ON THE ESTIMATION OF DISCRETE PROBABILITY DENSITIES FROM NOISY MEASUREMENTS

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ABSTRACT

The estimation of a discrete probability density from independent
observations is considered. For a wide class of noises, a method is given
for estimating a probability density when the measurements are corrupted by
independent additive noise. This method is shown to be consistent, and
several bounds on the error are given.

I. INTRODUCTION

The need for considering discrete data is often encountered in data
communications, digital signal processing, and other areas. In this paper
we consider discrete valued random variables, and we are concerned with
estimating the discrete probability density function. Measurements are
taken, and from these measurements a density function is obtained. However,
we assume that the measurements are imperfect. We derive the estimators,
establish the appropriate forms of convergence, and supply an abundance of
bounds on the errors.

Assume that we can observe $X_1, X_2, \ldots, X_n$, a sequence of independent
identically distributed random variables with the unknown discrete proba-
bility density $f$. An obvious way of estimating $f(x)$ is to use the empirical
density based on the $n$ observations. However, the estimation problem is
complicated if we can only observe $X_1 + Z_1, X_2 + Z_2, \ldots, X_n + Z_n$, where $X_1, Z_1,
X_2, Z_2, \ldots, X_n, Z_n$ are independent random variables and the $Z_i$'s, commonly
referred to as noise, have a common known discrete probability density
function $g$. For a wide class of densities $g$, a method is given to recover
$f$ which is shown to be strongly uniformly consistent, that is,

$$\limsup_{n, x} \frac{|f_n(x) - f(x)|}{n} = 0$$

with probability one (wp1), where $f_n$ is the estimate of $f$ with just $n$
observations.

II. PROPERTIES OF THE EMPIRICAL DENSITY

Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random
variables with a discrete probability density function $f$. Assume without
loss of generality that $f$ is supported on $\mathbb{Z}$, the set of integers. The
empirical density $f_n$ is defined by

$$f_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{X_i = x\}}, \quad x \in \mathbb{Z},$$

where $\mathbb{1}$ is the indicator function. Thus, $f_n$ is also a discrete probability
density.

We will briefly review some properties of $f_n$, starting with the point-
wise consistency. By the strong law of large numbers [1, p.239] we know
that $f_n(x) \to f(x)$ wp1 for all $x$. In fact, by Hoeffding's inequality [2],
for all $\epsilon > 0$,

$$P \left\{ \left| f_n(x) - f(x) \right| \geq \epsilon \right\} \leq 2 \exp(-2n\epsilon^2) .$$

Presented at the Fifteenth Annual Allerton Conference on Communication,
Control, and Computing, September 28-30, 1977; to be published in the
Proceedings of the Conference.
from which the strong consistency follows by the Borel-Cantelli lemma [1, p.228]. This bound is independent of \( f \) and \( x \). Since countable unions of null events are null, we immediately have the strong uniform consistency

\[
\lim_{n \to \infty} \sup_{x} |f_n(x) - f(x)| = 0 \quad \text{wp}.
\]

Now we consider some uniform error bounds. If \( F \) and \( F_n \) are the distribution functions corresponding respectively to \( f \) and \( f_n \), then

\[
P \left( \sup_{x} |f_n(x) - F(x)| \geq \varepsilon \right) \leq 2P \left( \sup_{x} |F_n(x) - F(x)| \geq \frac{\varepsilon}{2} \right)
\]

\[
\leq 2C_1 \exp \left( -2n \left[ \frac{\varepsilon}{2} \right]^2 \right),
\]

by an inequality of Dvoretzky, Kiefer, and Wolfowitz [3], where \( C_1 \) is a universal constant. In the Appendix it is shown that \( C_1 \approx 0.4 \). Recently, Singh [4] (see also, [5]) has shown that

\[
P \left( \sup_{x} |F_n(x) - F(x)| \geq \beta \right) \leq 4 \varepsilon^2 n \beta \exp \left( -2n \beta^2 \right)
\]

for \( n \beta^2 \geq 1 \). This implies that

\[
P \left( \sup_{x} |f_n(x) - f(x)| \geq \varepsilon \right) \leq 4 \varepsilon^2 n \varepsilon \exp \left( -\frac{\varepsilon^2}{2} \right)
\]

for \( n \varepsilon^2 \geq 4 \). Both bounds (1) and (2) are valid for all discrete densities \( f \) and, by the Borel-Cantelli lemma, each implies that

\[
\lim_{n \to \infty} \sup_{x} |f_n(x) - f(x)| = 0 \quad \text{wp}.
\]

Now consider the following two lemmas.

**Lemma 1:** Let \( f_n(x) \) be the empirical estimate of the discrete density \( f(x) \). Then

\[
P \left( \sum_{x=\infty}^{\infty} |f_n(x) - f(x)| \geq \varepsilon \right) \leq K_1 \exp (-K_2 n)
\]

where \( K_1, K_2 > 0 \) depend upon \( \varepsilon \) and \( f \) only.

**Proof:** Pick \( N \geq 1 \) such that \( \sum_{|x|>N} f(x) < \frac{\varepsilon}{6} \). Then

\[
P \left( \sum_{x=\infty}^{\infty} |f_n(x) - f(x)| \geq \varepsilon \right)
\]

\[
\leq P \left( \sum_{|x| \leq N} |f_n(x) - f(x)| \geq \frac{\varepsilon}{2} \right) + P \left( \sum_{|x| \geq N} |f_n(x) - f(x)| \geq \frac{\varepsilon}{2} \right)
\]

\[
\leq \sum_{|x| \leq N} P \left( |f_n(x) - f(x)| \geq \frac{\varepsilon}{4N+2} \right) + P \left( \sum_{|x|>N} f_n(x) - \sum_{|x|>N} f(x) \geq \frac{\varepsilon}{6} \right)
\]
\[ (4N+2) \exp \left[ -2n \left( \frac{\varepsilon}{4N+2} \right)^2 \right] + \exp \left[ -2n \left( \frac{\varepsilon}{6} \right)^2 \right] \]

\[ (4N+3) \exp \left[ -2n \left( \frac{\varepsilon}{4N+2} \right)^2 \right] \]

by Hoeffding's inequality [2]. Q.E.D.

**Lemma 2:** Let \((\Omega, \mathcal{B}, P)\) be a probability space. Let \(f_1, f_2, \ldots\) be densities on the integers for each fixed \(\omega\), and random variables \(\chi^i_n(\Omega, \mathcal{B}, P)\) for each fixed \(x\). We will write \(f_n(x, \omega)\) to make the dependency on \(\omega\) explicit. Let \(f\) be a density on the integers. If \(f_n(x, \omega) \overset{P}{\to} f(x)\) wp1 for all \(x\), then

\[ \sum_{x=-\infty}^{\infty} \sum_{\omega} |f_n(x, \omega) - f(x)| \overset{P}{\to} 0 \quad \text{wp1.} \quad (3) \]

**Proof:**

\[
\begin{align*}
\mathbb{P} \left\{ \omega: \lim_{n \to \infty} \sum_{x=-\infty}^{\infty} |f_n(x, \omega) - f(x)| \neq 0 \right\} & \leq \mathbb{E} \sup_x \sum_{\omega} |f_n(x, \omega) - f(x)| \\
\mathbb{P} \left\{ \omega: \sum_{x=-\infty}^{\infty} \lim_{n \to \infty} |f_n(x, \omega) - f(x)| \neq 0 \right\} & = \sum_{x=-\infty}^{\infty} \mathbb{P} \left\{ \omega: \lim_{n \to \infty} |f_n(x, \omega) - f(x)| \neq 0 \right\} \\
\sum_{x=-\infty}^{\infty} \mathbb{P} \left\{ \omega: \lim_{n \to \infty} |f_n(x, \omega) - f(x)| \neq 0 \right\} & \leq \sum_{x=-\infty}^{\infty} \mathbb{E} \left[ |f_n(x, \omega) - f(x)| \right].
\end{align*}
\]

Therefore, if \(f_n(x, \omega) \overset{P}{\to} f(x)\) wp1, then

\[ \sum_{x=-\infty}^{\infty} |f_n(x, \omega) - f(x)| \overset{P}{\to} 0 \quad \text{wp1.} \quad Q.E.D. \]

**III. ESTIMATION IN THE PRESENCE OF ADDITIVE NOISE**

Because of background noise, faulty equipment, or other practical problems, it may not be possible to observe \(X_1, X_2, \ldots, X_n\); but instead, we can observe \(Y_1, Y_2, \ldots, Y_n\), where

\[ Y_i = X_i + Z_i, \quad 1 \leq i \leq n, \]

and \(X_1, Z_1, X_2, Z_2, \ldots, X_n, Z\) are independent. The \(Z\) have a common known density \(g\) on the integers and the \(X_i\) have an unknown density \(f\) on the integers which we would like to estimate. The discrete probability density \(h\) of the \(Y_i\) is given by

\[ h(x) = \sum_{y=-\infty}^{\infty} f(x-y) g(y). \]

Assume that we can write

\[ f(x) = \sum_{y=-\infty}^{\infty} \xi_y h(x-y) = \sum_{y=-\infty}^{\infty} \xi_{x-y} h(y) \]
for some sequence \( \{ \xi_y \} \) of real numbers. Resubstitution gives

\[
f(x) = \sum_{y=-\infty}^{\infty} \xi_y \sum_{u=-\infty}^{\infty} g(u) f(x-y-u)
\]

which should hold for all \( x \) and all \( f \).

**Lemma 3:** Eq. (4) is valid for all \( x \) and all \( f \) if and only if

\[
\sum_{y=-\infty}^{\infty} \xi_y g(k-y) = \begin{cases} 
1 & \text{for } k = 0 \\
0 & \text{for } k \neq 0 
\end{cases}
\]

for all integers \( k \).

**Proof:** Clearly, if (5) holds, then (4) is valid for all \( x \) and all \( f \). Conversely, let \( f(0) = 1 \), and note that

\[
\sum_{y, u: y+u=0} \xi_y g(u) = 1 .
\]

Next, let \( f(0) = \frac{1}{2} = f(k), k \neq 0 \). Then (4) reads, for \( x = 0 \),

\[
\frac{1}{2} = \frac{1}{2} \sum_{y, u: y+u=0} \xi_y g(u) + \frac{1}{2} \sum_{y, u: y+u=-k} \xi_y g(u)
\]

\[
= \frac{1}{2} + \frac{1}{2} \sum_{y, u: y+u=-k} \xi_y g(u) ,
\]

from which (5) follows by the arbitrariness of \( k \). Q.E.D.

Deferring for the moment the question of how to determine the \( \xi_y \) from \( g \) so that (5) is satisfied, we return to the construction of an estimate of \( f \) assuming the knowledge of the \( \xi_y \). Let \( h_n \) be the empirical density for \( Y_1, Y_2, \ldots, Y_n \),

\[
h_n(x) = \frac{1}{n} \sum_{i=1}^{n} I_{\{ Y_i = x \}} ,
\]

which suggests the following estimate of \( f \):

\[
f_n(x) = \sum_{y=-\infty}^{\infty} \xi_y h_n(x-y) = \sum_{y=-\infty}^{\infty} \xi_{x-y} h_n(y)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \sum_{y=-\infty}^{\infty} \xi_{x-y} I_{\{ Y_i = y \}} = \frac{1}{n} \sum_{i=1}^{n} \xi_{x-Y_i} .
\]

Notice that for \( g \) with \( g(0) = 1 \) (and thus \( \xi_0 = 1, \xi_1 = 0, i \neq 0 \)), we get back the original empirical estimate of \( f \) because

\[
\xi_{x-Y_i} = I_{\{ Y_i = x \}} .
\]
The first question that arises is the question of the closeness of $\hat{f}_n$ to $f$. Notice that $f_n$ is not a probability density in general. Of course, $\tilde{f}_n$ defined by
\[
\hat{f}_n(x) = \begin{cases} 
1, & f_n(x) \geq 1 \\
 f_n(x), & 0 < f_n(x) < 1 \\
0, & f_n(x) \leq 0
\end{cases}
\]
is a strictly better density estimate than $f_n$. However, we will not further discuss this trivial modification of our estimate. Clearly, $f_n$ satisfies
\[
\sup_x |f_n(x) - f(x)| \leq \left( \sum_{y=-\infty}^{\infty} |\xi_y| \right) \sup_x |h_n(x) - h(x)| ,
\]
(7)

\[
\sup_x |f_n(x) - f(x)| \leq \left( \sup_y |\xi_y| \right) \sum_{x=-\infty}^{\infty} |h_n(x) - h(x)| ,
\]
and
\[
\sum_{x=-\infty}^{\infty} |f_n(x) - f(x)| \leq \left( \sum_{y=-\infty}^{\infty} |\xi_y| \right) \sum_{x=-\infty}^{\infty} |h_n(x) - h(x)| .
\]

Let
\[ C = \sum_{y=-\infty}^{\infty} |\xi_y| \]
and
\[ D = \sup_y |\xi_y| .
\]

Applying some of the results of the previous section, we have the following theorem.

**Theorem 1:** Let $f_n(x)$ be given by (6) and assume that $\{\xi_y\}$ satisfies (5). Then if $C$ is finite, we have
\[
P \left\{ \sup_x |f_n(x) - f(x)| \geq \varepsilon \right\} \leq \begin{cases} 
2 C_1 \exp \left[ -\frac{n \varepsilon^2}{2 C^2} \right], & \text{all } n \geq 1 \\
\frac{4 e \varepsilon}{C} \exp \left[ -\frac{n \varepsilon^2}{2 C^2} \right], & \text{all } n \geq \frac{4 C^2}{\varepsilon^2}
\end{cases}
\]
(9)

and
\[
P \left\{ \sum_{x=-\infty}^{\infty} |f_n(x) - f(x)| \geq \varepsilon \right\} \leq K_1 \exp(-K_2 n)
\]
(10)
where $K_1, K_2 > 0$ are constants depending upon $\varepsilon, h,$ and $C$. Also, if $D$ is finite, we have
\[ P \left\{ \sup_x |f_n(x) - f(x)| \geq \varepsilon \right\} \leq K_3 \exp \left( -K_4 n \right) \]  

(11)

where \( K_3, K_4 > 0 \) are constants depending upon \( \varepsilon, h, \) and \( D \).

The bounds in (9) are distribution-free but require that \( C < \infty \). Eq. (11), which is not distribution-free requires only that \( D < \infty \). The strong result (10) assumes finiteness of \( C \) and is not uniform over all densities.

Using the Borel–Cantelli lemma, we obtain the following result.

**Corollary 1**: Let \( f_n(x) \) be given by (6) and assume that \( \{\xi_y\} \) satisfies (5). If \( D = \sup_y |\xi_y| < \infty \), then

\[ \sum_{x=\infty}^\infty |f_n(x) - f(x)| \to 0 \quad \text{wp1.} \]

In the remainder of this paper we briefly discuss practical solutions to (5) and give some examples of sequences \( \{\xi_y\} \) for some common densities \( g \).

**Practical Considerations**: A solution to (5) can be obtained recursively if \( g \) is a single tailed density, that is, if there exists a \( K \) such that \( g(x) = 0 \) for all \( x > K \) or \( g(x) = 0 \) for all \( x < K \). For example, assume that \( g(K) > 0 \) and \( g(x) = 0 \) for all \( x < K \). Let \( \xi_y = 0 \) for \( y < -K \) and \( \xi_{-K} = 1/g(K) \). It is easy to see that the \( k = 0 \) equation of (5) holds and that the \( k = 1 \) equation results in

\[ \xi_{-K+1} \cdot g(K) + \xi_{-K} \cdot g(K+1) = 0, \]

from which we find \( \xi_{-K+1} \). Solving the \( k = 2 \) equation of (5) gives us \( \xi_{-K+2} \) and so on. Clearly, this is probably not the only solution to (5).

Consider the following simple example.

**Example 1**: Let \( g(0) = g(1) = 1/2 \) and \( g(x) = 0, x \neq 0, 1 \). Then (5) results in

\[ \xi_{-1} + \xi_0 = 2 \]

\[ ... = \xi_{-2} + \xi_{-1} = \xi_0 + \xi_1 = \xi_1 + \xi_2 = \ldots = 0. \]

If \( \xi_0 = \alpha \) and \( \xi_{-1} = 2 - \alpha \), then all solutions of (5) can be written as

\[ \xi_y = \alpha (-1)^y, \quad y \geq 0 \]

\[ \xi_{-y} = (2 - \alpha) (-1)^{y+1}, \quad y \geq 1 \]

where \( \alpha \) is any real number. For this case we note that \( D \) is finite while \( C \) is infinite.

**Example 2**: In Example 1 we have \( \sup_y |\xi_y| < \infty \), but this is not always the case. If \( g(-1) = g(1) = 1/4 \) and \( g(0) = 1/2 \), then it is straightforward to show that for any solution of (5) we have \( \sup_y |\xi_y| = \infty \).

Now we will give solutions for some well known densities \( g \).
Example 3: Poisson noise with parameter $\lambda > 0$: Let

$$g(x) = \frac{\lambda^x}{x!} e^{-\lambda}, \quad x \geq 0,$$

and it can be verified that a solution to (5) is given by

$$\xi_y = \begin{cases} 
\frac{(-\lambda)^y}{y!} e^\lambda, & y \geq 0 \\
0, & y < 0
\end{cases}.$$

It is easily seen that

$$\sum_{y=-\infty}^{\infty} |\xi_y| = e^{2\lambda} < \infty.$$

In [6] the estimation of a continuous probability density function from measurements corrupted by Poisson noise is considered.

Example 4: Geometric noise with parameter $\lambda > 1$: If

$$g(x) = (\lambda-1)/\lambda^{x+1}, \quad x \geq 0,$$

then a solution to (5) is given by

$$\xi_y = \begin{cases} 
\lambda/(\lambda-1), & y = 0 \\
-1/(\lambda-1), & y = 1 \\
0, & y \neq 0, 1
\end{cases}.$$

Example 5: Binomial noise with parameters $N$ and $p \neq 0, 1$: If

$$g(x) = \binom{N}{x} p^x (1-p)^{N-x}, \quad 0 \leq x \leq N,$$

then a solution to (5) is given by

$$\xi_y = \begin{cases} 
(-1)^y \binom{N+y-1}{y} (1-p)^{-N} \left(\frac{p}{1-p}\right)^y, & y \geq 0 \\
0, & y < 0
\end{cases}.$$

We notice that

$$\sum_{y=-\infty}^{\infty} |\xi_y| < \infty$$

if and only if $p < 1/2$, and that $|\xi_y| \uparrow \infty$ if $p > 1/2$ or if $N > 2$ and $p = 1/2$. Another solution to (5) is given by

$$\xi_{N-y} = \begin{cases} 
(-1)^y \binom{N+y-1}{y} p^{-N} \left(\frac{1-p}{p}\right)^y, & y \geq 0 \\
0, & y < 0
\end{cases}.$$
and it is easy to see that for each $p > 1/2$,

$$
\sum_{y=-\infty}^{\infty} |\xi_y| < \infty.
$$

**APPENDIX**

In this appendix we prove the following Lemma. The bound that we present results directly from the work of Dvoretzky, Kiefer, and Wolfowitz.

**Lemma:** If $F$ is any distribution function on $\mathbb{R}$ and $F_n$ is the empirical distribution function with $X_1, X_2, \ldots, X_n$ a sequence of independent random variables with distribution function $F$, then

$$
P \left\{ \sup_x |F_n(x) - F(x)| \geq \epsilon \right\} \leq C_1 \exp \left( -2n\epsilon^2 \right)
$$

where

$$
C_1 = 2 \left[ 1 + \frac{32}{3\sqrt{6\pi}} + \frac{4}{3\sqrt{3}} + \frac{\exp(40/9)}{8\sqrt{2e}} \right].
$$

Proof: We will use the notation of Dvoretzky, Kiefer, and Wolfowitz [3, pp.646-648]. In [3] they establish that

$$
1 - G_n(r) \leq 2 \left[ 1 - H_n(r) \right]
$$

where $r$ takes values in $(0, \sqrt{n})$. Expression (2.9),

$$
1 - H_n(r) = \left( 1 - \frac{r}{\sqrt{n}} \right)^n + r\sqrt{n} \sum_{j=[r\sqrt{n}]+1}^{n-1} Q_n(j,r)
$$

can be upper bounded as follows. Notice that

$$
\left( 1 - \frac{r}{\sqrt{n}} \right)^n < \exp \left( -2r^2 \right)
$$

(see (2.11)).

First, consider those $j$ for which $|j - \frac{n}{2}| \leq \frac{n}{4}$. We will show that

$$
Q_n(j,0) < c_2 n^{-3/2}
$$

for $c_2 = \frac{16}{3\sqrt{6\pi}}$. By an approximation of Feller [7] for $n!$ we have

$$
Q_n(j,0) = \binom{n}{j} \frac{j^j}{n^j} (n-j)^{n-j-1}
\leq \frac{1}{\sqrt{2\pi} \sqrt[j]{j(n-j)^3}} \exp \left[ \frac{1}{12n} - \frac{1}{12j+1} - \frac{1}{12(n-j)+1} \right].
$$
Notice that \( j(n-j)^3 \geq 27 \left( \frac{n}{4} \right)^4 \). Also,
\[
\exp \left[ \frac{1}{12n} - \frac{1}{12j+1} - \frac{1}{12(n-j)+1} \right] < \exp \left[ \frac{-1}{12j+1} \right] < 1.
\]

Thus,
\[
Q_n(j,0) < \frac{16}{3\sqrt{3\pi}} \frac{1}{n^{-3/2}}.
\]

Next, we know that
\[
\frac{1}{n} + \int_0^{\infty} \exp\left(-8 r^2 t^2\right) dt < 1 + \frac{\sqrt{2\pi}}{4}
\]
and (2.15) holds:
\[
\sqrt{n} \sum' Q_n(j,r) < 2 c_2 \left( 1 + \frac{\sqrt{2\pi}}{4} \right) \exp(-2 r^2) = \left( \frac{32}{3\sqrt{6\pi}} + \frac{4}{3\sqrt{3}} \right) \exp(-2 r^2).
\]

Now consider those \( j \) for which \( |j - \frac{n}{2}| > \frac{n}{4} \). It follows from the equation at the top of p.647 that
\[
Q_n(j,r) \leq Q_n(j,1) \exp\left[ -f(n,r,j) + f(n,1,j) \right]
\]
for \( r \geq 1 \), where \( f(n,r,j) \) is the negative of the exponent in (2.12). From (2.9) it follows that \( j \geq r\sqrt{n} \). Thus, it is easily seen that \( f(n,1,j) \leq 40/9 = c \). Therefore, we have that
\[
Q_n(j,r) \leq c Q_n(j,1) \exp\left[ -f(n,r,j) \right]
\]
for \( r \geq 1 \). Since \( f(n,r,j) \geq 2 r^2 + \frac{r^2}{64} \), we have that
\[
Q_n(j,r) \leq c Q_n(j,1) \exp(-2 r^2) \exp\left(-\frac{r^2}{64}\right).
\]
Using the fact that \( r \exp\left(-\frac{r^2}{64}\right) \) is maximized at \( r = \sqrt{2} / 16 \), we obtain
\[
Q_n(j,r) \leq \frac{c}{8r\sqrt{2e}} Q_n(j,1) \exp(-2 r^2).
\]

Thus,
\[
r\sqrt{n} \sum'' Q_n(j,r) \leq \frac{c}{8\sqrt{2e}} \sqrt{n} \exp(-2 r^2) \sum'' Q_n(j,1)
\]
\[
< \frac{c}{8\sqrt{2e}} \exp(-2 r^2).
\]
from (2.16).

Collecting bounds, we find that, for \( r \geq 1 \),

\[
1 - G_n(r) \leq 2 \left[ 1 + \frac{32}{3 \sqrt{6 \pi}} + \frac{4}{\sqrt{3}} + \frac{\exp(40/9)}{8 \sqrt{2} e} \right] \exp\left(-2 r^2\right).
\]

(Al)

For \( r < 1 \), the expression on the right hand side is greater than one, so that (Al) is valid for all \( r \).

Q.E.D.

ACKNOWLEDGEMENT

This research was supported by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant AFOSR-76-3062.

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