A lower bound on the size of an absorbing set in an arc-coloured tournament

L. Beaudou[a,*], L. Devroye[b], G. Hahn[c]

[a] *Université Clermont Auvergne, Clermont-Ferrand, France*
[b] *McGill University, Montréal, Québec, Canada*
[c] *Université de Montréal, Montréal, Québec, Canada*

Abstract

Bousquet, Lochet and Thomassé recently gave an elegant proof that for any integer n, there is a least integer $f(n)$ such that any tournament whose arcs are coloured with n colours contains a subset of vertices S of size $f(n)$ with the property that any vertex not in S admits a monochromatic path to some vertex of S. In this note we provide a lower bound on the value $f(n)$.

A *directed graph* or *digraph* D is a pair (V, A) where V is a set called the *vertex set* of D and A is a subset of V^2 called the *arc set* of D. When (u, v) is in A, we say there is an arc from u to v. A *tournament* is a digraph for which there is exactly one arc between each pair of vertices (in only one direction).

Given a colouring of the arcs of a digraph, we say that a vertex x *is absorbed* by a vertex y if there is a monochromatic path from x to y. A subset S of V is called an *absorbing set* if any vertex in $V \setminus S$ is absorbed by some vertex in S.

In the early eighties, Sands, Sauer and Woodrow [7] suggested the following problem (also attributed to Erdős in the same paper):

Problem (Sands, Sauer and Woodrow [7])**.** *For each n, is there a (least) positive integer $f(n)$ so that every finite tournament whose arcs are coloured with n colours contains an absorbing set S of size $f(n)$?*

This problem has been investigated in weaker settings by forbidding some structures (see [5], [3]), or by making stronger claims on the nature of the tournament (see [6]). Hahn, Ille and Woodrow [4] also approached the infinite case.

Recently, Pálvölgyi and Gyárfás [6] have shown that a positive answer to this problem would imply a strengthening of a former result from Bárány and Lehel [1] stating that any set X of points in $\mathbb{R}^d$ can be covered by $f(d)$ X-boxes

*Corresponding author
Email addresses: laurent.beaudou@uca.fr (L. Beaudou), lucdevroye@gmail.com (L. Devroye), hahn@iro.umontreal.ca (G. Hahn)

(each box is defined by two points in X). In 2017, Bousquet Lochet and Thomassé [2] gave a positive answer to the problem of Sands, Sauer and Woodrow.

Theorem 1 (Bousquet, Lochet and Thomassé [2])**.** *Function f is well defined and $f(n) = O(\ln(n) \cdot n^{n+2})$.*

In this note, we provide a lower bound on the value of $f(n)$.

Theorem 2. *For any integer n, let*

$$p = \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}.$$

There is a tournament arc-coloured with n colours, such that no set of size less than p absorbs the rest of the tournament, and thus $f(n) \geq p$.

Proof. Let n be an integer and let $\mathcal{P}$ be the family of all subsets of $\{1, \ldots, n-1\}$ of size exactly $\lfloor \frac{n-1}{2} \rfloor$. Note that $\mathcal{P}$ has size p.

For any integer m, let $V(m)$ be the set $\{1, \ldots, m\} \times \mathcal{P}$ and let $\mathcal{T}(m)$ be the probability space consisting of arc-coloured tournaments on $V(m)$ where the orientation of each arc is fairly determined (probability $1/2$ for each orientation), and the colour of an arc from (i, P) to (j, P') is n if $P = P'$, and $\min(P' \setminus P)$ otherwise. Note that if P and P' are distinct, then $P' \setminus P$ can not be empty since both sets have the same size.

For any set P in $\mathcal{P}$, we shall denote by $B(P)$ the set of vertices having P as second coordinate. We may call this set, the *bag of P*. A key observation is that the arcs coming into a bag share no common colour with the arcs leaving this bag. Moreover, the arcs contained in a bag use an additional distinct colour. As a consequence, a monochromatic path is either contained in a bag or of length 1.

Let us find an upper bound on the probability that such a tournament is absorbed by a set of size strictly less than p. Let S be a subset of $V(m)$ of size $p-1$. There must exist a set P in $\mathcal{P}$ such that S does not hit $B(P)$. For S to be absorbing, each vertex in $B(P)$ must have an outgoing arc to some vertex of S. Let x be a vertex in $B(P)$,

$$Pr(x \text{ is absorbed by } S) = 1 - \left(\frac{1}{2}\right)^{p-1}.$$

The events of being absorbed by S are pairwise independent for elements of $B(P)$. Then,

$$Pr(B(P) \text{ is absorbed by } S) = \left(1 - \left(\frac{1}{2}\right)^{p-1}\right)^m.$$

This is an upper bound on the probability for S to absorb the whole tournament. We may sum this for every possible choice of S.

$$Pr(\text{some } S \text{ is absorbing}) \leq \binom{mp}{p-1} \left(1 - \left(\frac{1}{2}\right)^{p-1}\right)^m.$$

Finally, by using the classic inequalities $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$ and $1 + x \leq e^x$, we obtain

$$Pr(\text{some } S \text{ is absorbing}) \leq \left(\frac{emp}{p-1}\right)^{p-1} e^{-m(\frac{1}{2})^{p-1}}.$$

When m tends to infinity, this last quantity tends to 0. So that, for m large enough, there is a tournament which is not absorbed by $p-1$ vertices. $\square$

Remark. By Stirling's approximation we derive that the bound obtained in Theorem 2 is of order $\frac{2^n}{\sqrt{n}}$.

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