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# EXACT CONVERGENCE RATE IN THE LIMIT THEOREMS OF ERDŐS–RÉNYI AND SHEPP

By Paul Deheuvels, Luc Devroye, and James Lynch

*Université Paris VI, McGill University, and the Pennsylvania State University*

The original Erdős–Rényi theorem states that $U_n/(\alpha k) \to 1$ almost surely for a large class of distributions, where $U_n = \sup_{0 \le i \le n-k}(S_{i+k} - S_i)$, $S_i = X_1 + \cdots + X_i$ is a partial sum of i.i.d. random variables, $k = \kappa(n) = [c \log n]$, $c > 0$, and $\alpha > 0$ is a number depending only upon $c$ and the distribution of $X_1$. We prove that the lim sup and the lim inf of $(U_n - \alpha k)/\log k$ are almost surely equal to $(2t^*)^{-1}$ and $-(2t^*)^{-1}$, respectively, where $t^*$ is another positive number depending only upon $c$ and the distribution of $X_1$. The same limits are obtained for the random variable $T_n = \sup_{1 \le i \le n}(S_{i+\kappa(i)} - S_i)$ studied by Shepp.

**1. Introduction.** We are concerned with the asymptotic behavior of

$$U_n = \sup_{0 \le i \le n-k} \{S_{i+k} - S_i\},$$

for $k = [c \log n]$, where $c > 0$, $S_0 = 0$, $S_i = X_1 + \cdots + X_i$, and $X_1, X_2, \ldots$ are independent, identically distributed random variables having moment generating function $\phi(t)$ and satisfying the conditions

(A) $E(X_1) = 0$;
(B) $X_1$ is nondegenerate, i.e., $P(X_1 = x) < 1$ for all $x$;
(C) $t_0 = \sup\{t;\ \phi(t) = E(e^{tX_1}) < \infty\} > 0$.

If $c$ is related to $\alpha$ via the equation

$$\exp\left(-\frac{1}{c}\right) = \inf_t \phi(t) e^{-t\alpha},$$

Erdős and Rényi (1970) proved that, for any $\alpha \in \{\phi'(t)/\phi(t),\ 0 < t < t_0\}$,

$$\lim_{n \to \infty} \frac{U_n}{\alpha k} = 1 \quad \text{almost surely.}$$

Earlier, Shepp (1964) had obtained a related theorem under the same conditions by showing that

$$\lim_{n \to \infty} \frac{T_n}{\alpha k} = 1 \quad \text{almost surely,}$$

for

$$T_n = \sup_{1 \le i \le n} \{S_{i+\kappa(i)} - S_i\},$$

where $\kappa(i) = [c \log i]$.

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These fundamental results were followed by a flurry of refinements and extensions. Among the refinements, we cite the work of S. Csörgő (1979) and of M. Csörgő and Steinebach (1981), the latter two of whom proved that

$$\frac{U_n}{\alpha k} = 1 + o(k^{-1/2}) \quad \text{almost surely.}$$

In this paper we show that the $o(k^{-1/2})$ term can be replaced by $O(k^{-1}\log k)$ and that this replacement is the best we can achieve. We also prove that the almost sure behavior of $T_n$ is identical to that of $U_n$.

Before stating our results in detail, we need to specify the range of values of $c$ and $\alpha$ that will be covered by our theorems. This will be done in Section 2. Section 3 presents a large deviation estimate applicable to our problem. Section 4 contains the proof of our main theorems.

**2. Properties of the moment generating function.** Let $X = X_1$ be a random variable satisfying conditions (ABC) and define

$$m(t) = \frac{\phi'(t)}{\phi(t)} = \frac{E(Xe^{tX})}{E(e^{tX})}.$$

Then $m(0) = 0$ and $m(\cdot)$ is strictly increasing on $[0, t_0)$ and continuously differentiable on $(0, t_0)$. Define further

$$A = \lim_{t \uparrow t_0} m(t) \quad \text{and} \quad c_0 = 1\bigg/\int_0^{t_0} tm'(t)\,dt.$$

Throughout, we will consider only the interval $[0, t_0)$ for $t$, where $0 < t_0 \le \infty$. Let $c = c(\alpha)$ and $\rho = \rho(\alpha)$ be defined by

$$\rho = \exp\left(-\frac{1}{c}\right) = \inf_t \phi(t)e^{-t\alpha}.$$

THEOREM 1. (1) *For any* $t \in (0, t_0)$, $m(t) \in (0, A)$. *Conversely, for any* $\alpha \in (0, A)$, *there exists a unique* $t^* = t^*(\alpha) \in (0, t_0)$ *such that* $m(t^*) = \alpha$;

(2) *For any* $\alpha \in (0, A)$,

$$\rho = \exp\left(-\frac{1}{c}\right) = \phi(t^*)e^{-t^*\alpha} \quad \text{and} \quad c \in (c_0, \infty);$$

(3) *For any* $c \in (c_0, \infty)$, *there exists a unique* $\alpha \in (0, A)$ *such that* $c = c(\alpha)$.

PROOF. First, on $(0, t_0)$, $\phi$ is continuously infinitely differentiable. Next, $\phi(0) = 1$ and $\phi$ is nondecreasing on $[0, t_0)$. This follows from Gurland's inequality [see Gurland (1967)] and the inequalities, for $s \ge t$,

$$E(e^{sX}) = E\left(e^{tX}(e^{tX})^{(s-t)/t}\right) \ge E(e^{tX})E(e^{(s-t)X}) \ge E(e^{tX}).$$

Furthermore, $\phi$ is convex because $\phi''(t) = E(X^2e^{tX}) \ge 0$.

The function $m$ is continuous and strictly increasing on $[0, t_0)$. This follows from the Cauchy–Schwarz inequality $E^2(Xe^{tX}) \le E(X^2e^{tX})E(e^{tX})$, which implies that $m'(t) = (\phi''(t)\phi(t) - \phi'^2(t))/\phi^2(t) \ge 0$. This last inequality has to be strict by (B).

We have $m(0) = \lim_{t\downarrow 0} m(t) = \phi'(0)/\phi(0) = E(X) = 0$. On the other hand, $A = \lim_{t\uparrow t_0} m(t) \le \operatorname{ess\,sup} X$. Here, equality occurs when $\operatorname{ess\,sup} X < \infty$, or when $t_0 = \infty$, $\operatorname{ess\,sup} X = \infty$, and we have an inequality in the other cases [see Petrov (1965), p. 288].

Consider now the equation $m(t) = \alpha$, and its solution $t^* = t^*(\alpha)$. For all $0 \le \alpha < A$, there is a unique solution in the range $0 \le t < t_0$. Conversely, as $t$ takes all values in $[0, t_0)$, $m(t)$ takes all values in $[0, A)$.

Next, $\log\phi(t) - t\alpha$ has first derivative $m(t) - \alpha$ and strictly positive second derivative $m'(t)$ on $[0, t_0)$. Thus it has a unique minimum on $[0, t_0)$ as the solution of the equation $m(t) = \alpha$. This proves (1), and allows us to write

$$\log\rho(\alpha) = \log\phi(t^*) - \alpha t^* = -\frac{1}{c}.$$

Since $m(t^*(\alpha)) = \alpha$, it follows that $t^{*\prime}(\alpha)m'(t^*(\alpha)) = 1$, and that

$$(\log\rho(\alpha))' = -t^*(\alpha), \qquad 0 \le \alpha < A.$$

Noting that $\rho(0) = \inf_t \phi(t) = \phi(0) = 1$, it follows that

$$\frac{1}{c} = -\int_0^\alpha (\log\rho(\theta))'\,d\theta = \int_0^\alpha t^*(\theta)\,d\theta = \int_0^{t^*(\alpha)} tm'(t)\,dt.$$

Clearly, $c$ is a continuous function of $\alpha \in (0, A)$, strictly increasing in $\alpha$, with $\lim_{\alpha\downarrow 0} c = \infty$, and $\lim_{\alpha\uparrow A} c = c_0 = 1/\int_0^{t_0} tm'(t)\,dt$. Thus, for every value $\alpha \in (0, A)$, there exists a unique value $c \in (c_0, \infty)$ and vice versa. This completes the proof of Theorem 1.

REMARK 1. In the sequel we shall make use of the fact that $\sigma^2(t) = m'(t) > 0$ on $(0, t_0)$. The function $\sigma$ is continuous on $[0, t_0)$, such that

$$\sigma^2(0) = \lim_{t\downarrow 0} \sigma^2(t) = E(X^2) \le \infty.$$

REMARK 2. We shall need also the fact that $(1/t)\log\phi(t)$ is strictly increasing in $t$. This last result is a consequence of Jensen's inequality, for $s > t$,

$$E^{1/t}(e^{tX}) \le E^{1/s}(e^{sX}),$$

with equality if and only $X$ is constant, which is not allowed by (B).

THEOREM 2. $c_0 = 0$ *in all cases except the following two*:

(i) $A < \infty$, $t_0 < \infty$. (*This covers a class of distributions with* $\operatorname{ess\,sup} X_1 = \infty$.) *In that case*, $c_0 = 1/(At_0 - \log\phi(t_0))$;

(ii) $A < \infty$, $t_0 = \infty$. (*This occurs if and only if* $\operatorname{ess\,sup} X_1 < \infty$.) *In that case, we have* $A = \operatorname{ess\,sup} X_1$, $P(X_1 = A) > 0$, *and* $c_0 = -1/\log P(X_1 = A)$.

PROOF. It is easy to show that if $\operatorname{ess\,sup} X < \infty$, then $t_0 = \infty$ and $A < \infty$. Conversely, by Petrov (1965), if $t_0 = \infty$ and $\operatorname{ess\,sup} X = \infty$, then $A = \infty$.

It follows that $\operatorname{ess\,sup} X < \infty$ if and only if $A < \infty$ and $t_0 = \infty$. In that case, we have $A = \operatorname{ess\,sup} X$.

Let us now characterize the distributions for which $c_0 = 1/\int_0^A t^*(\alpha)\,d\alpha = 0$. Since $t^*$ is an increasing function of $\alpha$, we see that $c_0 = 0$ if $A = \infty$. Let us assume that $A < \infty$. There are then two cases:

(i) $t_0 < \infty$. In that case, $\int_0^A t^*(\alpha)\,d\alpha \le At_0 < \infty$ and hence $c_0 \neq 0$. We have here

$$\frac{1}{c_0} = -\lim_{\alpha \uparrow A} \log \rho(\alpha) = \lim_{\alpha \uparrow A} (\alpha t^* - \log \phi(t^*)) = At_0 - \log \phi(t_0).$$

(ii) $t_0 = \infty$. By the remarks above, we must have then $A = \operatorname{ess\,sup} X < \infty$. In that case, we have $c_0 < \infty$. Furthermore,

$$\int_0^A t^*(\alpha)\,d\alpha = \int_0^\infty tm'(t)\,dt$$

$$= \int_0^\infty (A - m(t))\,dt = \int_0^\infty \frac{E((A - X)e^{tX})}{E(e^{tX})}\,dt = -\int_0^\infty \frac{E(Ye^{tY})}{E(e^{tY})}\,dt,$$

where $Y = X - A$.

Put $\zeta(t) = E(e^{tY})$. We have

$$\int_0^A t(\alpha)\,d\alpha = -\int_0^\infty \frac{\zeta'(t)}{\zeta(t)}\,dt = -\lim_{t \uparrow \infty} \log \zeta(t)$$

$$= -\log P(Y = 0) = -\log P(X = A).$$

Here, we have used the fact that by the dominated convergence theorem, $E(e^{tY}) \to P(Y = 0)$ as $t \uparrow \infty$. This proves Theorem 2.

REMARK 3. A number of authors have apparently ignored the fact that there exist distributions for which $A \neq \operatorname{ess\,sup} X_1$ and yet fulfill condition (i) of Theorem 2. By taking a density decreasing as $e^{-x}x^{-3}$, we get $A < \operatorname{ess\,sup} X_1 = \infty$, and $t_0 < \infty$, as sought.

**3. Application of Petrov's large-deviation theorem.** In this section, we use the hypotheses and notation of Sections 1 and 2.

THEOREM 3 (Petrov, 1965).

$$P(S_n \ge n\alpha) \sim \frac{\psi(t^*)}{\sqrt{n}} \exp\left(-\frac{n}{c}\right) = \frac{\psi(t^*)}{\sqrt{n}} \exp(n(\log \phi(t^*) - t^*\alpha)),$$

*uniformly for* $\alpha \in [\varepsilon, \min(A - \varepsilon, 1/\varepsilon)]$, *where* $\varepsilon > 0$ *is arbitrary, and* $\psi(t^*) > 0$ *is a finite number depending upon* $t^*$ *and the distribution of* $X_1$ *only.*

*For nonlattice distributions, one can take* $\psi(t^*) = (t^*\sigma(t^*)\sqrt{2\pi})^{-1}$, *while for lattice distributions with span* $H$, *one can take*

$$\psi(t^*) = \frac{H}{1 - e^{-Ht^*}} \cdot \frac{1}{\sigma(t^*)\sqrt{2\pi}}.$$

REMARK 4. Cramér (1938) proved a similar result for more restricted classes of random variables and Bahadur and Ranga Rao (1960) obtained another result that comes close to Theorem 3 [see Nagaev (1979) for a general discussion of large deviation results].

We will repeatedly use the following corollaries of Theorem 3.

COROLLARY 1. *Let* $\alpha \in (0, A)$ *and let* $y_n$ *be a sequence of numbers satisfying* $ny_n^2 \to 0$ *as* $n \to \infty$. *Then, uniformly over all sequences* $z_n$ *with* $|z_n| \le |y_n|$, *we have*

$$P(S_n \ge n(\alpha + z_n)) \sim \frac{\psi(t^*)}{\sqrt{n}} \exp\left(-\frac{n}{c}\right) \exp(-nz_n t^*).$$

PROOF. The proof is based upon Theorem 3 jointly with the following observations taken from Section 2: $t^* = t^*(\alpha)$ is a continuous function of $\alpha$, and thus $\psi(t^*)$ is a continuous function of $\alpha$ too. The derivative of $-(1/c) = \log \rho(\alpha)$ with respect to $\alpha$ is $-t^*$.

COROLLARY 2. *For all* $\varepsilon \in \mathbb{R}$, $\alpha \in (0, A)$, *we have*

$$\frac{\psi(t^*) + o(1)}{k^{1/2 + (\pm 1/2 + \varepsilon)}} \le nP\left(S_k \ge \alpha k + \left(\pm \frac{1}{2} + \varepsilon\right)\frac{\log k}{t^*}\right) \le \frac{e^{1/c}\psi(t^*) + o(1)}{k^{1/2 + (\pm 1/2 + \varepsilon)}}.$$

PROOF. It follows directly from Corollary 1 and the observation that $e^{(k+1)/c} \ge n \ge e^{(k/c)}$.

**4. The main theorems.** In the remainder of this section, we will need the following increasing sequence of integers:

$$n_j = \inf\{n; [c \log n] = j\}.$$

It is clear that $k = \kappa(n) = j$ for $n_j \le n < n_{j+1}$.

LEMMA 1.

(i) $$\limsup_{n \to \infty} (U_n - \alpha k)/\log k \le \frac{1}{2t^*} \quad \textit{almost surely};$$

(ii) $$\limsup_{n \to \infty} (T_n - \alpha k)/\log k \le \frac{1}{2t^*} \quad \textit{almost surely}.$$

PROOF. (i) For $n_j \le n < n_{j+1}$, we know that $k = j$, and $U_n \le U_{n_{j+1}-1}$. Thus, for $\varepsilon > 0$,

$$P\left(U_n \ge \alpha k + \left(\frac{1}{2} + \varepsilon\right)\frac{\log k}{t^*} \text{ i.o.}\right) \le P\left(U_{n_{j+1}-1} \ge \alpha j + \left(\frac{1}{2} + \varepsilon\right)\frac{\log j}{t^*} \text{ i.o. (in } j)\right).$$

By Corollary 2, since $j = [c \log(n_{j+1} - 1)]$,

$$P\left(U_{n_{j+1}-1} \ge \alpha j + \left(\frac{1}{2} + \varepsilon\right)\frac{\log j}{t^*}\right) \le n_{j+1} P\left(S_j \ge \alpha j + \left(\frac{1}{2} + \varepsilon\right)\frac{\log j}{t^*}\right)$$
$$= O(j^{-1-\varepsilon}),$$

which is summable in $j$. The result follows by Borel–Cantelli.

(ii) Likewise, for $n_j \le n < n_{j+1}$, we have $T_n \le T_{n_{j+1}-1} = \sup_{0 \le i \le j} \Lambda_i$, where $\Lambda_i = \sup_{n_i \le l < n_{i+1}} \{S_{l+\kappa(l)} - S_l\}$. It follows that, for any $\varepsilon > 0$,

$$P\left(\Lambda_j \ge \alpha j + \left(\frac{1}{2} + \varepsilon\right)\frac{\log j}{t^*} \text{ i.o.}\right) = 0 \Rightarrow P\left(T_n \ge \alpha k + \left(\frac{1}{2} + \varepsilon\right)\frac{\log k}{t^*} \text{ i.o.}\right) = 0.$$

By Corollary 2,

$$P\left(\Lambda_j \ge \alpha j + \left(\frac{1}{2} + \varepsilon\right)\frac{\log j}{t^*}\right) \le (n_{j+1} - n_j) P\left(S_j \ge \alpha j + \left(\frac{1}{2} + \varepsilon\right)\frac{\log j}{t^*}\right)$$
$$= O(j^{-1-\varepsilon}),$$

which is summable in $j$. The result follows by Borel–Cantelli.

LEMMA 2.

(i) *For any* $\varepsilon > 0$, $P\left((U_n - \alpha k)/\log k \le -\dfrac{1}{2t^*} + \varepsilon\right) \to 1$;

(ii) *For any* $\varepsilon > 0$, $P\left((T_n - \alpha k)/\log k \le -\dfrac{1}{2t^*} + \varepsilon\right) \to 1$.

PROOF. (i) By Corollary 2, we have, for any $\varepsilon > 0$,

$$P\left((U_n - \alpha k)/\log k \ge \frac{-1}{2t^*} + \varepsilon\right)$$
$$\le nP\left(S_k \ge \alpha k + \left(-\frac{1}{2} + \varepsilon t^*\right)\frac{\log k}{t^*}\right) \to 0, \quad \text{hence result.}$$

(ii) For $n_j \le n < n_{j+1}$, we have, using $\varepsilon/t^*$ in place of $\varepsilon$,

$$P\left((T_n - \alpha k)/\log k \ge \frac{-1}{2t^*} + \frac{\varepsilon}{t^*}\right) \le P\left((T_{n_{j+1}-1} - \alpha j)/\log j \ge \frac{-1}{2t^*} + \frac{\varepsilon}{t^*}\right).$$

Proceeding as in the proof of Lemma 1, let $\Theta_j = T_{n_{j+1}-1} = \sup_{0 \le i \le j} \Lambda_i$. By Lemma 1(ii), for any $\varepsilon > 0$, $P(\Theta_j \ge \alpha j + (1/2 + \varepsilon)((\log j)/t^*)) \to 0$. It follows by change of index from $j$ to $j - [j^{\varepsilon/2}]$ that

$$P\left(\Theta_{j-[j^{\varepsilon/2}]} \ge \alpha j + \left(-\frac{1}{2} + \varepsilon\right)\frac{\log j}{t^*}\right) \to 0.$$

Finally by Corollary 2, we have

$$P\left(\sup_{j-[j^{\varepsilon/2}] < i \le j} \Lambda_i \ge \alpha j + \left(-\frac{1}{2} + \varepsilon\right)\frac{\log j}{t^*}\right)$$
$$\le \sum_{j-[j^{\varepsilon/2}] < i \le j} P\left(\Lambda_i \ge \alpha i + \left(-\frac{1}{2} + \varepsilon\right)\frac{\log i}{t^*}\right) = O(j^{-\varepsilon/2}) \to 0.$$

This suffices for the proof.

COROLLARY 3.

(i) $$\liminf_{n\to\infty} (U_n - \alpha k)/\log k \le -\frac{1}{2t^*} \quad \textit{almost surely};$$

(ii) $$\liminf_{n\to\infty} (T_n - \alpha k)/\log k \le -\frac{1}{2t^*} \quad \textit{almost surely}.$$

PROOF. This statement uses the observations that for any sequence of events $A_n$ with $P(A_n) \to 1$, we must have $P(A_n \text{ i.o.}) = 1$. It follows directly from Lemma 2.

Having established the easy halves of our main results, we now turn to the more complicated parts. We shall make use of the following lemmas.

LEMMA 3 (Chung and Erdős, 1952). *For arbitrary sequences of events $A_1, \ldots, A_n$, we have*

$$P\left(\bigcup_{i=1}^n A_i\right) \ge \frac{\left(\sum_{i=1}^n P(A_i)\right)^2}{\sum_{i=1}^n P(A_i) + \sum_{i\ne j} P(A_i \cap A_j)}.$$

LEMMA 4. *Let $1 \le i \le k$ and let $S_i = X_1 + \cdots + X_i$, $S'_{k-i} = X_{i+1} + \cdots + X_k$, and $S''_i = X_{k+1} + \cdots + X_{k+i}$. Then for any $x$ and $q$ and for any $t \in (0, t^*)$, we have*

$$P(S_i + S'_{k-i} \ge x,\ S'_{k-i} + S''_i \ge x) \le (\phi(t^*))^{k-i} e^{-t^*q} + P(S_k \ge x)(\phi(t))^i e^{-t(x-q)}.$$

PROOF OF LEMMA 4. Note first that $P(S_n \ge \alpha n + u) \le E(e^{t(S_n - n\alpha - u)}) = (\phi(t)e^{-t\alpha})^n e^{-tu}$. From there we get, for any $0 < t \le t^*$, by Jensen's inequality,

$$P(S_n \ge s) \le (\phi(t))^n e^{-ts} \quad \text{and} \quad P(S_n \ge s) \le (\phi(t^*))^n e^{-t^*s}.$$

Next, we have

$$\begin{aligned} &P(S_i + S'_{k-i} \ge x,\ S'_{k-i} + S''_i \ge x) \\ &\quad \le P(S'_{k-i} \ge q) + P(S''_i \ge x - q) P(S_i + S'_{k-i} \ge x) \\ &\quad \le (\phi(t^*))^{k-i} e^{-t^*q} + (\phi(t))^i e^{-t(x-q)} P(S_k \ge x) \end{aligned}$$

as sought.

LEMMA 5. *For any $\varepsilon > 0$, we have with $\varepsilon' = \varepsilon/t^*$:*

(i) $$P\left((U_n - \alpha k)/\log k \ge \frac{-1}{2t^*} - \varepsilon'\right) \to 1;$$

(ii) $$P\left((T_n - \alpha k)/\log k \ge \frac{-1}{2t^*} - \varepsilon'\right) \to 1.$$

PROOF. (i) We will use Lemma 3 for events

$$A_i = \left\{ S_{i+k} - S_i \geq \alpha k + \left( -\frac{1}{2} - \varepsilon \right) \frac{\log k}{t^*} \right\}, \quad 0 \leq i \leq n - k.$$

Noting that $A_i$ and $A_j$ are independent when $|i-j| > k$ and that $P(A_i) = P(A_0)$, we have

$$\sum_{i=0}^{n-k} P(A_i) + \sum_{i \neq j} P(A_i \cap A_j)$$
$$= \left( \sum_{i=0}^{n-k} P(A_i)^2 \right) + \sum_{1 \leq |i-j| \leq k} \big( P(A_i \cap A_j) - P(A_i)P(A_j) \big)$$
$$+ \sum_{i=0}^{n-k} \big( P(A_i) - P^2(A_i) \big).$$

It follows that we need only show that, for $\varepsilon > 0$,

$$(*) \qquad nP(A_0) \to \infty,$$

and

$$(**) \qquad n \sum_{i=1}^{k} P(A_0 \cap A_i) = o\big( (nP(A_0))^2 \big).$$

By Corollary 2, $(\psi(t^*) + o(1))k^{\varepsilon} \leq nP(A_0) \leq (e^{1/c}\psi(t^*) + o(1))k^{\varepsilon}$. Hence $(*)$ is satisfied and $(**)$ amounts to

$$(**) \qquad n \sum_{i=1}^{k} P(A_0 \cap A_i) = o(k^{2\varepsilon}).$$

Put in the inequality of Lemma 4, $x = \alpha k + (-1/2 - \varepsilon)((\log k)/t^*)$, and $q = \alpha k - (i/t^*)\log \phi(t^*) + (2/t^*)\log i$. Let $t \in (0, t^*)$ be fixed. We have by Lemma 4 that

$$\begin{aligned} P(A_0 \cap A_i) &= P(S_i + S'_{k-i} \geq x,\ S'_{k-i} + S''_i \geq x) \\ &\leq (\phi(t^*))^{k-i} e^{-t^* q} + P(A_0)(\phi(t))^i e^{-t(x-q)} \\ &= i^{-2} e^{-k/c} + P(A_0) k^{(t/t^*)(\varepsilon + 1/2)} i^{2t/t^*} e^{-\theta i} \\ &\leq i^{-2} e^{-k/c} + P(A_0) k^{\varepsilon + 5/2} e^{-\theta i}, \end{aligned}$$

where $\theta = t((1/t^*)\log \phi(t^*) - (1/t)\log \phi(t)) > 0$ by Remark 2.

Let $l = [k^{\varepsilon/2}]$. We have

$$\begin{aligned} n \sum_{i=1}^{k} P(A_0 \cap A_i) &\leq 2nlP(A_0) + n \sum_{i=l+1}^{k-l} P(A_0 \cap A_i) \\ &\leq 2nlP(A_0) + e^{1/c} \sum_{i=l+1}^{\infty} i^{-2} + nP(A_0) k^{\varepsilon + 5/2} \frac{e^{-\theta l}}{1 - e^{-\theta}} \\ &= k^{\varepsilon/2} nP(A_0) O(1) = O(k^{3\varepsilon/2}), \end{aligned}$$

which suffices for $(**)$, since $nP(A_0) = O(k^{\varepsilon})$.

(ii) Let again $\Lambda_j = \sup_{n_j \le i < n_{j+1}}\{S_{i+j} - S_i\}$, and note for further use that $T_n \ge \Lambda_j$ for $n \ge n_{j+1}$. Let $N = n_{j+1} - n_j \le e^{(j+2)/c}$, $N \ge e^{j/c}(e^{1/c} - 1) - 1 \sim e^{j/c}(e^{1/c} - 1)$. We obtain by the same arguments as above, with $j$ replacing $k$ in the definition of $A_i$,

$$P\left(\Lambda_j \ge \alpha j + \left(-\frac{1}{2} - \varepsilon\right)\frac{\log j}{t^*}\right) = P\left(\bigcup_{i=1}^{N} A_i\right) \to 1,$$

whenever

$$(*) \qquad NP(A_0) \to \infty,$$

and

$$(**) \qquad N\sum_{i=1}^{j} P(A_0 \cap A_i) = o\big((NP(A_0))^2\big).$$

We have just proved that $(*)$ and $(**)$ hold if $\varepsilon > 0$. This implies that, as $j \to \infty$

$$P\left(T_n \ge \alpha j + \left(-\frac{1}{2} - \varepsilon\right)\frac{\log j}{t^*}\right) \ge P\left(\Lambda_j \ge \alpha j + \left(-\frac{1}{2} - \varepsilon\right)\frac{\log j}{t^*}\right) \to 1,$$

uniformly in $n_{j+1} \le n < n_{j+2}$. This, in turn, implies that

$$P\left(T_n \ge \alpha k + \left(-\frac{1}{2} - 2\varepsilon\right)\frac{\log k}{t^*}\right) \to 1,$$

which completes the proof of Lemma 5, $\varepsilon > 0$ being arbitrary.

The first of the two main theorems of this paper follows:

THEOREM 4. *For any $\alpha \in (0, A)$ or, equivalently, for any $c = c(\alpha) \in (c_0, \infty)$, we have*

$$\text{(i)} \qquad (U_n - \alpha k)/\log k \to -\frac{1}{2t^*} \quad \textit{in probability};$$

$$\text{(ii)} \qquad (T_n - \alpha k)/\log k \to -\frac{1}{2t^*} \quad \textit{in probability}.$$

PROOF. Combine Lemmas 2 and 5.

We proceed with the sequence of lemmas directed toward the second of our main theorems.

LEMMA 6.

$$\text{(i)} \qquad \limsup_{n\to\infty} (U_n - \alpha k)/\log k \ge \frac{1}{2t^*} \quad \textit{almost surely};$$

$$\text{(ii)} \qquad \limsup_{n\to\infty} (T_n - \alpha k)/\log k \ge \frac{1}{2t^*} \quad \textit{almost surely}.$$

PROOF. Let $R_j = \sup_{n_j \le n < n_{j+1} - j}\{S_{n+j} - S_n\}$. Since $k = \kappa(n) = j$ when $n_j \le n < n_{j+1}$, it is straightforward that $R_j \le \min\{T_m, U_m\}$, for $n_j \le m < n_{j+1}$. It follows that

$$P\left(T_m \ge \alpha k + \left(\frac{1}{2} - \varepsilon\right)\frac{\log k}{t^*} \text{ i.o.}\right) = P\left(U_m \ge \alpha k + \left(\frac{1}{2} - \varepsilon\right)\frac{\log k}{t^*} \text{ i.o.}\right) = 1$$

for any $\varepsilon > 0$, if

$$P\left(R_j \ge \alpha j + \left(\frac{1}{2} - \varepsilon\right)\frac{\log j}{t^*} \text{ i.o.}\right) = 1$$

for any $\varepsilon > 0$. Since the $R_j$'s form a sequence of independent random variables, the latter probability is one if and only if

$$\sum_j P\left(R_j \ge \alpha j + \left(\frac{1}{2} - \varepsilon\right)\frac{\log j}{t^*}\right) = \sum_j P_j = \infty.$$

Next, we note that $P_j = P(\bigcup_{i=1}^N A_i)$, where $A_i = \{S_{i+j} - S_i \ge x\}$, $x = \alpha j + (1/2 - \varepsilon)((\log j)/t^*)$, and $N = n_{j+1} - n_j - j \le e^{(j+2)/c}$, $N \ge e^{j/c}(e^{1/c} - 1) - j - 2 \sim e^{j/c}(e^{1/c} - 1)$.

By Lemma 3, it follows that

$$P_j = P\left(\bigcup_{i=1}^N A_i\right) \ge \frac{\left(NP(A_0)\right)^2}{NP(A_0) + \left(NP(A_0)\right)^2 + 2N\sum_{l=1}^{j} P(A_0 \cap A_l)}.$$

By Corollary 2 and our bounds on $N$, we note that, for some appropriate constants $c_1 > 0$ and $c_2 > 0$, we have

$$\left(c_1 + o(1)\right)j^{-1+\varepsilon} \le NP(A_0) \le \left(c_2 + o(1)\right)j^{-1+\varepsilon}.$$

Summarizing and simplifying, we obtain

$$P_j \ge (1 + o(1))\frac{c_1^2 j^{-2(1-\varepsilon)}}{c_2 j^{-(1-\varepsilon)} + 2N\sum_{l=1}^{j} P(A_0 \cap A_l)}.$$

We will show further on that there exists a constant $c_3 > 0$ such that

$$N\sum_{l=[j^{\varepsilon/2}]}^{j} P(A_0 \cap A_l) \le \left(c_3 + o(1)\right)j^{-1}.$$

But this is all we need, because

$$\begin{aligned} N\sum_{l=1}^{j} P(A_0 \cap A_l) &\le \left(c_3 + o(1)\right)j^{-1} + NP(A_0)j^{\varepsilon/2} \\ &\le \left(c_3 + o(1)\right)j^{-1} + \left(c_2 + o(1)\right)j^{-1+(3\varepsilon/2)} \\ &\sim c_2 j^{-1+3\varepsilon/2}. \end{aligned}$$

From there, we conclude that

$$P_j = P\left(\bigcup_{i=1}^{N} A_i\right) \geq (1 + o(1)) \frac{c_1^2}{2c_2} j^{-1+(\varepsilon/2)},$$

which is not summable in $j$ and we are done.

To bound $P(A_0 \cup A_l)$, we use Lemma 4 with $x = \alpha j + (1/2 - \varepsilon)((\log j)/t^*)$, $k = j$, $i = l$, $q = \alpha j - (l/t^*)\log \phi(t^*) + (1 + 2/\varepsilon)((\log l)/t^*)$, the constants $t$, $t^*$, and $\theta$ are as in the proof of Lemma 5. This gives

$$P(A_0 \cap A_l) \leq e^{-j/c} l^{-(1+(2/\varepsilon))} + P(A_0) e^{-\theta l} j^{(t/t^*)(\varepsilon - 1/2)} l^{(t/t^*)(1+2/\varepsilon)}.$$

If we sum over all $l \geq [j^{\varepsilon/2}]$ and multiply by $N \leq e^{(j+2)/c}$, we see that

$$\begin{aligned} N \sum_{l=[j^{\varepsilon/2}]}^{j} P(A_0 \cap A_l) &\leq e^{2/c} \Bigg\{ \left( [j^{\varepsilon/2}]^{2/\varepsilon} \right)^{-1} \\ &\qquad + (c_2 + o(1)) j^{-1+\varepsilon} \frac{\exp(-\theta j^{\varepsilon/2})}{1 - e^{-\theta}} j^{(t/t^*)(1/2+\varepsilon+2/\varepsilon)} \Bigg\} \\ &\leq (c_3 + o(1)) j^{-1}, \end{aligned}$$

for some positive constant $c_3$.

This concludes the proof of Lemma 6.

In view of Lemma 1, Corollary 3, and Lemma 6, only one piece of the puzzle is missing, i.e.,

LEMMA 7.

(i) $$\liminf_{n\to\infty} (U_n - \alpha k)/\log k \geq \frac{-1}{2t^*} \quad \textit{almost surely};$$

(ii) $$\liminf_{n\to\infty} (T_n - \alpha k)/\log k \geq \frac{-1}{2t^*} \quad \textit{almost surely}.$$

PROOF. For $n_j \leq n < n_{j+1}$, we know that $U_n \geq U_{n_j}$ and $T_n \geq T_{n_j}$. Thus, by the Borel–Cantelli lemma, we are done if we can show that, for all $\varepsilon > 0$,

$$\sum_{j=1}^{\infty} P\left( U_{n_j} < \alpha j - \left(\frac{1}{2} + \varepsilon\right) \frac{\log j}{t^*} \right) < \infty$$

$$\text{and} \quad \sum_{j=1}^{\infty} P\left( T_{n_j} < \alpha j - \left(\frac{1}{2} + \varepsilon\right) \frac{\log j}{t^*} \right) < \infty.$$

Consider the set $J_j$ of all integers of the form $r[j^{\varepsilon/2}]$, $r = 0, 1, 2, \ldots$. For integer $l$, let us also define the quantity, for each fixed $j$,

$$Q_l = \sup_{2lj \leq i < (2l+1)j;\ i \in J_j} \{S_{i+j} - S_i\}.$$

It is noteworthy that $Q_0, Q_1, Q_2, \dots$ are independent random variables for each $j$ and that

$$U_{n_j} \geq \sup_{0 \leq l \leq L} Q_l,$$

where $L$ is the largest integer such that $(2L+1)j - 1 \leq n_j - j$, i.e.,

$$L = \left[\frac{1}{2}\left(\frac{n_j - j + 1}{j} - 1\right)\right] \geq \frac{n_j - 2j}{2j} - 1 = \frac{n_j - 4j}{2j} \sim \frac{n_j}{2j}.$$

By all this, we have

$$P_j' = P\left(U_{n_j} < \alpha j - \left(\frac{1}{2} + \varepsilon\right)\frac{\log j}{t^*}\right) \leq \prod_{l=0}^{L} P\left(Q_l < \alpha j - \left(\frac{1}{2} + \varepsilon\right)\frac{\log j}{t^*}\right)$$
$$\leq \exp\left(-(1 + o(1))\frac{n_j}{2j} P\left(Q_0 \geq \alpha j - \left(\frac{1}{2} + \varepsilon\right)\frac{\log j}{t^*}\right)\right),$$

where we used the independence of the $Q_l$'s and the inequality $1 - u \leq e^{-u}$. Let $N_l$ be the number of indices $i$ in the intersection $J_j \cap \{2lj, \dots, (2l+1)j - 1\}$. This number $N_l$ satisfies, uniformly in $l$, $N_l \sim j^{1-\varepsilon/2}$ as $j \to \infty$ (and this is why we need only consider $Q_0$). Put, in the sequel, $N = N_0$.

By a simple Bonferroni inequality

$$P\left(Q_0 \geq \alpha j - \left(\frac{1}{2} + \varepsilon\right)\frac{\log j}{t^*}\right) \geq NP\left(S_j \geq \alpha j - \left(\frac{1}{2} + \varepsilon\right)\frac{\log j}{t^*}\right)$$
$$-2N \sum_{r=1}^{N-1} P\left(S_j \geq \alpha j - \left(\frac{1}{2} + \varepsilon\right)\frac{\log j}{t^*},\right.$$
$$\left. S_{j + r[j^{\varepsilon/2}]} - S_{r[j^{\varepsilon/2}]} \geq \alpha j - \left(\frac{1}{2} + \varepsilon\right)\frac{\log j}{t^*}\right).$$

By Corollary 2, the first term in this lower bound is larger than

$$(1 + o(1)) j^{1-\varepsilon/2} e^{-j/c} \psi(t^*) j^{\varepsilon} \sim \psi(t^*) j^{1+\varepsilon/2} e^{-j/c}.$$

We will show that the second term in the lower bound is $o$(first term), so that $P_j'$ is not greater than

$$\exp\left(-(1 + o(1))\frac{n_j}{2} j^{\varepsilon/2} \psi(t^*) e^{-j/c}\right) \leq \exp\left(-(1 + o(1))\frac{1}{2}\psi(t^*) j^{\varepsilon/2}\right),$$

which is summable in $j$ by the integral test. This proves (i).

Let $x = \alpha j - (1/2 + \varepsilon)((\log j)/t^*)$ and $m = r[j^{\varepsilon/2}]$. Then we need only show that

$$(*) \qquad \sum_{r=1}^{N-1} P(S_j \geq x, S_{j+m} - S_m \geq x) = O(j^{\varepsilon} e^{-j/c}).$$

To do so, we will once again use Lemma 4, with $t$, $t^*$, and $\theta$ defined as in the proof of Lemma 5, and with the formal replacements

$$k = j, \qquad i = m = r[j^{\varepsilon/2}], \qquad q = \alpha j - r[j^{\varepsilon/2}]\frac{1}{t^*}\log \phi(t^*) + \frac{2}{t^*}\log(r[j^{\varepsilon/2}]).$$

The $r$th term of $(*)$ is bounded from above by

$$e^{-j/c}\left(r\left[j^{\varepsilon/2}\right]\right)^{-2} + P(S_j \geq x)\, j^{(t/t^*)(1/2+\varepsilon)}\left(r\left[j^{\varepsilon/2}\right]\right)^{(2t/t^*)} e^{-\theta r[j^{\varepsilon/2}]},$$

which, taking into account that $r[j^{\varepsilon/2}] \leq j$, $\varepsilon < \frac{1}{2}$ (without loss of generality) and $t/t^* < 1$, is by Corollary 2 not greater that

$$e^{-j/c} r^{-2} j^{-\varepsilon}(1 + o(1)) + (e^{1/c}\psi(t^*) + o(1)) e^{-j/c} j^{3+\varepsilon} e^{-\theta r[j^{\varepsilon/2}]},$$

where the "$o(1)$" terms are uniform in $r \geq 1$.

Summing over $r$ gives the bound

$$(**)\quad (1 + o(1)) e^{-j/c}\left(\frac{\pi^2}{6j^{\varepsilon}} + \frac{e^{1/c}\psi^{\varepsilon}(t^*) j^{3+\varepsilon} e^{-\theta N[j^{\varepsilon/2}]}}{1 - \exp\left(-\theta\left[j^{\varepsilon/2}\right]\right)}\right) = o(e^{-j/c}) = o(j^{\varepsilon} e^{-j/c}),$$

as requested. This completes the proof of (i).

The proof of (ii) is based on the same arguments as the proof of (i), but with slight modifications. We first replace $Q_l$ by $Q_l'$, defined by

$$Q_l' = \sup_{2lj \leq i < (2l+1)j;\ i \in J_j} \left\{S_{i+\kappa(i)} - S_i\right\}.$$

We have evidently, for any $0 \leq M \leq L$,

$$T_{n_j} \geq \sup_{M \leq l \leq L} Q_l'.$$

Next, we choose $M = o(L)$ such that, for any $i \in \{2Mj, \ldots, (2L + 1)j\}$, we have

$$j - \frac{\log j}{t^*} o(1) \leq \kappa(i) \leq j,$$

where the "$o(1)$" term is uniform in $i$ as $j \to \infty$. We can take here $M = [L/\log\log j]$. We get then, by the same arguments as above

$$P_j'' = P\left(T_{n_j} < \alpha j - \left(\frac{1}{2} + \varepsilon\right)\frac{\log j}{t^*}\right) \leq \exp\left(-\sum_{l=M}^{L} P\left(Q_l' \geq \alpha j - \left(\frac{1}{2} + \varepsilon\right)\frac{\log j}{t^*}\right)\right).$$

By Bonferroni, we have

$$P\left(Q_l' \geq \alpha j - \left(\frac{1}{2} + \varepsilon\right)\frac{\log j}{t}\right) \geq \sum_{i \in I_l} P(S_{\kappa(i)} \geq x)$$
$$- \sum_{r \neq s \in I_l} P\left(S_{r+\kappa(r)} - S_r \geq x,\ S_{s+\kappa(s)} - S_s \geq x\right),$$

where $I_l = J_j \cap \{2lj, \ldots, (2l + 1)j - 1\}$.

By Corollaries 1 and 2 we have evidently, for $j$ large enough,

$$P(S_{\kappa(i)} \geq x) \geq (\psi(t^*) + o(1)) e^{-j/c} j^{3\varepsilon/4},$$

uniformly in $i \in I_l$, so that

$$P(S_{\kappa(i)} \geq x) \geq \frac{e^{1/c}\psi(t^*) + o(1)}{e^{j/c} j^{-\varepsilon/2}} \geq (1 + o(1)) P\left(S_j \geq \alpha j - \left(\frac{1}{2} + \frac{\varepsilon}{2}\right)\frac{\log j}{t^*}\right).$$

Likewise, by Lemma 4 and the argument we have just used to prove (i), we deduce from $(**)$ that

$$\sum_{r \neq s \in I_l} P\left(S_{r+\kappa(r)} - S_r \geq x,\, S_{s+\kappa(s)} - S_s \geq x\right) = o\left(\sum_{i \in I_l} P\left(S_{\kappa(i)} \geq x\right)\right).$$

It follows that

$$P_j'' \leq \exp\left(-(L-M)(1+o(1))P\left(Q_0 \geq \alpha j - \left(\frac{1}{2} + \frac{\varepsilon}{2}\right)\frac{\log j}{t^*}\right)\right).$$

Since $L - M = L(1 + o(1)) = (n_j/(2j))(1 + o(1))$, the expression above yields the same upper bound as was obtained for $P_j'$ with $\varepsilon/2$ replacing $\varepsilon$. But we have proved that $\sum_j P_j' < \infty$ for all $\varepsilon > 0$. This concludes the proof of Lemma 7.

The theorem that gives the exact convergence rate for the Erdős–Rényi and Shepp limit theorems has now been proved:

THEOREM 5. *For any $\alpha \in (0, A)$, or equivalently, for any $c = c(\alpha) \in (c_0, \infty)$, we have*

(i) $$\limsup_{n\to\infty} (U_n - \alpha k)/\log k = \frac{1}{2t^*} \quad \textit{almost surely};$$

(ii) $$\liminf_{n\to\infty} (U_n - \alpha k)/\log k = -\frac{1}{2t^*} \quad \textit{almost surely}.$$

*In statements* (i)–(ii), *$U_n$ can be replaced by $T_n$.*

PROOF. Combine Corollary 3 with Lemmas 1, 6, and 7.

REMARK 5. It can be seen that the methods we have used can be extended to the case where $k = \kappa(n)$ is a nondecreasing sequence such that $\kappa(n) - c\log n = o(\log\log n)$.

REMARK 6. What happens when $c \in (0, c_0)$, corresponding to cases (i)–(ii) of Theorem 2, will be discussed in forthcoming papers (Deheuvels, 1985 and Deheuvels–Devroye, 1983 and 1985). Related results concerning Erdős–Rényi laws for moving quantiles are to be found in Deheuvels and Steinebach (1984).

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PAUL DEHEUVELS
INSTITUT DE STATISTIQUE
LSTA UNIVERSITÉ PARIS VI
4 PLACE JUSSIEU
75230 PARIS CEDEX 05
FRANCE

LUC DEVROYE
SCHOOL OF COMPUTER SCIENCE
MCGILL UNIVERSITY
805 SHERBROOKE STREET WEST
MONTREAL, PQ
CANADA H3A 2K6

JAMES LYNCH
DEPARTMENT OF STATISTICS
THE PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK
PENNSYLVANIA 16802