SIMULATING BESSEL RANDOM VARIABLES

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Abstract. In this paper, we discuss efficient exact random variate generation for the Bessel distribution. The expected time of the algorithm is uniformly bounded over all choices of the parameters, and the algorithm avoids any computation of Bessel functions or Bessel ratios.

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Introduction

A random variable X on the non-negative integers is said to be a Bessel random variable with parameters $\nu > -1$ and $a > 0$ if

$$p_n \stackrel{\text{def}}{=} \mathbb{P}\{X = n\} = \frac{(a/2)^{2n+\nu}}{I_\nu(a) n! \Gamma(n+\nu+1)} \ , \ n \geq 0 \ ,$$

where

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+\nu}}{n!\Gamma(n+\nu+1)}$$

is the first type of modified Bessel function. It arises in a natural way in the theory of stochastic processes (Pitman and Yor, 1982), and is related to many other distributions, including multivariate and randomized gamma distributions and the von Mises-Fisher distribution (Yuan and Kalbfleisch, 2000). Yuan and Kalbfleisch (2000) describe a number of properties of this distribution but note that *simulating a Bessel distribution is generally difficult.* They suggest a truncated normal approximation when the mean is large and table sampling when the mean is small. The purpose of this note is to derive a simple exact algorithm with expected time uniformly bounded over the entire parameter space.

Main properties

In this section we review the main properties that will be useful for us. We define the Bessel quotient

$$R_\nu(x) = \frac{I_{\nu+1}(x)}{I_\nu(x)} \ .$$

We begin with bounds for the Bessel quotient, as taken from Amos (1974) and Yuan and Kalbfleisch (2000).

LEMMA 1 (INEQUALITIES FOR THE BESSEL QUOTIENT). *For all $\nu > -1$ and $x > 0$,*

$$\frac{x}{\nu+1+\sqrt{x^2+(\nu+1)^2}} \leq R_\nu(x) \leq \frac{x}{\nu+\sqrt{x^2+\nu^2}} \ .$$

For fixed x, $R_\nu(x)$ is nonincreasing in ν.

LEMMA 2 (YUAN AND KALBFLEISCH, 2000). *If X is Bessel (ν, a), then*

$$\mu \stackrel{\text{def}}{=} \mathbb{E}\{X\} = \frac{aR_\nu(a)}{2} \ ,$$

and

$$\chi^2 \stackrel{\text{def}}{=} \mathbb{V}\{X\} = \mu + \frac{1}{4}a^2 R_\nu(a)(R_{\nu+1}(a) - R_\nu(a)) \ .$$

Furthermore, the distribution is unimodal and has a unique mode or two modes at consecutive integers, and one mode is always located at

$$m \stackrel{\text{def}}{=} \left\lfloor \frac{\sqrt{a^2+\nu^2}-\nu}{2} \right\rfloor .$$

LEMMA 3. *The Bessel distribution is log concave, that is, p_{n+1}/p_n is decreasing in n.*

PROOF. Note that for $n \ge 0$, $p_{n+1}/p_n = (a/2)^2/(n+1)(n+\nu+1)$, and that this is a decreasing function of n. $\square$

LEMMA 4 (SPECIAL CASES). *The Bessel $(1/2, a)$ distribution is given by*

$$p_n = \frac{(a/2)^{2n+\nu}\sqrt{\pi a}}{\sqrt{2}\sinh(a)\, n!\Gamma(n+\nu+1)} \ , \ n \ge 0 \ .$$

The Bessel $(3/2, a)$ distribution is given by

$$p_n = \frac{(a/2)^{2n+\nu}\sqrt{\pi a^3}}{\sqrt{2}(a\cosh(a)-\sinh(a))\, n!\Gamma(n+\nu+1)} \ , \ n \ge 0 \ .$$

LEMMA 5. *Let p_n describe a unimodal distribution on the integers with mode at m, and let σ^2 be the second moment centered at the mode, $\sum_n (n-m)^2 p_n$. Then, if $p_m \le 1/3$, we have*

$$\sigma^2 p_m^2 \ge 1/648 \ .$$

PROOF. Assume without loss of generality that $m = 0$. Define $p = \sum_{n>0} p_n$, $q = \sum_{n<0} p_n$, $r = p_0$. It is clear that

$$\sigma^2 = \sum_{n>0} n^2 p_n + \sum_{n<0} n^2 p_n \ge r \sum_{s \ge n > 0} n^2 + r \sum_{-t \le n < 0} n^2$$

where $s = \lfloor p/r \rfloor$ and $t = \lfloor q/r \rfloor$. Thus, because $\sum_{n=1}^{s} n^2 = s^3/3 + s^2/2 + s/6$, we have

$$\sigma^2 \ge r(s^3/3 + s^2/2 + s/6) + r(t^3/3 + t^2/2 + t/6) \ge r(s^3/3 + t^3/3) \ .$$

If $p \geq r$, then $s \geq p/2r$. If $q \geq r$, then $t \geq q/2r$. Thus,

$$\begin{aligned}\sigma^2 &\geq 1_{[p \geq r]} \left(\frac{p^3}{24r^2} \right) + 1_{[q \geq r]} \left(\frac{q^3}{24r^2} \right) \\ &\geq 1_{[\max(p,q) \geq r]} \frac{\max(p,q)^3}{24r^2} \\ &\geq 1_{[(1-r)/2 \geq r]} \frac{(1-r)^3}{192r^2} \\ &\geq 1_{[r \leq 1/3]} \frac{1}{648r^2} \ . \ \square\end{aligned}$$

Lemma 6. *For all discrete log-concave distributions with mode at m, we have, for all n,*

$$p_n \leq p_m \min\left(1, \exp\left(1 - p_m |n-m|\right)\right) \ .$$

Furthermore,

$$p_n \leq p_m \min\left(1, \exp\left(1 - q |n-m|\right)\right) \ ,$$

where $q = \min\left(\frac{1}{\sigma\sqrt{648}}, \frac{1}{3}\right)$.

Proof. Devroye (1987) derived a general inequality for all discrete log-concave distributions with mode at m:

$$p_n \leq p_m \min\left(1, \exp\left(1 - p_m |n-m|\right)\right) \ , \ \text{all } n.$$

Assume $p_m \leq 1/3$. Then by Lemma 5, $p_m \geq 1/\sigma\sqrt{648}$, which shows that $p_m \geq q$. This concludes the proof. $\square$

Lemma 7. *For all log-concave density functions,* $p_m \sigma \leq \sqrt{28 + 4e}$.

PROOF. Note that, by Lemma 6,

$$\begin{aligned}
\sigma^2 &\le \sum_k k^2 p_m \min\left(1, e^{1-p_m|k|}\right) \\
&= 2\sum_{k>0} k^2 p_m \min\left(1, e^{1-p_m k}\right) \\
&\le 2\sum_{3/p_m>k>0} k^2 p_m + 2\sum_{k\ge 3/p_m} k^2 e p_m e^{-p_m k} \\
&\le 2p_m\left((1/3)(3/p_m)^3 + (1/2)(3/p_m)^2 + (1/6)(3/p_m)\right) + 2\sum_{k\ge 3/p_m} k^2 e p_m e^{-p_m k} \\
&= 18/p_m^2 + 9/p_m + 1 + 2\int_{x\ge 3/p_m - 1} x^2 e p_m e^{-p_m x}\, dx \\
&\le \frac{28}{p_m^2} + \frac{4e}{p_m^2}\int_{t\ge 0} \frac{t^2 e^{-t}}{2}\, dt\ .
\end{aligned}$$

Thus, $p_m^2\sigma^2 \le 28 + 4e$. □

LEMMA 8. *For the Bessel* (ν, a) *distribution, if we set* $A = \sqrt{a^2+\nu^2}$, $B = \sqrt{a^2+(\nu+1)^2}$, *then*

$$\sigma^2 \le Q \stackrel{\text{def}}{=} \frac{a^2}{2(\nu+A)} + \left(1 + \frac{a^2(1+B-A)}{2(\nu+A)(\nu+1+B)}\right)^2\ .$$

PROOF. Note that

$$\begin{aligned}
\sigma^2 &= \chi^2 + (m-\mu)^2 \\
&= \mu + \frac{1}{4}a^2 R_\nu(a)(R_{\nu+1}(a) - R_\nu(a)) + \left(\lfloor \frac{\sqrt{a^2+\nu^2}-\nu}{2} \rfloor - \frac{aR_\nu(a)}{2}\right)^2 \\
&\le \mu + \left(\frac{\sqrt{a^2+\nu^2}-\nu}{2} + \theta - \frac{a^2}{2(\nu+\zeta+\sqrt{a^2+(\nu+\zeta)^2})}\right)^2 \\
&\stackrel{\text{def}}{=} I + II\ ,
\end{aligned}$$

by Lemmas 1 and 2, where $\zeta \in \{0,1\}$, $\theta \in [0,1]$. Consider each term separately. Clearly,

$$I = \mu = aR_\nu(a)/2 \le \frac{a^2}{2(\nu+\sqrt{a^2+\nu^2})}\ .$$

Also, rewrite II as

$$II = \left(\theta + \frac{a^2}{2(\nu+\sqrt{a^2+\nu^2})} - \frac{a^2}{2(\nu+\zeta+\sqrt{a^2+(\nu+\zeta)^2})}\right)^2\ .$$

For $\zeta = 0$, we have $II \leq \theta^2 \leq 1$. For $\zeta = 1$, by Lemma 1, the last term in the brackets is smaller than the middle term, so that we may bound as follows:

$$II \leq \left(1 + \frac{a^2 \left(1 + \sqrt{a^2 + (\nu+1)^2} - \sqrt{a^2 + \nu^2}\right)}{2(\nu + \sqrt{a^2 + \nu^2})(\nu + 1 + \sqrt{a^2 + (\nu+1)^2})}\right)^2$$

and combining these bounds proves the Lemma. □

Random variate generation

Random variate generators can be designed based upon which functions are available. In our case, three non-standard functions are involved, Γ (in the computation of p_n), $I_\nu(a)$ (in the computation of p_n), and $R_\nu(a)$ (in the computation of μ and σ^2). The gamma function is in most standard libraries, so we assume that it is available at unit cost. See pages 489–493 of Devroye (1986) on how to avoid computing the gamma function in rejection algorithms. So, we will present three rejection algorithms, one in which all three functions above are available, one in which only Γ and R_ν are available, and one in which only Γ is available.

We begin with rejection based on the bound of Lemma 6. Suppose that $p_{m+k} \leq g(x)$ for all $k - 1/2 \leq x \leq k + 1/2$ and all $x \in \mathbb{R}$, where g is a nonnegative integrable function (hence, proportional to a density). Then, a random variate with probability vector $\{p_n\}$ can be generated as follows:

```
repeat
    generate a uniform [0,1] random variate U.
    generate Y with density proportional to g.
    X ← round(Y).
until Ug(Y) ≤ p_{m+X}.
return m + X.
```

This algorithm will be used here with

$$g(x) \stackrel{\text{def}}{=} \min\left(p_m \ , \ p_m e^{1 - p_m(|x| - 1/2)}\right) .$$

The validity of g as a dominating function follows from Lemma 6. Observe that g is a mixture of a rectangular function on $[-w/p_m, w/p_m]$ (of integral $2w$ where $w = 1 + p_m/2$) and two antisymmetric exponential tails outside $[-w/p_m, w/p_m]$ (of integral 2). When g is used in the rejection algorithm, the rejection constant (or, expected number of iterations before halting) is $2w + 2 \ = \ 4 + p_m$ Devroye, 1987). We summarize as follows:

```
w ← 1 + p_m/2 (computed once)
repeat
    generate iid uniform [0,1] random variates U, W and a random sign S.
    if U ≤ w/(1+w) then Y ← Vw/p_m (where V is uniform [0,1])
                   else Y ← (w+E)/p_m (where E is exponential)
    X ← S round(Y).
until W min(1, e^{w - p_m Y}) ≤ p_{m+X}/p_m.
return m + X.
```

Note that p_m is needed once, and that p_{m+X}/p_m does not require the evaluation of any Bessel function. Only the Γ function is needed.

The Bessel function $I_\nu(a)$ can be computed in various ways, but none is efficient. It could be based on a numerical approximation of this integral, given in Abramowitz and Stegun (1965, p. 376):

$$I_\nu(a) = \frac{(a/2)^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)} \int_{-1}^{1} (1-t^2)^{\nu-1/2} e^{at}\, dt \ .$$

It could also be based on the defining series. However, none is satisfactory. Luckily, the computation of $I_\nu(a)$ can be avoided altogether. Based on the last inequality of Lemma 6, we obtain the following rejection algorithm, with

$$g(x) \stackrel{\text{def}}{=} \min\left(p_m \ , \ p_m e^{1-q(|x|-1/2)}\right) \ ,$$

where

$$q = \min\left(\frac{1}{\sigma\sqrt{648}}, \frac{1}{3}\right)$$

and $\sigma^2 = \chi^2 + (m-\mu)^2$ is as in Lemma 2. Note that σ and q can be computed using the function R_ν only. That function can be written as a continued fraction (Amos, 1974):

$$R_\nu(a) = \frac{1}{2(\nu+1)/a+} \frac{1}{2(\nu+2)/a+} \frac{1}{2(\nu+3)/a+\cdots}$$

which is a fast convergent and stable way of computation. Observe that g is a mixture of a rectangular function on $[-(1/2+1/q), (1/2+1/q)]$ (of integral $p_m + 2p_m/q$) and two antisymmetric exponential tails outside $[-(1/2+1/q), (1/2+1/q)]$ (of integral p_m/q each). When g is used in the rejection algorithm, the rejection constant (or, expected number of iterations before halting) is $p_m + 4p_m/q$. We summarize as follows:

```
q ← min(1/(σ√648), 1/3)
repeat
    generate iid uniform [0,1] random variates U,W and a random sign S.
    if U ≤ (1+2/q)/(1+4/q) then Y ← V(1/2+1/q) (where V is uniform [0,1])
                               else Y ← 1/2+1/q+E/q (where E is exponential)
    X ← S round(Y).
until W min(1, e^{1+q/2−qY}) ≤ p_{m+X}/p_m.
return m+X.
```

That this algorithm takes uniformly bounded time (if R_ν is computable at unit cost uniformly over all parameter values and arguments) follows from the fact that $p_m + 4p_m/q$ is uniformly bounded. This in turn follows from the fact that $p_m\sigma \le \sqrt{28+4e}$ for all discrete log-concave distributions (Lemma 7).

We finally turn to an algorithm that avoids even R_ν. Using Lemma 8, we see that

$$q \ge q^* \stackrel{\text{def}}{=} \min\left(\frac{1}{Q\sqrt{648}}, \frac{1}{3}\right)$$

where Q is as in Lemma 8. The computation of Q involves only addition, multiplication, division, and square root. It is clear then that we may use rejection with

$$g(x) \stackrel{\text{def}}{=} \min\left(p_m\ ,\ p_m e^{1-q^*(|x|-1/2)}\right)\ .$$

The last algorithm shown above, with q replaced by q^*, is valid too. The expected number of iterations before halting is $p_m + 4p_m/q^*$.

The double Poisson algorithm

Consider ν integer. We may generate independent pairs (X, Y) of Poisson $(a/2)$ random variates until for the first time $X - Y = \nu$. At that point, the random variate Y is distributed as a Bessel (ν, a) random variate (Yuan and Kalbfleisch, 2000, p. 439). We will refer to this method as the double Poisson algorithm. Poisson variates can be generated in expected time uniformly bounded in the parameters (see Devroye (1986), Hörmann (1993, 1994), Stadlober (1990), and Ahrens and Dieter (1991)). If we set $\rho_n = \mathrm{P}\{X = n\} = e^{-a/2}(a/2)^n/n!$, then the expected number of iterations before halting is

$$\frac{1}{\mathrm{P}\{X-Y=\nu\}} = \frac{1}{\sum_{n=0}^{\infty}\rho_n\rho_{n+\nu}} = \frac{1}{e^{-a}I_\nu(a)}\ .$$

This is only acceptable for moderate values of a and ν. On the other hand, given a fast Poisson source, the method may be surprisingly efficient in practice as long as ν is of the order of or smaller than $\sqrt{a}$.

Generator for the von Mises distribution

The von Mises distribution has density function

$$\frac{e^{\kappa \cos\theta}}{2\pi I_0(\kappa)} \ , \ |\theta| \le \pi \ ,$$

where $\kappa > 0$ is a parameter. A uniformly fast algorithm for this distribution was derived by Best and Fisher (1979), with alternate methods proposed later by Dagpunar (1990), Barabesi (1993) and Wood (1994). However, Yuan and Kalbfleisch (2000) point out that a von Mises random variable can be generated in yet another way:

```
generate X ← Bessel (0,κ).
generate B ← beta (X+1/2,1/2).
generate S, a random sign.
generate U uniform on [0,1].
if U < 1/(1+exp(−2κ√B))
     then return θ = S arccos(√B)
     else return θ = S arccos(−√B)
```

The only difficulty with this algorithm is that it requires a Bessel and a beta random variate. Note however that for $a \ge 1/2$, a beta $(a, 1/2)$ random variate can be obtained in one line of code as

$$1 - \left(1 - U_1^{\frac{2}{2a-1}}\right)\cos^2(2\pi U_2)$$

where U_1, U_2 are independent uniform $[0,1]$ random variables (Devroye, 1996, based in part upon a formula of Ulrich, 1984).

Generator for the randomized gamma distribution

A randomized gamma distribution of the second type has three positive parameters, a, c, s. It is the distribution of sG_{a+X+2Y} where X is Poisson $(c/2s)$, and Y is Bessel $(a-1, c/2s)$ and independent of X, and G_u denotes a gamma random variate with parameter u, i.e., a random variate with density $x^{u-1}e^{-x}/\Gamma(u)$ on the positive halfline. The density of the randomized gamma distribution is proportional to

$$e^{-sx}\left(I_{a-1}(\sqrt{cx})\right)^2 \ , \ x > 0$$

(Yuan and Kalbfleisch, 2000). Clearly, we can generate this in constant average time by using uniformly fast Poisson, gamma and Bessel generators. Uniformly fast gamma generators are described in Cheng and Feast (1980) and Devroye (1996) and the references found there. For more recent work, see also the methods in Ahrens (1995), Leydold (2000), Hörmann and Leydold (2000), or Evans and Swartz (1996).

Generator for the standard squared Bessel bridge process

This process on $[0,1]$, denoted by $\xi(t)$, conditional on $\xi(0)=a$, $\xi(1)=b$, and with parameter $\nu>-1$, is studied by Pitman and Yor (1982). If we know that $\xi(s)=x$, then $\xi(t)$ for $0\le s<t\le 1$ can be obtained as follows (Yuan and Kalbfleisch, 2000): let Y be Bessel $(\nu, \sqrt{bx}/(1-s))$, and let Z be independent of Y and Poisson (λ) where

$$\lambda = \frac{1}{2(1-s)}\left(\frac{(1-t)x}{(t-s)}+\frac{(t-s)b}{1-t}\right) .$$

Then return

$$\xi(t) = \frac{1-s}{2(t-s)(1-t)} G_{\nu+Y+2Z+1} .$$

By using recursively finer and finer partitions, one can fill the entire interval $[0,1]$ with realizations. The expected cost is linear in the number of points generated provided that uniformly fast gamma and Bessel generators are available.

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