Transversals in Trees

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1 Farley's problem and its solution

A *transversal* in a rooted tree is any set of nodes that meets every path from the root to a leaf. We let $c(T,k)$ denote the number of transversals of size k in a rooted tree T. If T has n nodes and $n \geq 2$, then

$$\begin{aligned} c(T,n) &= 1, \\ c(T,n-1) &= n, \end{aligned}$$

and

$$\binom{n-1}{k-1} \leq c(T,k) \leq \binom{n}{k} \quad \text{for all } k = 1,2,\ldots,n-2. \tag{1}$$

The $n-2$ upper bounds in (1) are attained simultaneously if and only if T is the path (the tree with precisely one leaf); the $n-2$ lower bounds in (1) are attained simultaneously if and only if T is the star (the tree where all leaves are children of the root). Jonathan David Farley asked how high can these lower bounds be raised if each node of T has at most two children; he offered a creative interpretation of this question in [7, 8]. In this section, we give an answer.

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Harary and Schwenk [9] call a *caterpillar* a tree (an unrooted one) where removal of all nodes of degree one produces a path. Abusing this usage a little, we will call a caterpillar any rooted tree where removal of all leaves produces a rooted tree with precisely one leaf. By a *full caterpillar of degree* d, we will mean a caterpillar where each internal node, except possibly the lowest one, has precisely d children. (A *d-ary tree* means a rooted tree where each node has at most d children and the order of these children is irrelevant.)

Theorem 1. *Let n and d be positive integers such that $d < n$; let T be any d-ary tree on n nodes and let $T^{\min}$ be the full caterpillar of degree d on n nodes. Then $c(T,k) \geq c(T^{\min},k)$ for all $k = 1,2,\ldots,n$.*

In the special case where $d = 2$, Farley proved inequalities $c(T,k) \geq c(T^{\min},k)$ with (essentially) $k = 1,2,\ldots,1+\lfloor \log_2 n \rfloor$ by an argument different from ours.

Given rooted trees T, T' on n nodes, we will write $T \succ T'$ to mean that $c(T,k) \geq c(T',k)$ for all $k = 1,2,\ldots,n$ and that $c(T,k) > c(T',k)$ for at least one of these values of k. With this notation, a refinement of Theorem 1 can be stated as follows.

Theorem 2. *Let n and d be positive integers such that $d < n$ and let $T^{\min}$ be the full caterpillar of degree d on n nodes. If T is a d-ary tree on n nodes, then $T \succ T^{\min}$ or else $T = T^{\min}$.*

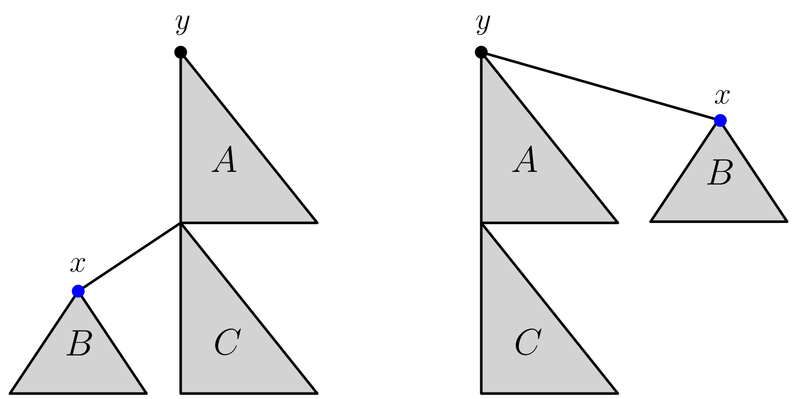
Our proof of Theorem 2 relies on two ways of altering a rooted tree T so that the resulting tree succeeds T in the partial order $\succ$. We shall describe these alterations in terms of the *parent function* of a rooted tree that assigns to each node z of the tree its parent $p(z)$ — except when z is the root, in which case $p(z)$ is undefined.

Lemma 1. *Let T be a rooted tree defined by parent function p. Let x and y be nodes of T such that y is a proper ancestor of $p(x)$. Let T' be the rooted tree defined by parent function p' such that*

$$p'(z) = \begin{cases} p(z) & \text{if } z \neq x, \\ y & \text{if } z = x. \end{cases}$$

Then $T \succ T'$.

This operation is illustrated in Figure 1.



(a) y is a proper ancestor of $p(x)$ in T. (b) y becomes the parent of x in T'.

Figure 1: From a subtree of T to a subtree of T' in Lemma 1.

Proof. If z is a leaf of T, then z is a leaf of T' and every node on the path from the root to z in T' lies on the path from the root to z in T. It follows that every transversal in T' is a transversal in T, and so $c(T,k) \geq c(T',k)$ for all k. To see that $c(T,k) > c(T',k)$ for at least one k, consider the set that consists of $p(x)$ and all leaves of T that are not descendants of $p(x)$: this set is a transversal in T but not in T'. $\square$

Lemma 2. *Let T be a rooted tree defined by parent function p. Let x and y be nodes of T such that x is not a leaf and y is a leaf which is a proper descendant of a sibling of x. Let T' be the rooted tree defined by parent function p' such that*

$$p'(z) = \begin{cases} p(z) & \textit{if } p(z) \neq x, \\ y & \textit{if } p(z) = x. \end{cases}$$

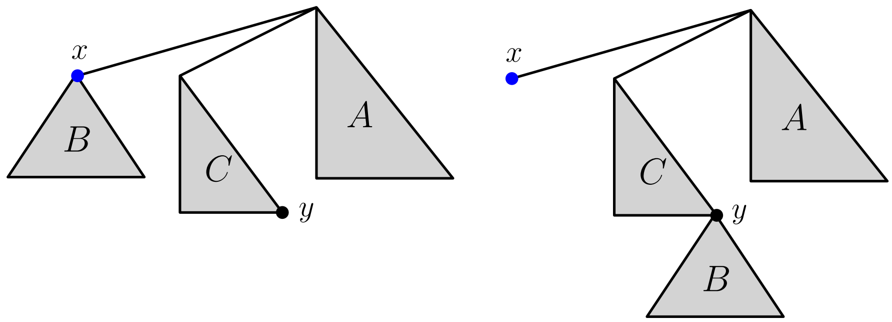
Then $T \succ T'$.

This operation is illustrated in in Figure 2.

Proof. Given any set S' of nodes in T', define

$$f(S') = \begin{cases} S' & \text{if } S' \text{ meets the path from the root to } y, \\ S' - \{x\} \cup \{y\} & \text{otherwise.} \end{cases}$$

By this definition, $f(S')$ always meets the path from the root to y; this path is the same in T' and T. Every leaf of T distinct from y is a leaf of T'; unless this leaf is a descendant of x, the path from the root to it is again the



(a) y is a proper descendant of a sibling of x in T.

(b) Children of x in T become children of y in T'.

Figure 2: From a subtree of T to a subtree of T' in Lemma 2.

same in the two trees and, in particular, x does not lie on this path; since $f(S') \supset S' - \{x\}$, it follows that

(i) if S' is a transversal in T', then $f(S')$ meets every path in T from the root to a leaf that is not a descendant of x.

To see that

(ii) if S' is a transversal in T', then S' meets every path in T from the root to a leaf that is a descendant of x,

note that S' meets the path from the root to x and that this path is the same in T' and T. Next, we claim that

(iii) if S' is a transversal in T' such that $f(S') \neq S'$, then $f(S')$ meets every path in T from a child of x to a leaf.

To justify (iii), note that S', being a transversal in T', meets every path in T' from the root to a leaf and that, since $f(S') \neq S'$, it does not meet the path from the root to y; it follows that S' meets every path in T' from a child of y to a leaf. Since every path in T from a child of x to a leaf is a path in T' from a child of y to a leaf, we conclude that S' meets every such path; now (iii) follows as $f(S') \supset S' - \{x\}$. In addition, we note that

(iv) if S' is a transversal in T' such that $f(S') \neq S'$, then $x \in S', y \notin S'$

(since S' does not meet the path from the root to y, it does not include y and it does not meet the path from the root to the parent of x; but then, since it meets the path from the root to x, it must include x) and that

(v) if S' is a transversal in T' such that $f(S') \neq S'$, then $f(S')$ is not a transversal in T'

(because it does not meet the path from the root to x, which is a leaf of T').

From (i), (ii), (iii), (iv), we see that f maps every transversal in T' to a transversal of the same size in T; from (iv) and (v), we see that it maps distinct transversals in T' to distinct transversals in T. It follows that $c(T,k) \geq c(T',k)$ for all k. To see that $c(T,k) > c(T',k)$ for at least one k, consider the set that consists of $p(y)$ and all leaves of T that are not descendants of $p(y)$: this set S is a transversal in T, but there is no transversal S' in T' such that $f(S') = S$. □

Proof of Theorem 2. Consider any d-ary tree T on n nodes. Assuming that there is no d-ary tree T' on n nodes such that $T \succ T'$, we shall prove that T is the full caterpillar of degree d. Lemma 2 guarantees that no two internal nodes of T are siblings, which means that T is a caterpillar; in turn, Lemma 1 guarantees that each internal node of T, except possibly the lowest one, has precisely d children. □

Let us point out that the number of transversals of size k in the caterpillar $T^{\min}$ featured in Theorem 1 is easy to calculate:

$$c(T^{\min},k) = \begin{cases} \displaystyle\sum_{i=0}^{s} \binom{n-1-id}{k-1-i(d-1)} + 1 & \text{if } k = n-s-1, \\ \displaystyle\sum_{i=0}^{s} \binom{n-1-id}{k-1-i(d-1)} & \text{otherwise.} \end{cases}$$

To see this, let $r_0, r_1, \ldots, r_s$ denote the path produced when all leaves are deleted from $T^{\min}$: r_0 is the root, each r_i with $0 \leq i < s$ has $d-1$ leaves and r_{i+1} as children, and r_s has between 1 and d leaves as children. Note that a transversal in $T^{\min}$ includes no r_i at all if and only if it consists of all the leaves of $T^{\min}$. Given a transversal S in $T^{\min}$ that includes at least one r_i, consider the smallest subscript i such that $r_i \in S$; note that S must contain the $i(d-1)$ leaves that are children of $r_0, r_1, \ldots, r_{i-1}$ and that its remaining elements consist of r_i and an arbitrary set of its descendants.

Imposing an upper bound on the number of children of every node is a way of staying clear of the tree that attains simultaneously the $n-2$ lower bounds in (1), one where all children of the root are leaves. Another way to stay clear of this tree is to impose an upper bound on the number of leaves. There is a corresponding analogue of Theorem 2 and this analogue also follows directly from our two lemmas.

Theorem 3. *Let n and m be positive integers such that $m < n$ and let T^0 be the caterpillar with n nodes, where the root has m children and every node other than the root has at most one child. If T is a rooted tree with n nodes and at most m leaves, then $T \succ T^0$ or else $T = T^0$.*

Proof. Consider any rooted tree T with n nodes and at most m leaves. Assuming that there is no rooted tree T' with n nodes and at most m leaves such that $T \succ T'$, we shall prove that $T = T^0$. Lemma 2 guarantees that no two internal nodes of T are siblings, which means that T is a caterpillar; in turn, Lemma 1 guarantees that no internal node of T other than the root has two or more children. $\square$

We close this section by pointing out that the number of transversals of size k in the caterpillar T^0 featured in Theorem 3 is also easy to calculate:

$$c(T^0, k) = \begin{cases} \dbinom{n-1}{k-1} & \text{if } k < m, \\ \dbinom{n-1}{k-1} + \dbinom{n-m}{k-m+1} & \text{if } k \geq m. \end{cases}$$

To see this, note that precisely $\binom{n-1}{k-1}$ transversals of size k include the root and that every transversal that does not include the root consists of $m-1$ leaves that are children of the root and at least one node on the path from the root to the m-th leaf.

2 Typical number of transversals

As T ranges through all d-ary trees on n nodes, the number $c(T, k)$ of transversals of size k attains its minimum at the full caterpillar, which we will denote as $T_d^{\min}(n)$ from now on. How close is this minimum to values of $c(T, k)$ that are typical for d-ary trees T on n nodes? This is the question that we address in this section.

The ratio $c(T, k)/\binom{n}{k}$ is the probability that a randomly chosen set of k of the n nodes of T is a transversal. It is intuitively obvious that this probability increases with k; one formal way of capturing this intuition consists of counting in two different ways the number N of pairs (A, B) such that A is a tranversal of size k, B is a transversal of size $k+1$, and $A \subset B$ (choosing A first, we get $N = c(T,k) \cdot (n-k)$ and choosing B first, we get $N \leq c(T, k+1) \cdot (k+1)$).

We will study $c(T,k)/\binom{n}{k}$ indirectly by focussing on the probability $\xi(T,p)$ of obtaining a transversal when we choose each node of T independently with probability p. These two quantities are related by the identity

$$\xi(T,p) \;=\; \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \cdot \frac{c(T,k)}{\binom{n}{k}} \;. \tag{2}$$

Since the sequence of the probabilities $\binom{n}{k} p^k (1-p)^{n-k}$ has a sharp peak when k is around pn, it may seem intuitively obvious that $c(T,k)/\binom{n}{k}$ is well approximated by $\xi(T, k/n)$.

To provide a rigorous justification of this intuition, we will bound $c(T,k)/\binom{n}{k}$ in terms of $\xi(T,p)$. For this purpose, we first note that identity (2) implies

$$\xi(T,p) \;\leq\; \frac{c(T,k)}{\binom{n}{k}} + \sum_{i=k+1}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

and

$$\xi(T,p) \;\geq\; \frac{c(T,k)}{\binom{n}{k}} \cdot \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i} \;.$$

Using the well-known inequality [2, 4, 3, 1, 10]

$$\sum_{i \geq (p+t)n} \binom{n}{i} p^i (1-p)^{n-i} \leq \exp\left(-2t^2 n\right)$$

and its companion

$$\sum_{i \leq (p-t)n} \binom{n}{i} p^i (1-p)^{n-i} \leq \exp\left(-2t^2 n\right),$$

we conclude that

$$\xi(T, k/n - t) - \exp\left(-2t^2 n\right) \;\leq\; \frac{c(T,k)}{\binom{n}{k}} \;\leq\; \frac{\xi(T, k/n + t)}{1 - \exp\left(-2t^2 n\right)} \;. \tag{3}$$

We have already seen that the star minimizes all $c(T,k)$ and that the path maximizes all $c(T,k)$. Similarly, we have

$$p+(1-p)p^{n-1} \le \xi(T,p) \le 1-(1-p)^n \quad \text{whenever } n \ge 2$$

for all trees T on n nodes, with the lower bound attained by the star and the upper bound attained by the path (the lower bound follows from observing that one transversal consists of the root alone and another transversal consists of all the leaves; the upper bound follows from observing that the empty set is not a transversal). Now we are going to tighten these bounds under the assumption that, for a prescribed integer d greater than one, T is a full d-ary tree (meaning that each of its internal nodes has precisely d children).

Let C_d^ω denote the *complete d-ary tree* where all leaves have depth ω. (The *depth* of a node in a rooted tree means the length of the path from the root to this node: the root has depth 0, its children have depth 1, and so on.) Given an integer d such that $d \ge 2$ and a real number p such that $0 \le p \le 1$, let us write

$$\ell_d(p) \;=\; \frac{p}{1-(1-p)p^{d-1}}$$

and let $u_d(p)$ denote the smallest nonnegative root of the polynomial

$$p - x + (1-p)x^d.$$

In particular,

$$u_2(p) = \begin{cases} \dfrac{p}{1-p} & \text{if } 0 \le p \le 1/2, \\ 1 & \text{if } 1/2 \le p \le 1. \end{cases}$$

Theorem 4. *For every integer d such that $d \ge 2$, every real number p such that $0 \le p \le 1$, and every full d-ary tree T on n nodes, we have*

$$\xi(T_d^{\min}(n),p) \;\le\; \xi(T,p) \;\le\; \xi(C_d^n,p).$$

As n increases, these bounds tend to the limits $\lim_{n\to\infty} \xi(T_d^{\min}(n),p) = \ell_d(p)$ *and* $\lim_{n\to\infty} \xi(C_d^n,p) \;=\; u_d(p)$.

Proof. The lower bound on $\xi(T,p)$ follows from our Theorem 1; the upper bound follows from the observation that every transversal in T is a transversal in C_d^n. Since

$$\xi(T_d^{\min}(n),p) = p+(1-p)p^{n-1} \quad \text{whenever } 2 \le n \le d+1$$

and

$$\xi(T_d^{\min}(n), p) = p + (1-p)p^{d-1}\xi(T_{n-d}^{\min}, p) \quad \text{whenever } n \geq d+2,$$

induction on n shows that, for all n greater than 1,

$$\xi(T_d^{\min}(n), p) = p\,\frac{1-((1-p)p^{\,d-1})^{\,t}}{1-(1-p)p^{\,d-1}} + p^{n-t}(1-p)^t \ \text{ with } t = \lfloor (n-2)/d \rfloor + 1.$$

Finally, we have $\xi(C_d^0, p) = p$ and

$$\xi(C_d^\omega, p) = p + (1-p)\xi(C_d^{\omega-1}, p)^d \quad \text{whenever } \omega \geq 1.$$

Since $\xi(C_d^\omega, 0) = 0 = u_d(0)$ and $\xi(C_d^\omega, 1) = 1 = u_d(1)$ for all ω, we may assume that $0 < p < 1$. Now let us write $f(x) = p + (1-p)x^d$, so that $\xi(C_d^\omega, p) = f^\omega(p)$ for all ω. By definition of $u_d(p)$ and since $f(0) > 0$, we have

$$0 < x < u_d(p) \ \Rightarrow\ x < f(x);$$

since the polynomial $f(x) - x$ is convex in the interval $[0,1]$ and since $u_d(p)$ and 1 (possibly identical) are among its roots, we have

$$u_d(p) < x < 1 \ \Rightarrow\ f(x) < x;$$

since f is increasing in the interval $[0,1]$, we have

$$\begin{aligned} 0 < x < u_d(p) \quad &\Rightarrow \quad f(x) < f(u_d(p)) = u_d(p), \\ u_d(p) < x < 1 \quad &\Rightarrow \quad u_d(p) = f(u_d(p)) < f(x). \end{aligned}$$

It follows that the sequence $\xi(C_d^0, p), \xi(C_d^1, p), \xi(C_d^2, p), \ldots$ is monotone (increasing if $p < u_d(p)$ and decreasing if $u_d(p) < p$), and so it tends to a limit in $(0,1)$; this limit must be a fixed point of f, and so it equals $u_d(p)$. $\square$

We have noted that the full caterpillar $T_d^{\min}(n)$ of degree d on n nodes satisfies

$$c(T_d^{\min}(n), k) = \sum_{i=0}^{s} \binom{n-1-id}{k-1-i(d-1)} + \delta(k, n-s-1)$$

with $s = \lfloor (n-2)/d \rfloor$ and δ the Kronecker delta. A part of Theorem 4 yields a cleaner asymptotic formula (where, as usual, $f \sim g$ means $\lim f/g = 1$):

Corollary 1. *If k/n tends to a limit as n tends to infinity, then*

$$c(T_d^{\min}(n), k) \sim \binom{n}{k} \frac{k}{n-(n-k)(k/n)^{\,d-1}}.$$

Proof. Set $T = T_d^{\min}(n)$ and (for instance) $t = n^{-1/3}$ in (3); then appeal to Theorem 4 for the value of $\lim_{n\to\infty} \xi(T_d^{\min}(n), k/n)$. $\square$

Corollary 2. *Given a function $\omega : \mathbf{N} \to \mathbf{N}$ that tends to infinity with n, let $\mathcal{F}_d(n)$ denote the set of all full d-ary trees T on n nodes where all leaves have depth at least $\omega(n)$. If k/n tends to a limit as n tends to infinity, then*

$$\min\{c(T,k) : T \in \mathcal{F}_d(n)\} \sim \max\{c(T,k) : T \in \mathcal{F}_d(n)\} \sim \binom{n}{k} u_d\left(\frac{k}{n}\right).$$

Proof. If $T \in \mathcal{F}_d(n)$, then every transversal in $C_d^{\omega(n)}$ is a transversal in T and every transversal in T is a transversal in C_d^n. It follows that

$$\xi(C_d^{\omega(n)}, p) \le \xi(T,p) \le \xi(C_d^n, p);$$

by Theorem 4, both the lower and the upper bound tend to $u_d(p)$ as $n \to \infty$; we conclude that

$$\lim_{n\to\infty} \min\{\xi(T,p) : T \in \mathcal{F}_d(n)\} = \lim_{n\to\infty} \max\{\xi(T,p) : T \in \mathcal{F}_d(n)\} = u_d(p).$$

Then we set again (for instance) $t = n^{-1/3}$ in (3). $\square$

In a number of naturally arising classes of rooted trees, the minimum depth of a leaf grows logarithmically with the number of vertices. For instance, Devroye [5] proved that almost all random binary search trees have this property. These trees are not, in general, full 2-ary trees; however, every 2-ary tree on n nodes can be extended into a full 2-ary tree on $2n+1$ nodes by adding $n+1$ leaves; in this sense, random binary search trees provide a class of full 2-ary trees that can be taken for $\mathcal{F}_d(n)$ in Corollary 2 with $d = 2$. A similar comment, with varying values of d, applies to other classes of *split trees*, introduced by Devroye [6].

With $d = 2$, Corollary 1 states that

$$c(T_2^{\min}(n), k) \sim \binom{n}{k} \frac{k}{n - k + k^2/n}$$

and Corollary 2 states that

$$T \in \mathcal{F}_2(n) \quad \Rightarrow \quad c(T,k) \sim \begin{cases} \binom{n}{k} \dfrac{k}{n-k} & \text{if } k \le n/2, \\ \binom{n}{k} & \text{if } k \ge n/2; \end{cases}$$

it follows that

$$T \in \mathcal{F}_2(n) \quad \Rightarrow \quad \frac{c(T,k)}{c(T_2^{\min}(n),k)} \sim \begin{cases} 1 + \dfrac{k^2}{n(n-k)} & \text{if } k \leq n/2, \\ 1 + \dfrac{(n-k)^2}{nk} & \text{if } k \geq n/2. \end{cases}$$

In particular, the limit of ratio $c(T,k)/c(T_2^{\min}(n),k)$ with $T \in \mathcal{F}_2(n)$ increases in the interval $0 \leq k \leq n/2$ from 1 at $k = 0$ to its maximum $3/2$ at $k = n/2$ and then it decreases in the interval $n/2 \leq k \leq n$ until it reaches 1 again at $k = n$.

When $d \geq 3$, we have no explicit formula for $u_d(p)$, but at least we can prove that the ratio $u_d(p)/\ell_d(p)$ is unimodal:

Theorem 5. *For every integer d such that $d \geq 2$, the ratio $u_d(p)/\ell_d(p)$ increases in the interval $[0,(d-1)/d]$ from its limit 1 at $p = 0$ to its maximum*

$$1 + \frac{1}{d-1}\left(1 - \left(\frac{d-1}{d}\right)^{d-1}\right)$$

at $p = (d-1)/d$ and then it decreases in the interval $[(d-1)/d, 1]$ until it reaches 1 again at $p = 1$.

Proof. Write

$$p^* = (d-1)/d$$

and consider the function $g : [p^*, +\infty) \to \mathbf{R}$ defined by

$$g(x) = x^{d-1} - x^d.$$

Since $g'(x) = (d-1)x^{d-2} - dx^{d-1}$, this is a decreasing function; its inverse h is a decreasing function and it maps the interval $(-\infty, g(p^*)]$ onto the interval $[p^*, +\infty)$; for every y in the domain of h, the value of $h(y)$ is the largest real root of the polynomial $x^d - x^{d-1} + y$. Since a nonzero r is a root of the polynomial $x^d - x^{d-1} + (1-p)p^{d-1}$ if and only if p/r is a root of the polynomial $p - x + (1-p)x^d$, it follows that

$$u_d(p) = \frac{p}{h((1-p)p^{d-1})},$$

and so

$$\frac{u_d(p)}{\ell_d(p)} = \frac{1 - (1-p)p^{d-1}}{h((1-p)p^{d-1})}.$$

In particular, $\lim_{p\to 0} u_d(p)/\ell_d(p) = 1/h(0) = 1$.

Since the function defined by $p \mapsto (1-p)p^{d-1}$ increases in the interval $[0, p^*]$ and maps it onto the interval $[0, g(p^*)]$, proving that the ratio $u_d(p)/\ell_d(p)$ increases in the interval $[0, p^*]$ reduces to proving that the ratio $(1-y)/h(y)$ increases in the interval $[0, g(p^*)]$. For this purpose, note first that $g'(x)$ decreases in the interval $[p^*, 1]$: more precisely, $g'(x)$ decreases in the interval $[(d-2)/d, +\infty)$ and we have $p^* > (d-2)/d$. Since $h'(y) = 1/g'(h(y))$ and since $h(y)$ decreases in the interval $[0, g(p^*)]$, we conclude that $h'(y)$ decreases in the interval $[0, g(p^*)]$. In particular, as $h'(0) = 1/g'(1) = -1$, we have $h'(y) \le -1$ for all y in $[0, g(p^*)]$; now it follows first that $h(y) \le 1-y$ for all y in $[0, g(p^*)]$ and then that

$$\left(\frac{1-y}{h(y)}\right)' = \frac{-h(y)-(1-y)h'(y)}{h(y)^2} \ge \frac{-h(y)+(1-y)}{h(y)^2} \ge 0$$

for all y in $[0, g(p^*)]$.

Finally, if $p^* \le p \le 1$, then p belongs to the domain of g, and so

$$u_d(p) = \frac{p}{h((1-p)p^{d-1})} = \frac{p}{h(g(p))} = 1,$$

which implies

$$\frac{u_d(p)}{\ell_d(p)} = 1 + \frac{(1-p)(1-p^{d-1})}{p}.$$

□

In closing, let us return to $d = 2$. A *binary tree* is a 2-ary tree where each child is labeled as a 'left child' or a 'right child' in such a way that siblings always get distinct labels. For every function k of n that takes values in $\{1, 2, \ldots, n\}$, the average of $c(T, k)$ over all binary trees is

$$(1+o(1))\binom{n}{k}\frac{2k}{\sqrt{(n-k)^2+4k^2}}$$

and the average of $c(T, k)$ over all full binary trees is

$$(1+o(1))\binom{n}{k}\frac{k}{\sqrt{(n-k)^2+k^2}}.$$

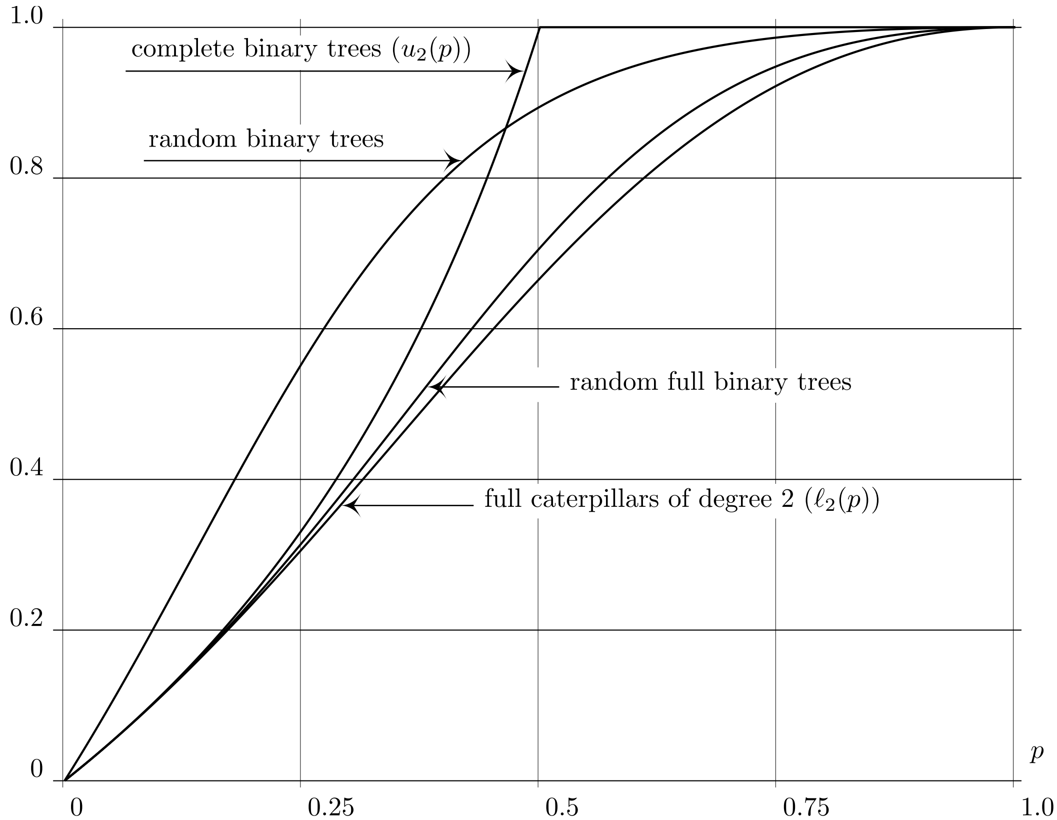


Figure 3: Limiting probabilities of cutting the root from all the leaves as a function of p, the probability of removing an individual node.

Analytic proofs of these two results are beyond the scope of the present paper; we shall content ourselves with commenting on the difference between the average $c(T, k)$ of full binary trees on n nodes and the average $c(T, k)$ of full 2-ary trees on n nodes. Many full 2-ary trees may be labelled in many different ways as full binary trees. For instance, there are precisely two full 2-ary trees on 7 nodes: the full caterpillar $T_2^{\min}(7)$ and the the complete 2-ary tree C_2^2. Since four distinct binary trees are isomorphic to $T_2^{\min}(7)$ as 2-ary trees and only one binary tree is isomorphic to C_2^2 as a 2-ary tree, the caterpilar contributes 80% of its (low) values of $c(T_2^{\min}(7), k)$ in computing the average $c(T, k)$ of full binary trees on 7 nodes, as opposed to only 50% in computing the average $c(T, k)$ of full 2-ary trees on 7 nodes.

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References

[1] G. Bennett, Probability inequalities for the sum of independent random variables, *Journal of the American Statistical Association* **57** (1962), 33 – 45.

[2] S. Bernstein, On a modification of Chebyshev's inequality and of the error formula of Laplace, *Section Mathématique des Annales Scientifiques des Institutions Savantes de l'Ukraine*, **1** (1924), 38 – 49. (Russian)

[3] H. Chernoff, A measure of asymptotic effiiency of tests of a hypothesis based on the sum of observations, *Annals of Mathematical Statistics* **23** (1952), 493 – 507.

[4] C.C. Craig, On the Tchebychef inequality of Bernstein, *Annals of Mathematical Statistics* **4** (1933), 94 – 102.

[5] L. Devroye, Branching processes in the analysis of the heights of trees, *Acta Informatica* **24** (1987), 277 – 298.

[6] L. Devroye, Universal limit laws for depths in random trees, *SIAM Journal on Computing* **28** (1999), 409 – 432.

[7] J.D. Farley, Breaking Al Qaeda cells: a mathematical analysis of counterterrorism operations (a guide for risk assessment and decision making), *Studies in Conflict and Terrorism* **26** (2003), 399 – 411.

[8] J.D. Farley, *Toward a Mathematical Theory of Counterterrorism,* The Proteus Monograph Series, Volume 1, Issue 2, December 2007.

[9] F. Harary and A.J. Schwenk, Trees with hamiltonian square, *Mathematika* **18** (1971), 138 – 140.

[10] W. Hoeffding, Probability inequalities for sums of bounded random variables, *Journal of the American Statistical Association* **58** (1963), 13 – 30.