $T = T-a_n(X)$. If $U < T$, go to 1.1.

1.4 Go to 1.2

It is clear that S1 is a special case of S0 because $f \leq h$ and

$$1 + \sum_{j=1}^{k} (-1)^j a_j(x) \leq \frac{f(x)}{h(x)} \leq 1 + \sum_{j=1}^{k+1} (-1)^j a_j(x) \ , \ k \text{ odd}.$$

S0 and S1 can be considered as generalizations of the acceptance/rejection method with squeezing, with the special feature that f or f/h need never be computed. S0 requires the evaluation of h, however. For recent detailed descriptions of the acceptance/rejection method, see Vaduva (1977), Tadikamalla (1978), or Tadikamalla and Johnson (1981). For the squeeze method, see Schmeiser and Lal (1980).

Different special cases are often encountered in practice. For example, if

$$f(x) = h(x) \exp(-a_1(x) + a_2(x)\ldots)$$

where $h \geq 0$ is integrable, and $a_n(x) \downarrow 0$ for all x , then X can be generated by

Algorithm S2 (The Exponential Series Method).

2.1 Generate X with density ch (c is a constant), generate an independent exponential random variate E , set n = 0 and T = 0.

2.2 n = n+1 , T = T + $a_n(X)$. If E ≥ T , exit with X.

2.3 n = n+1 , T = T - $a_n(X)$. If E < T , go to 2.1.

2.4 Go to 2.2.

Some examples are given in Sections 3 and 4, and the average time taken by algorithm S1 is analyzed in Section 2. Experimental timings are included and show that for some distributions where f contains trigonometric and/or exponential/logarithmic functions some savings can be obtained via our methods unless random variates for the distribution in question can be obtained in a very simple fast way by other means (e.g., as for the exponential distribution). An example is included for which the series method seems the only feasible method of random variate generation (i.e., the Kolmogorov-Smirnov distribution).

2. ANALYSIS OF ALGORITHM S1.

Let X be a random variate generated by algorithm S1, and let N_i be the number of times step 1.i in the algorithm was executed, $1 \le i \le 4$. We will show that when $a_0 \equiv 1$, and a_n and h satisfy (4) and $a_1(x) \le 1$,

(i) $P(N_2 < \infty) = 1$ and

(ii) $$E(N_2) = \sum_{i=1}^{\infty} i \int h(x)\,(a_{2i-2}(x) - a_{2i}(x))dx.$$

Since $N_i \le N_2$ for all i , it is clear that the properties of N_2 are essential for the study of the time taken by S1. Note that (i) is necessary in order for S1 to halt with probability one. It is possible however that $E(N_2) = \infty$.

Proof.

(i) is true in view of (4) and steps 1.2 and 1.3 of S1. For (ii), let (X,U) be a pair of random variates generated in step 1.1. Define event A by "(X,U) will be accepted; i.e., $Uh(X) \le f(X)$". Now

$$P(N_2=i\,|\,A) = \int h(x)(a_{2i-2}(x) - a_{2i-1}(x))dx = p_i$$

and

$$P(N_2=i\,|\,\bar{A}) = \left[\int h(x)\,(a_{2i-1}(x) - a_{2i}(x))dx\right] \Big/ \left[\int h(x)dx - 1\right] = q_i ,$$

where $\bar{A}$ denotes the complement of A . Since on the average $\int h(x)dx$ pairs (X,U) are needed, we have

$$E(N_2) = \sum_i i p_i + \left[\int h(x)dx - 1\right] \sum_i i q_i ,$$

from which (ii) follows.

3. THE EXPONENTIAL AND RAAB-GREEN DISTRIBUTIONS.

The exponential distribution. It is known that for all odd k ,

$$\sum_{j=0}^{k-1} (-1)^j \frac{x^j}{j!} \geq e^{-x} \geq \sum_{j=0}^{k} (-1)^j \frac{x^j}{j!} , \quad x > 0.$$

For the generation of exponential random variates, we can use SO with a well-chosen function h . We choose h from the family of densities

$$\frac{na^n}{(x+a)^{n+1}} , \quad x > 0 , \ n \geq 1 \text{ integer} ,$$

where $a > 0$ is a parameter. Note that this is the density of

$$a(U^{-\frac{1}{n}} - 1)$$

when U is a uniform (0,1) random variable. It is also the density of $a \max^{-1}(U_1,\ldots,U_n) - 1)$ when $U_1,\ldots,U_n$ are independent and identically distributed uniform (0,1) random variables. Since $e^{-x}(x+a)^{n+1}$ is maximal when $(n+1) = x+a$, i.e. $x = n + 1 - a$, we see that

$$e^{-x} \leq \frac{\gamma\, na^n}{(x+a)^{n+1}}$$

where $\gamma = (\frac{n+1}{e})^{n+1} \frac{e^a a^{-n}}{n}$. Now, $e^a a^{-n}$ is minimal when $a = n$. Thus, choosing $a = n$, $\gamma \frac{(n+1)^{n+1}}{n^{n+1}} \frac{e^n}{e^{n+1}} = (1 + \frac{1}{n})^{n+1} \frac{1}{e} \to 1$ as $n \to \infty$. In particular, for $n = 1,2,3$ we obtain $\frac{4}{3} \cong 1.471518$, $\frac{(3/2)^3}{e} \cong 1.241592$, $\frac{(4/3)^4}{e} \cong 1.162679$. Since the rejection rate decreases with n , but the time needed to obtain a random variate with density $1/(1+x/n)^{n+1}$ (as $n(\max^{-1}(U_1,\ldots,U_n) - 1)$) increases with n , we expect that the best performance is obtained for some small n (the value $n = 2$ will be suggested below). The algorithm version for the exponential distribution with mean 1 is:

0.1.E. Generate $n \geq 2$ independent uniform (0,1) random variates $U_1,\ldots,U_n$. Let $U = \max(U_1,\ldots,U_n)$. If $U_1 = U$, interchange U_2 and U_1 . Set $X = n(1/U - 1)$, $T = \gamma(\frac{U_1}{U})U^{n+1} - 1 = \gamma U_1 U^n - 1$ where $\gamma = (\frac{n+1}{n})^{n+1} \cdot \frac{1}{e}$.

[Note: U_1/U is uniform (0,1) and independent of U ; $U^{n+1} = 1/(1 + \frac{X}{n})^{n+1}$]. If $T > 0$, go to 0.1.E.

Set $j = 0$, $P = 1$.

0.2.E. $j = j+1$. $P = PX/j$. $T = T+P$. If $T \le 0$, exit with X .

0.3.E. $j = j+1$. $P = PX/j$. $T = T-P$. If $T > 0$, go to 0.1.E.

0.4.E. Go to 0.2.E.

Several remarks here will illustrate some dangers of the series method.

Remark 1. The computation of $E(N_2)$ is usually not possible by analytical means. However, much can still be said about this average. Clearly, $E(N_1) = \gamma$. Also, $E(N_4) = \gamma\, E(N_4^*)$ where N_4^* is the number of visits of step 4 before an exit (step 2) or a return to step 1 (see step 3). For the proof of this, let $N_4(r)$ ($N_4(a)$) be the number of visits of step 4 given ultimate rejection (ultimate acceptance). Then,

$$E(N_4) = (E(N_1) - 1)E(N_4(r)) + E(N_4(a))$$

$$= \frac{(1-1/\gamma)\ E(N_4(r))}{1/\gamma} + \frac{(1/\gamma)E(N_4(a))}{1/\gamma} = \gamma\, E(N_4^*) \ .$$

But

$$P(N_4^* \ge i) \le \int_0^\infty \frac{1}{(1+\frac{x}{n})^{n+1}} \cdot \min\left(1, \frac{x^{2i}(2i)!}{1/(1+\frac{x}{n})^{n+1}}\right) dx$$

$$\le \int_0^c \frac{x^{2i}}{(2i)!}\, dx + \int_c^\infty \frac{1}{(1+x/n)^{n+1}}\, dx$$

$$= \frac{c^{2i+1}}{(2i+1)!} + \frac{1}{(1+\frac{c}{x})^n}$$

for any $c > 0$. If we take $c = (\frac{2i+1}{e})^{\frac{2i+1}{2i+1+n}} \sim \frac{2i}{e}$, then the upper bound becomes asymptotic to

$$\frac{1}{(2i)^n\sqrt{2\pi 2i}} + (\frac{ne}{2i})^n \sim (\frac{ne}{2i})^n \text{ as } i \to \infty \ ,$$

by use of Stirling's formula. Thus, $E(N_4^*) = \sum_1^\infty P(N_4^* \ge i) < \infty$ for $n \ge 2$. For $n = 1$, with some work, one can show that $E(N_4^*) = \infty$ (step 1 has to be replaced by: generate U_1, U_2 independent uniform (0,1) random variates; set $X = 1/U_1 - 1$, $T = \gamma\, U_2\, U_1^2 - 1$ where $\gamma = 4/e$. If $T > 0$, go to 0.1.E. Set $j = 0$, $P = 1$.). This is due to the fact that the tail of the dominating function $1/(x+1)^2$ is too heavy, and that the series approximation of e^{-x} is too slow in the tail.

Remark 2. If one uses the given algorithm with $n = 2$, one will run into overflow problems in steps 2 and 3 since X is too large with too high a probability. To guard against this, step 1 could be extended as follows:

if $X > \alpha$ (a threshold): exit with X when $T+1 < \exp(-X)$, and go to 0.1.E. otherwise .

The average time (in microseconds) varies from 30.7 ($\alpha=0$) down to 26.2 ($\alpha=1$) and monotonically back up to 30.1 ($\alpha=5$). The case $\alpha=0$ corresponds to a complete bypass of steps 2, 3, 4 in the algorithm, and the case $\alpha \to \infty$ corresponds to no bypass. The average time (in microseconds) for the random variate generator -log(U) where U is uniform (0,1) is much lower: 17.8 . (All timings here and below were done on an AMDAHL V7 computer using 10000 observations and FORTRAN coding. The uniform random variate generator was taken from the Super-Duper random number package (see Dudewicz and Ralley (1981) for the code, and Marsaglia, Ananthanarayanan and Paul (1973) for additional explanation).

Remark 3. A similar experiment to that described in Remark 2 was carried out by the author for the normal distribution: the tails were taken care of by rejection from the Rayleigh density $x e^{-x^2/2}$ properly truncated, and the main body was treated using the series method, with rejection from the uniform density. The average computer time was mdiway between the average time for the polar method and the average times for the algorithms of Kinderman and Ramage (1976) and Marsaglia and Bray (1964). The space requirements were also about midway between those of the polar method and the Kinderman-Ramage, Marsaglia-Bray algorithms. This situates the method in an area of the time/space map (figure B of Kinderman and Ramage (1976)) practically by itself. No method known to us is both shorter and faster.

The Raab-Green distribution. Consider the density

$$f(x) = \frac{1+\cos x}{2\pi} \quad -\pi < x < \pi .$$

$$= \frac{1}{\pi}\left(1 - \frac{x^2}{2.2!} + \frac{x^4}{2.4!} - \cdots\right) , \quad -\pi < x < \pi .$$

Thus, f can be put into form (3) with $h(x) = \frac{1}{\pi}$, $-\pi < x < \pi$, and $a_n(x) = \frac{x^{2n}}{2.(2n)!}$. It is easy to check that $a_n(x) \downarrow 0$ as $n \to \infty$, since

$$\frac{a_{n+1}(x)}{a_n(x)} = \frac{x^2}{(2n+2)(2n+1)} \leq \frac{\pi^2}{12} < 1 .$$

Density f was suggested by Raab and Green (1961) as an approximation for the normal density. Algorithm S1 is, for the Raab-Green density:

1.1RG. Generate two independent uniform (0,1) random variates U and V . Set $X = \pi(2V-1)$, $n = 0$, $T = 0$, $P = \frac{1}{2}$.

1.2RG. $n = n+1$, $P = PX^2/[(2n)(2n-1)]$, $T = T+P$. If $U \geq T$, exit with X .

1.3RG. $n = n+1$, $P = PX^2/[(2n)\ (2n-1)]$, $T = T-P$. If $U < T$, go to 1.1RG.

1.4RG. Go to 1.2RG.

It is easy to see that $E(N_1) = 2$. However, we may apply the following alias principle (Walker (1977); see also Kronmal and Peterson (1979)) or band rejection method (Payne (1977)): generate (X,U) uniformly in $[-\frac{\pi}{2}, +\frac{\pi}{2}]x[0,1]$. If $\frac{U}{\pi} \le f(X)$, exit with X , and otherwise, exit with π sign X - X . X will have density f because (for $0 < x < \frac{\pi}{2}$) we have $\frac{1}{\pi} - f(x) = f(\pi-x)$. Thus, algorithm S1 can, for the Raab-Green density, be improved to:

1.1RGA. Generate two independent uniform (0,1) random variates U and V . Set $X = \frac{\pi}{2}(2V-1)$, $n = 0$, $T = 0$, $P = \frac{1}{2}$.

1.2RGA. $n = n+1$, $P = PX^2/[(2n)\ (2n-1)]$, $T = T+P$. If $U \ge T$, exit with X .

1.3RGA. $n = n+1$, $P = PX^2/[(2n)\ (2n=1)]$, $T = T-P$. If $U < T$, exit with π sign X - X .

1.4RGA. Go to 1.2RGA.

Then we will have $N_1 = 1$ (no rejections), and furthermore

$$P(N_2>1) = 2 \int_0^{\pi/2} \frac{1}{\pi} \frac{x^4}{48}\, dx = \frac{\pi^4}{3840} \cong 0.0254 .$$

In other words, not only is no cosine evaluation necessary with this algorithm, but step 4 is reached only about 2.54% of the time.

The last algorithm RGA takes 18.6 μs per random variate on the average. If steps 2-4 are replaced by the direct method "If $2U \le 1 + \cos X$, exit with X . Otherwise, exit with π sign X-X", then it takes 20.9 μs per random variate on the average. Thus in this example it is desirable to use the series method rather than the direct method.

4. THE KOLMOGOROV-SMIRNOV DISTRIBUTION.

The Kolmogorov-Smirnov distribution function

$$F(x) = \sum_{n=-\infty}^{\infty} (-1)^n e^{-2n^2x^2} , \quad x > 0 , \tag{6}$$

appears as the limit distribution of the Kolmogorov-Smirnov test statistic (Kolmogorov (1933), Smirnov (1939), Feller (1948)). No simple procedure for inverting F is known, hence the inversion method is likely to be slow. The density f corresponding to F is

$$f(x) = 8 \sum_{n=1}^{\infty} (-1)^{n+1} n^2 x\, e^{-2n^2x^2} , \quad x > 0 , \tag{7}$$

which is of form (3) when

$$h(x) = 8xe^{-2x^2} \quad , x > 0 \ ,$$

$$a_n(x) = (n+1)^2 \, e^{-2x^2[(n+1)^2-1]} \quad , x > 0 \tag{8}$$

It is known (for the equivalence of (7) and (9) see Whittaker and Watson (1963) or Byrd and Friedman (1964)) that F and f can also be written as

$$F(x) = \frac{\sqrt{2\pi}}{x} \sum_{n=1}^{\infty} e^{-(2n-1)^2\pi^2/8x^2} \quad , x > 0 \ , \tag{9}$$

and

$$f(x) = \frac{\sqrt{2\pi}}{x} \sum_{n=1}^{\infty} \left[\frac{(2n-1)^2\pi^2}{4x^3} - \frac{1}{x}\right] e^{-(2n-1)^2\pi^2/8x^2} \quad , x > 0 \ . \tag{10}$$

This also follows the format of (3), but now with

$$h(x) = \frac{\sqrt{2\pi}\,\pi^2}{4x^4} \, e^{-\pi^2/(8x^2)} \tag{11}$$

$$a_n(x) = \begin{cases} \dfrac{4x^2}{\pi^2} \, e^{-(n^2-1)\pi^2/8x^2} \quad , n \geq 1 \ , \ n \text{ odd} \ , \ x > 0 \ , \\ (n+1)^2 \, e^{-[(n+1)^2-1]\pi^2/8x^2} \quad , n \geq 1 \ , \ n \text{ even} \ , \ x > 0 \ . \end{cases}$$

Lemma 1. The terms $a_n(x)$ in (8) are monotone $\downarrow$ for $x > \sqrt{1/3}$. The terms $a_n(x)$ in (11) are monotone $\downarrow$ for $x < \pi/2$.

Proof. For (8), we have $\log(a_{n-1}(x)/a_{n_2}(x)) = -2\log(1+n^{-1}) + 2(2n+1)x^2 \geq -2n^{-1} + 2(2n+1)x^2 \geq -2 + 6x^2 > 0$. For (11), when n is even, we have $a_n(x)/a_{n+1}(x) = (n+1)^2\pi^2/4x^2 \geq \pi^2/4x^2 > 1$. Also, $\log(a_{n-1}(x)/a_n(x)) = -\log((n+1)^2\pi^2/4x^2) + n\pi^2/2x^2 = ny - 2\log(n+1) - \log(y/2)$ (where $y = \pi^2/2x^2$). The last expression is increasing in y for $y \geq 2$ and all $n \geq 2$. Thus it is not smaller than $2n-2\log(n+1) \geq 0$. This concludes the proof of Lemma 1.

The monotonicity condition (4) necessary to apply algorithm S1 is satisfied for (8) on $(\sqrt{1/3}, \infty)$ and for (11) on $(0, \pi/2)$. The algorithm that we propose to generate a random variate X from The Kolmogorov-Smirnov f is a combination of S1 and the mixture method:

Algorithm S1M.

1.0M. (Preparation.) Let c be a constant in $(\sqrt{1/3}, \pi/2)$ ($c = 0.75$ is suggested), and let $p = F(c)$.

1.1M. Generate a uniform (0,1) random variate U . If $U > p$, go to 1.2M.
Otherwise, exit with a random variate X from density

$$f_1(x) = f(x)/p \ , \ 0 < x < c \ .$$

To generate X , use S1 with (10) and (11).

1.2M. Exit with a random variate X from density

$$f_2(x) = f(x)/(1-p) \ , \ c < x \ ,$$

where X is generated by the alternating series method S1 applied to (7) and (8).

For details of the application of S1 to densities f_1 and f_2 we make use of the following lemmas.

Lemma 2. If G is a random variable with the truncated gamma $(\frac{3}{2})$ density $c_1\sqrt{y}\ e^{-y}$, $y \geq c' = \pi^2/8c^2$, then $X = \pi/\sqrt{8G}$ has density

$$c_2\ x^{-4}\ e^{-\pi^2/8x^2} \ , \ x \leq c \ . \tag{12}$$

(Here c_1 and c_2 are normalization constants).

Proof. The Jacobian of the transformation $y = \pi^2/8x^2$ is $4\pi/(8y)^{3/2}$. If X has density (12), then $G = \pi^2/8X^2$ has density $c_1\sqrt{y}\ e^{-y}$, $y \geq c'$.

Lemma 3. If E is an exponential random variable, then $X = \sqrt{c^2+E/2}$ has density

$$c_3\ x\ e^{-2x^2} \ , \ x \geq c \ , \tag{13}$$

where c_3 is a normalization constant.

Proof. The distribution function of (13) is $1 - \exp(-2(x^2-c^2))$, $x \geq c$.
That of E is $1 - \exp(-x)$, $x \geq 0$.

We propose the following algorithms for the generation of random variates from f_1 and f_2 , respectively. The constant c is picked as in S1M and $c' = \pi^2/8c^2$.

(f_1) 1. 1A. Generate two independent exponential random variates E_0 and E_1 . Set $E_0 = E_0/(1-(2c')^{-1})$, $E_1 = 2E_1$, $G = c'+E_0$.

1B. If $E_0^2 > c'E_1(G+c')$, go to 1D.

1C. If $G/c' - 1 - \log(G/c') > E_1$, go to 1A.

1D. $X = \pi/\sqrt{8G}$, $T = 0$, $Z = (2G)^{-1}$, $P = e^{-G}$, $n = 1$, $Q = 1$.

Generate a uniform (0,1) random variate U .

2. $T = T+ZQ$. If $U \geq T$, exit with X .

3. $n = n+2$, $Q = P^{n^2-1}$, $T = T-n^2Q$. If $U < T$, go to 1.

4. Go to 2.

(f_2) 1. Generate an exponential random variate E and an independent uniform (0,1) random variate U . Set X = $\sqrt{c^2+E/2}$, T ← 0 , n ← 1 , Z = $\exp(-2X^2)$.

2. n = n+1 , T = T+$n^2Z^{n^2-1}$. If U ≥ T , exit with X .

3. n = n+1 , T = T=$n^2Z^{n^2-1}$. If U < T , go to 1.

4. Go to 2.

The steps in both algorithms are numbered as in S1. In step 1 of the algorithm for f_1, a random variate X is generated that has density h (11) restricted to (0,c) . The algorithm uses rejection (step 1C) with squeezing (step 1B) (for details, see Dagpunar (1978) and Devroye (1980)). The computation of Z in step 1 of the f_2 algorithm requires exponentiation. This may be avoided some of the time by accepting X in step 1 "quickly" when U ≥ 4 $\exp(-6c^2)$. Similarly, we may accept X in step 1D of the f_1 algorithm when U ≥ $4c^2/\pi^2$, before Z or P are computed. In Table 1 values are given for these quick acceptance probabilities and for p = F(c) as a function of c . Both probabilities are close to 0.80 when c = 0.75 . A complete FORTRAN program with discussion is given at the end of this section.

TABLE 1. Values for Quick Acceptance Probabilities and p ≡ F(c) as Functions of c.

c	$1-4e^{-6c^2}$	$1-4c^2/\pi^2$	p=F(c)	Average Time per Variate (μs*)
0.60	0.54	0.85	0.136	39
0.65	0.68	0.83	0.208	36
0.70	0.79	0.80	0.289	34
0.75	0.86	0.77	0.373	34
0.80	0.914	0.74	0.456	36
0.85	0.948	0.71	0.535	38
0.90	0.969	0.67	0.607	41
0.95	0.982	0.63	0.673	44
1.00	0.9901	0.59	0.730	48
1.05	0.9946	0.55	0.780	53

* 1 μs = 1 microsecond = 10^{-6} seconds.

Related limit distributions. The empiric distribution function $F_n(x)$ for a sample $X_1,\ldots,X_n$ of independent identically distributed random variables is defined by $F_n(x) = \sum_{i=1}^{n} n^{-1}[X_i \le x]$ where I is the indicator function. If X_1 has distribution function $F(x)$, then the following statistics have been proposed for goodness-of-fit tests by various authors:

$$K_n^+ = \sqrt{n} \sup_x F_n(x) - F(x), \quad K_n^- = \sqrt{n} \sup_x F(x) - F_n(x)$$

(the asymmetrical Kolmogorov-Smirnov statistics);

$K_n = \max(K_n^+, K_n^-)$ (the Kolmogorov-Smirnov statistic);

$V_n = K_n^+ + K_n^-$ (Kuiper's statistic);

$W_n^2 = n\int (F_n(x) - F(x))^2 dF(x)$ (von Mises' statistic);

$U_n^2 = n\int (F_n(x) - F(x) - \int (F_n(y) - F(y))dF(y))^2 dF(x)$

(Watson's statistic);

$A_n^2 = n\int \frac{(F_n(x)-F(x))^2}{F(x)\,(1-F(x))}\, dF(x)$ (the Anderson-Darling statistic).

For surveys of the properties and applications of these and other statistics, see Darling (1955), Barton and Mallows (1965), Sahler (1958), and Shapiro (1980). All the statistics mentioned here have limit distributions. The limit random variables will be denoted by K_∞^+, K_∞^-, K_∞, etc. In Table 2, the characteristic functions of these limit distributions are given. In all but one case they can be written as a countable product of characteristic functions of gamma random variables. Therefore the following conclusions about the generation of random variates from these distributions follow:

(1) $2K_\infty^{+2}$ and $2K_\infty^{-2}$ are exponentially distributed. For exponential random variate generation algorithms, see Maclaren, Marsaglia and Bray (1964), Sibuya (1961), Marsaglia (1961) and Ahrens and Dieter (1972).

(2) V_∞ is distributed as $K_\infty(1) + K_\infty(2)$, the sum of the two independent random variables distributed as K_∞.

(3) U_∞ is distributed as K_∞/π^2.

(4) The sum of two independent W_∞^2 random variables $(W_\infty^2(1)+W_\infty^2(2))$ is distributed as K_∞/π^2, but no simple function of K_∞ that would give a random variable distributed as W_∞^2 is known to us.

Note that the explicit form of the limit distribution of V_∞ is relatively simple (see Kuiper (1960), but A_∞^2 and W_∞^2 have complicated limit distributions (see Anderson and Darling ((1952)).

TABLE 2. Characteristic Functions of Some Limit Distributions for Tests of Fit.

Random Variable	Characteristic Function	Relevant References
K_∞	$\prod_{j=1}^{\infty} (1-it/2j^2)^{-1}$	Kolmogorov (1933), Smirnov (1939), Feller (1948)
K_∞^{+2} , K_∞^{-2}	$(1-it/2)^{-1}$	Smirnov (1939), Feller (1948)
V_∞	$\prod_{j=1}^{\infty} (1-it/2j^2)^{-2}$	Kuiper (1960)
W_∞^2	$\prod_{j=1}^{\infty} (1-2it/\pi^2 j^2)^{-1/2}$	Smirnov (1937), Anderson and Darling (1952)
U_∞^2	$\prod_{j=1}^{\infty} (1-it/2\pi^2 j^2)^{-1}$	Watson (1961, 1962)
A_∞^2	$\prod_{j=1}^{\infty} (1-it/j(j+1))^{-1/2}$	Anderson and Darling (1952)

```
      REAL FUNCTION SMIR(L)
C
C
C  THIS SUBPROGRAM PRODUCES VARIATES FROM THE KOLMOGOROV-SMIRNOV
C  LIMIT DISTRIBUTION FUNCTION.
C
C  SOURCE:  LUC DEVROYE "THE SERIES METHOD FOR RANDOM VARIATE
C           GENERATION AND ITS APPLICATION  TO THE KOLMOGOROV-
C           SMIRNOV DISTRIBUTION," AMERICAN JOURNAL OF MATHEMATICAL
C           AND MANAGEMENT SCIENCES.
C  AT EACH CALL, THE ARGUMENT CAN BE GIVEN THE VALUE 0 AS IN
C  THE STATEMENT X = SMIR (0).
C  THE PROGRAM USES THE SUBPROGRAMS UNI AND REXP FOR THE GENERA-
C  TION OF UNIFORM (0,1) AND EXPONENTIAL RANDOM VARIATES.
C  THESE SUBPROGRAMS ARE PART OF THE SUPER-DUPER RANDOM NUMBER
C  GENERATOR PACKAGE OF MCGILL UNIVERSITY.
C
C  THE CONSTANTS IN THE PROGRAM DEPEND UPON C.  ALL THE VALUES IN
C  THE PROGRAM ARE FOR C=0.75.  FOR OTHER VALUES OF C IN THE
C  RANGE SQRT (1./3.)< C <3.14159265/2.  RECALCULATE THEM AS
   FOLLOWS :
C          P= VALUE OF KOLMOGOROV-SMIRNOV DISTRIBUTION FUNCTION
C             AT C
C          P1=1./P
C          P2=1./(1.-P)
C          CSQRE=C*C
C          P28C=PIE**2/(8*C*C), PIE=3.14159265
C          PINV=1./P28C
C          ALPHA=4.*EXP(-6.*C*C)
C          BETA=4*C*C/PIE**2
C          B=1./(1.-4*C*C/PIE**2)

   DATA C,P,P1,P2,CSQRE,P28C,PINV,ALPHA,BETA,B/0.75,0.3728330,
  + 2.682166,1.594471,0.5625,2.193245.0.4559454,0.1368725,
  + 0.2279727,1.295291/
1  DEC=UNI (0)
   IF(DEC.LT.P)GOTO59
```

```
C
C  GENERATE VARIATE FROM TAIL OF RAYLEIGH DENSITY
C
   V=(DEC-P)*P2
   GOTO21
2  V=UNI(0)
21 SMIR=CSQRE+0.5*REXP(0)
   SMIRSQ=SMIR+SMIR
C
C  CONSECUTIVE ACCEPTANCE/REJECTION STEPS
C
   IF (V.LT.ALPHA)GOTO4
3  SMIR=SQRT(SMIR)

   RETURN
4  IF (SMIRSQ.GT.174.) GOTO3
   T=EXP(-SMIRSQ)
   K=1
   NUM=0
   SUM=0
5  NUM=NUM+K+K+1
   K=K+1
   (IF(NUM*SMIRSQ.GT.174.)GOTO3
   SUM=SUM+(NUM+1)*T**NUM
   IF(V.GE.SUM)GOTO3
   NUM=NUM+K+K+1
   K=K+1
   IF(NUM*SMIRSQ.GT.174)GOTO3
   SUM=SUM=(NUM+1)*T**NUM
   IF(V.LT.SUM)GOTO2
   GOTO5
C
C  GENERATE VARIATE DISTRIBUTED AS INVERSE OF SQUARE ROOT OF TAIL
   OF CHI-SQUARE DENISTY WITH 3 DEGREES OF FREEDOM
C
59 V=DEC*P1
   GOTO7
6  V=UNI(0)
7  E=REXP(0)*B
   E1=E1+E1
   Y=P28C+E
   IF(E*E.LE.E1*P28C*(Y+P28C))GOTO8
   IF(PINV*Y-1.-ALOG(PINV*Y).GT.E1)GOTO6
   SMIR=1.1107206/SQRT(Y)
   Z=0.5/Y
   SUM=1.-Z
C
C  CONSECUTIVE ACCEPTANCE/REJECTION STEPS
C
   IF(V.LT.SUM)RETURN
   K=3
   KSQRE=8
   IF(Y.GT.21.7)RETURN
   T=EXP(-Y)
9  TU=T**KSQRE
   SUM=SUM+(KSQRE+1)*TU
   IF(V.GT.SUM)GOTO6
   SUM=SUM-Z*TU
   IF(V.LE.SUM)RETURN
   K=K+2
   KSQRE=K*K-1
   IF(Y*KSQRE.GT.174.)RETURN
   GOTO9
   END
```

Subprogram SMIR requires a uniform random variate generator (UNI), and an exponential random variate generator (REXP). Both UNI and REXP are part of the SUPER-DUPER random number generator package developed by Marsaglia, Anantharayanan and Paul (1973) at McGill University. The IBM Assembler code can be found in Dudewicz and Ralley (1981). It can also be obtained directly from McGill University. The average time required per variate changes with the parameter c (see Table 1), and on McGill University's AMDAHL V7 computer, c = 0.75 seems to be the best choice. The sequence of random variates produced by SMIR was also submitted to a Kolmogorov-Smirnov goodness-of-fit test for sample size 5000 (the p-values obtained were 0.12, 0.24 and 0.37). To compare the speed of SMIR (34 μs/variate on the average) with that of other algorithms, note that the statement X = -ALOG(UNI(0)) takes on the average 10.5 μs, and that the fastest FORTRAN coded gamma generators (see Tadikamalla and Johnson((1981))take about 21 microseconds per variate for large values of the shape parameter.

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