ON SIMULATION AND PROPERTIES OF THE STABLE LAW

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Abstract. The stable distribution, in its many parametrizations, is central to many stochastic processes. Many random variables that occur in the study of Lévy processes are related to it. Good progress has been made recently for simulating various quantities related to the stable law. In this note, we survey exact random variate generators for these distributions. Many distributional identities are also reviewed.

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Introduction

The unilateral stable random variable S_α of parameter $\alpha \in (0,1)$ has Laplace transform

$$\mathbb{E}\left\{e^{-\lambda S_\alpha}\right\} = e^{-\lambda^\alpha}, \lambda \geq 0.$$

Its properties are well-known—see, e.g., Zolotarev (1986). A simple random variate generator for S_α has been suggested by Kanter (1975), who used an integral representation of Zolotarev (1966) (see Zolotarev (1986, p. 74)), which states that the distribution function of $S_\alpha^{\alpha/(1-\alpha)}$ is given by

$$\frac{1}{\pi}\int_0^\pi e^{-\frac{A(u)}{x}}\,du,$$

where A is Zolotarev's function:

$$A(u) \stackrel{\text{def}}{=} \left\{\frac{(\sin(\alpha u))^\alpha (\sin((1-\alpha)u)^{1-\alpha}}{\sin u}\right\}^{\frac{1}{1-\alpha}}.$$

Zolotarev's integral representation implies that

$$S_\alpha \stackrel{\mathcal{L}}{=} \left(\frac{A(U)}{E}\right)^{\frac{1-\alpha}{\alpha}},$$

where U is uniform on $[0,\pi]$ and E is exponential with mean one. Here $\stackrel{\mathcal{L}}{=}$ denotes equality in distribution. This is Kanter's method.

Since then, a similar generator has been proposed for all stable random variables by Chambers, Mallows and Stuck (1976), which was again based on Zolotarev's integral representation of stable distributions. However, clever combinations of unilateral stable random variables can be used to generate any stable random variable. The purpose of this survey is to convince the readers that the exact simulation of the stable distribution can be achieved using simple distributional properties of the stable family. We use this occasion to clarify the literature and to discuss random variate generation for distributions that are related to the stable law. These include the Mittag-Leffler distribution, polynomially and exponentially tilted unilateral stable distributions, the weakly stable distribution, and several distributions defined by Bertoin, Fujita, Roynette and Yor (2006). One of the principal tools is the Mellin transform.

The strictly stable distribution

The stable distribution has many definitions and parametrizations that may seem confusing at first. Zolotarev (1986) has forms called (A), (B), (M) and (C), all with different parameters. For the purpose of simulation and for the present discussion, it is important to single out the strictly stable distribution, Zolotarev's form (C) (1986, p. 17). It will become apparent that a thorough understanding of it is helpful. Other parametrizations can be dealt with by minor manipulation, often just a scale and translation transformation.

The classical parameters are the main shape parameter $\alpha \in (0,2]$, and the asymmetry parameter $\beta \in [-1,1]$. Zolotarev defines two related parameters,

$$\theta = \begin{cases} \beta & \text{if } \alpha \leq 1, \\ \beta\frac{\alpha-2}{\alpha} & \text{if } \alpha > 1, \end{cases}$$

and

$$\rho = \frac{1+\theta}{2}.$$

One can describe the strictly stable distribution uniquely using any pair (α,β), (α,θ) or (α,ρ). While the range of θ and ρ is $[-1,1]$ and $[0,1]$, respectively, this full range can only be attained when $\alpha \le 1$. For $\alpha > 1$, we have $\alpha\rho \le 1$ and $\alpha(1-\rho) \le 1$, and $|\theta| \le 2/\alpha - 1$. We will write $S_{\alpha,\beta}$, $S_{\alpha,\theta}$, or $S_{\alpha,\rho}$, depending upon the situation. When not explicitly stated, the (α,ρ) parametrization is understood. The characteristic function φ of $S_{\alpha,\theta}$ is defined by

$$\log\varphi(t) = -|t|^\alpha \exp\left(-\frac{i\pi\theta\alpha\,\mathrm{sign}\,t}{2}\right).$$

Equivalently, $S_{\alpha,\rho}$ is defined by

$$\log\varphi(t) = -(it)^\alpha \exp\left(-i\pi\rho\alpha\,\mathrm{sign}\,t\right).$$

Some special cases:

(i) The extreme strictly stable laws are obtained when $|\beta| = 1$. The unilateral stable law has parameters $\alpha < 1$ and $\beta = \theta = \rho = 1$. We will write S_α.

(ii) The symmetric strictly stable law corresponds to $\beta = \theta = 0$ and $\rho = 1/2$. We write $S_{\alpha,1/2}$. Its characteristic function is

$$\varphi(t) = e^{-|t|^\alpha}.$$

(iii) For $\alpha = 2$, $\theta = 0$, $\rho = 1/2$, $\beta \in [-1,1]$, the characteristic function of the strictly stable law is $\exp(-t^2)$, which is identical to that of $\sqrt{2}N$, N being standard gaussian. Thus,

$$S_{2,1/2} \stackrel{\mathcal{L}}{=} \sqrt{2}\,N.$$

(iv) The Cauchy law (with density $1/(\pi(1+x^2))$) has characteristic function $\exp(-|t|)$, and this corresponds to $\alpha = 1$ and $\beta = \theta = 0$, $\rho = 1/2$. Writing C for a Cauchy random variable, one notices that

$$C \stackrel{\mathcal{L}}{=} S_{1,1/2}.$$

Remark 1. A scale factor. Zolotarev's form (C) defines the strictly stable law via

$$\log\varphi(t) = -\lambda|t|^\alpha \exp\left(-\frac{i\pi\theta\alpha\,\mathrm{sign}\,t}{2}\right),$$

where $\lambda > 0$ is a scale parameter. It is easy to see that φ is the characteristic function of

$$\lambda^{1/\alpha} S_{\alpha,\theta}. \ \square$$

Remark 2. Zolotarev's form (B), weakly stable law. Zolotarev's form (B) includes the strictly stable law, and is in fact identical to his form (C) for $\alpha \neq 1$. The characteristic function is

$$\log \varphi(t) = \begin{cases} -|t|^\alpha \exp\left(-\frac{i\pi\theta\alpha \operatorname{sign} t}{2}\right), & \text{if } \alpha \neq 1, \\ -|t|\pi/2 - i\beta t \log|t|, & \text{if } \alpha = 1, \end{cases}$$

where $\beta \in [-1,1]$ in the case $\alpha = 1$. This law is referred to as the <u>weakly table law</u>. The random variable with parameters $\alpha = 1$ and $\beta \in [-1,1]$ is denoted by W_β. □

Let us use accents to denote independent copies of random variables. The symmetric difference

$$S_{\alpha,\rho} - S'_{\alpha,\rho}$$

has log characteristic function

$$\log \varphi(t) = -2|t|^\alpha \cos\left(\frac{\pi\theta\alpha}{2}\right).$$

This is distributed as

$$\left(2\cos\left(\frac{\pi\theta\alpha}{2}\right)\right)^{\frac{1}{\alpha}} S_{\alpha,1/2}.$$

For any constant $\delta \in [0,1]$,

$$\delta^{\frac{1}{\alpha}} S_{\alpha,\beta} + (1-\delta)^{\frac{1}{\alpha}} S'_{\alpha,\beta} \stackrel{\mathcal{L}}{=} S_{\alpha,\beta},$$

where $\stackrel{\mathcal{L}}{=}$ denotes equality in distribution. This is the remarkable property that uniquely characterizes the strictly stable family. Finally, we have a simple mirroring property:

$$S_{\alpha,\beta} \stackrel{\mathcal{L}}{=} -S_{\alpha,-\beta}, S_{\alpha,\rho} \stackrel{\mathcal{L}}{=} -S_{\alpha,1-\rho}, S_{\alpha,\theta} \stackrel{\mathcal{L}}{=} -S_{\alpha,-\theta}.$$

Because of this, the distributions of stable laws could just be studied on the positive halfline.

The shifted Cauchy distribution.

Writing C for a Cauchy random variable and U for a uniform $[0,1]$, it is easy to verify that

$$C \stackrel{\mathcal{L}}{=} S_{1,1/2} \stackrel{\mathcal{L}}{=} \tan(\pi U) \stackrel{\mathcal{L}}{=} \tan\left(\pi\left(U - \frac{1}{2}\right)\right).$$

The <u>shifted Cauchy</u> C_ρ, $\rho \in [0,1]$, is is defined by

$$C_\rho \stackrel{\text{def}}{=} S_{1,\rho} \stackrel{\mathcal{L}}{=} -\cos(\pi\rho) + \sin(\pi\rho)\, C \stackrel{\mathcal{L}}{=} \sin\left(\frac{\pi\theta}{2}\right) + \cos\left(\frac{\pi\theta}{2}\right) C.$$

In other words, the strictly stable laws with $\alpha = 1$ are all shifted Cauchy random variables. We note that $C_\rho \stackrel{\mathcal{L}}{=} -C_{1-\rho}$. Observe also the degenerate cases $S_{1,0} \equiv -1$ and $S_{1,1} \equiv 1$. Also, $C_{1-\rho} \stackrel{\mathcal{L}}{=} C_\rho + 2\cos(\pi\rho)$. Using $C \stackrel{\mathcal{L}}{=} 1/\tan(\pi U) \stackrel{\mathcal{L}}{=} \tan(\pi U)$, trigonometric manipulation leads to

$$-S_{1,\rho} \stackrel{\mathcal{L}}{=} \frac{\sin(\pi(U+\rho))}{\sin(\pi U)} \stackrel{\mathcal{L}}{=} \frac{\sin(\pi(U-\rho))}{\sin(\pi U)} \stackrel{\mathcal{L}}{=} \frac{\cos(\pi(U-\rho))}{\cos(\pi U)} \stackrel{\mathcal{L}}{=} \frac{\cos(\pi(U+\rho))}{\cos(\pi U)}.$$

For later reference, the density of C_ρ or $S_{1,\rho}$ is

$$\frac{\sin(\pi\rho)}{\pi} \frac{1}{x^2 + 2x\cos(\pi\rho) + 1}, x \in \mathrm{R},$$

and parametrized with θ, it is

$$\frac{\cos(\pi\theta/2)}{\pi} \frac{1}{x^2 - 2x\sin(\pi\theta/2) + 1}, x \in \mathbb{R}.$$

The distribution function of the latter is

$$\frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x - \sin(\pi\theta/2)}{\cos(\pi\theta/2)}\right),$$

which provides another explanation of the distributional identities above.

Remark 3. Weakly stable distribution. It is easy to see that the weakly stable random variable W_0 is distributed as $(\pi/2)C$. We will return to the weakly stable laws in a later section. □

If a random variable X_a for $|a| < 2$ has a density proportional to $1/(1 + x^2 + ax)$ on $\mathbb{R}$, then $1/X_a \stackrel{\mathcal{L}}{=} X_a$ and $-X_a \stackrel{\mathcal{L}}{=} X_{-a}$. If its support is the positive halfline, then we have $1/X_a \stackrel{\mathcal{L}}{=} X_a$. From this, we deduce that

$$C_\rho \stackrel{\mathcal{L}}{=} \frac{1}{C_\rho} \stackrel{\mathcal{L}}{=} -C_{1-\rho} \stackrel{\mathcal{L}}{=} -\frac{1}{C_{1-\rho}},$$

or,

$$S_{1,\rho} \stackrel{\mathcal{L}}{=} \frac{1}{S_{1,\rho}} \stackrel{\mathcal{L}}{=} -S_{1,1-\rho} \stackrel{\mathcal{L}}{=} -\frac{1}{S_{1,1-\rho}},$$

with $0 \le \rho \le 1$. It is also easy to see from the ratio of sine representation above that $\mathbb{P}\{S_{1,\rho} \ge 0\} = \rho$. If we define the random variable X_+ as X conditioned on $X \ge 0$, and X_- as X conditioned on $X \le 0$, and if B_ρ is Bernoulli (ρ), then

$$\begin{aligned} S_{1,\rho} &\stackrel{\mathcal{L}}{=} \begin{cases} (S_{1,\rho})_+ & \text{with probability } \rho, \\ (S_{1,\rho})_- & \text{with probability } 1-\rho \end{cases} \\ &\stackrel{\mathcal{L}}{=} \begin{cases} (S_{1,\rho})_+ & \text{with probability } \rho, \\ -(S_{1,1-\rho})_+ & \text{with probability } 1-\rho \end{cases} \\ &\stackrel{\mathcal{L}}{=} (S_{1,\rho})_+ B_\rho - (S_{1,1-\rho})_+ (1 - B_\rho). \end{aligned}$$

The multitude of relationships within the shifted Cauchy family explains many results in chapter 3 of Zolotarev (1986).

It is useful to introduce Lamperti's law (Lamperti, 1958), which is central in the study of occupation times of stochastic processes. A <u>Lamperti random variable</u> L_ρ of parameter $\rho \in (0, 1)$ has density

$$f(x) = \frac{\sin(\pi\rho)}{\pi\rho} \frac{1}{x^2 + 2x\cos(\pi\rho) + 1} x \ge 0.$$

Supported on the positive halfline, its tail decreases as $1/x^2$. In fact, $L_{1/2} \stackrel{\mathcal{L}}{=} |C|$, showing its relationship to the Cauchy law. Clearly, $L_\rho \stackrel{\mathcal{L}}{=} (C_\rho)_+ \stackrel{\mathcal{L}}{=} (S_{1,\rho})_+$. By earlier remarks,

$$L_\rho \stackrel{\mathcal{L}}{=} \frac{1}{L_\rho}.$$

A random variate can be generated by several methods (James (2010b), Zolotarev (1986, pages 83 and 198-199)), which are all based on the relationship with shifted Cauchy random variables. Consider the distribution function

$$F(x) = 1 - \frac{1}{\pi\rho} \arctan\left(\frac{\sin(\pi\rho)}{\cos(\pi\rho) + x}\right).$$

Taking the derivative of F with respect to x gives the Lamperti density. The inversion method thus shows that, with U uniform $[0, 1]$,

$$L_\rho \stackrel{\mathcal{L}}{=} \frac{\sin(\pi\rho)}{\tan((1-U)\pi\rho)} - \cos(\pi\rho) = \frac{\sin(U\pi\rho)}{\sin((1-U)\pi\rho)}.$$

Note from this that when $\rho \to 0$, $L_\rho \stackrel{\mathcal{L}}{\to} U/(1-U)$, a random variable with density $1/(x+1)^2, x \geq 0$. Furthermore, $L_1 \equiv 1$, a degenerate law, and $L_{1/2} \stackrel{\mathcal{L}}{=} \tan(\pi U/2) \stackrel{\mathcal{L}}{=} |C|$.

The Mellin transform.

The Mellin transform of a function f defined over the positive halfline is the complex-variable function $f^*(s)$ defined by the integral

$$f^*(s) = \int_0^\infty f(x) x^{s-1}\, dx.$$

See, e.g., Titschmarsh (1937) or Flajolet and Sedgewick (2009). When f is the density of a positive random variable X, then $f^*(s) = \mathbb{E}\{X^{s-1}\}$. Following Zolotarev (1957, 1959, 1981, 1986), it is convenient to shift the complex argument by one and to widen the definition to all random variables. Thus, for those s for which the integral exists, we define Mellin transform $M(s)$ by

$$M(s) = \mathbb{E}\left\{X^s 1_{[X>0]}\right\}.$$

The distribution of a generic X is uniquely determined by the Mellin transforms of $-X$ and X, or by the Mellin transforms of $|X|$ and X. For example, the Mellin transform of $S_{\alpha,\rho}$ (Zolotarev, 1981, p. 117) for all strictly stable random variables is

$$\frac{\sin(\pi\rho s)}{\sin(\pi s)} \frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)},$$

which is valid for $-1 < \Re(s) < \alpha$, where $\Re(s)$ denotes the real part of s.

Mellin transforms are heavily used in analytic combinatorics and in the study of harmonic sums (Flajolet and Sedgewick, 2009). They are also useful for determining distributional identities. For example, within the strictly stable family of distributions, there are quite a few fundamental identities on the composition and combination of member distributions. We refer to this collection as a "calculus" of family of distributions. Textbook examples include the gamma and beta distributions.

Chapters 2 and 3 of Zolotarev (1986) provide many distributional identities for the strictly stable family. Most of the relationships are immediate consequences of computations that involve Mellin transforms. For example, besides the Mellin transform of $S_{\alpha,\rho}$ shown above, Zolotarev gives the Mellin transforms of $|S_{\alpha,\rho}|$, i.e.,

$$\frac{\sin(\pi\rho s) + \sin(\pi(1-\rho)s)}{\sin(\pi s)} \frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)} = \frac{\cos(\pi\theta s/2)}{\cos(\pi s/2)} \frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)},$$

and of $(S_{\alpha,\rho})_+$, which is simply $1/\rho$ times that of $S_{\alpha,\rho}$:

$$\frac{\sin(\pi\rho s)}{\rho\sin(\pi s)}\,\frac{\Gamma(1-s/\alpha)}{\Gamma(1-s)}.$$

Verification of product identities for positive random variables is generally done by comparing Mellin transforms. It is useful because the method of moments is not always applicable.

Let us consider the important case $\alpha = 1$. The Mellin transform of $L_\rho = (S_{1,\rho})_+$, evaluated at $s = it$, $t \in \mathbb{R}$, is

$$\frac{\sin(\pi it\rho)}{\rho\sin(\pi it)}.$$

The well known identity for gamma functions, valid for all complex z,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

implies that for $t \in \mathbb{R}$,

$$\Gamma(1+it)\Gamma(1-it) = \frac{\pi it}{\sin(\pi it)}.$$

A double application of this shows that

$$\frac{\sin(\pi it\rho)}{\rho\sin(\pi it)} = \frac{\Gamma(1+it)\Gamma(1-it)}{\Gamma(1+it\rho)\Gamma(1-it\rho)}.$$

Since $\Gamma(1-it/\rho)/\Gamma(1-it)$ is the Mellin transform (at $s = it$) of $S_{\rho,1} = S_\rho$ for $\rho \in [0,1]$, we see with little work that

$$(S_{1,\rho})_+ \stackrel{\mathcal{L}}{=} L_\rho \stackrel{\mathcal{L}}{=} \left(\frac{S_\rho}{S'_\rho}\right)^\rho,$$

where S_ρ and S'_ρ are independent unilateral stable random variables of parameter ρ. This remarkable distributional identity is well-known (Zolotarev, 1986, p. 205; see also Bertoin et al (2006) and James (2006a)).

As a second application of the Mellin transform, setting $\alpha = 1$, we note that $|S_{1,\rho}|$ has Mellin transform

$$\frac{\cos(\pi\theta s/2)}{\cos(\pi s/2)}$$

and that $(S_{1,\rho})_+ \stackrel{\mathcal{L}}{=} (C_\rho)_+ \stackrel{\mathcal{L}}{=} L_\rho$ has Mellin transform

$$\frac{\sin(\pi\rho s)}{\rho\sin(\pi s)}.$$

The case $\rho = 1/2$ is of special interest, since $|C| \stackrel{\mathcal{L}}{=} (S_{1,1/2})_+ \stackrel{\mathcal{L}}{=} (C_{1/2})_+ \stackrel{\mathcal{L}}{=} L_{1/2}$:

$$\frac{2\sin(\pi s/2)}{\sin(\pi s)} = \frac{1}{\cos(\pi s/2)}.$$

In particular, with $s = it$, the Mellin transform becomes

$$\frac{2}{\exp(\pi t/2)+\exp(-\pi t/2)} = \frac{1}{\cosh(\pi t/2)}.$$

It is well-known the equality in distribution between X and Y may be established by showing for example that $\mathbb{E}\{X^r\} = \mathbb{E}\{Y^r\}$ for all $r \geq 1$, r integer, assuming that the moments uniquely define the distribution. A sufficient condition is Carleman's (see, e.g., Shohat and Tamerkin (1943), Akhiezer (1965) or Stoyanov (2000)):

$$\sum_{r=1}^{\infty} \frac{1}{(\mathbb{E}\{X^r\})^{\frac{1}{2r}}} = \infty.$$

This is especially useful when we want to verify distributional identities that are products, e.g., $X \stackrel{\mathcal{L}}{=} YZ$, with Y, Z independent. This method suffers, however, from various drawbacks: the moments must be finite for all r, and, more importantly, we can't conclude much without Carleman's condition.

A more general tool for checking the distributional identity $X \stackrel{\mathcal{L}}{=} Y$ is provided by characteristic functions (see, e.g., Kawata, 1972): is

$$\varphi(t) \stackrel{\text{def}}{=} \mathbb{E}\left\{e^{itX}\right\} = \psi(t) \stackrel{\text{def}}{=} \mathbb{E}\left\{e^{itY}\right\}$$

for all t? Since characteristic functions always exist and uniquely define distributions, there are no exceptions to the rule. However, for checking whether $X \stackrel{\mathcal{L}}{=} YZ$, this amounts to the verification of

$$\varphi(t) \stackrel{\text{def}}{=} \mathbb{E}\left\{e^{itX}\right\} = \mathbb{E}\left\{e^{itYZ}\right\} = \mathbb{E}\left\{\psi(tZ)\right\}.$$

This is rarely a useful route. Characteristic functions are much more adapted to checking distributional identities for sums, because $X+Y$ has characteristic function $\varphi(t)\psi(t)$ when X and Y are independent. For positive random variables, we can verify $X \stackrel{\mathcal{L}}{=} YZ$ by checking that $\log X \stackrel{\mathcal{L}}{=} \log Y + \log Z$, a much easier proposition, provided that we have the characteristic functions of all the log-variables. This is a well-trodden path, see, e.g., Kotlarski (1965). The characteristic function of $\log X$ is

$$\mathbb{E}\left\{e^{it\log X}\right\} = \mathbb{E}\left\{X^{it}\right\},$$

which looks like the r-th moment of X, with r replaced by it. One should be careful with this replacement, but for finite moments, this is often valid. In particular, if $M(s)$ is the Mellin transform of X, then

$$\mathbb{E}\left\{e^{it\log X}\right\} = M(it).$$

This is a powerful tool for studying distributional identities involving products and quotients of random variables, see, e.g. Epstein (1948) and Springer (1979). For example, if X_1 ands X_2 are positive random variables, and $\log X_1$ and $\log X_2$ have characteristic functions φ_1 and φ_2, then $\log(X_1X_2)$ has characteristic function $\varphi_1\varphi_2$. If $\log X$ has characteristic function φ, then $\log(aX^r)$ has characteristic function $a^{it}\varphi(rt)$. In particular, $\log(1/X)$ has characteristic function $\varphi(-t)$. Finally, X is distributed as $1/X$ if and only if the characteristic function of $\log X$ is real and symmetric. It is relatively easy to compute the characteristic functions for most log-variables discussed in this survey.

The strictly stable law: $\alpha < 1$**.**

By the product rule for Mellin transforms, we see that for $\alpha \le 1$ and $0 \le \rho \le 1$,

$$S_{\alpha,\rho} \stackrel{\mathcal{L}}{=} S_{1,\rho} \times S_\alpha \stackrel{\mathcal{L}}{=} C_\rho \times S_\alpha.$$

This property is called a decoupling of the parameters. Setting $\rho = 1/2$, we see that

$$S_{\alpha,1/2} \stackrel{\mathcal{L}}{=} CS_\alpha.$$

For simulation, these relationships should be the point of departure for all strictly stable laws with $\alpha \le 1$. Moreover, the fact that all these distributions are simultaneously scale mixtures of shifted Cauchy random variables and scale mixtures of unilateral strictly stable distributions is quite useful in the derivation of further properties.

Note also that

$$(S_{\alpha,\rho})_+ \stackrel{\mathcal{L}}{=} (S_{1,\rho})_+ \times S_\alpha \stackrel{\mathcal{L}}{=} (C_\rho)_+ \times S_\alpha \stackrel{\mathcal{L}}{=} L_\rho \times S_\alpha \stackrel{\mathcal{L}}{=} \left(\frac{S'_\rho}{S_\rho}\right)^\rho \times S_\alpha \stackrel{\mathcal{L}}{=} \left(\frac{S_{\alpha\rho}}{S_\rho}\right)^\rho,$$

where we used the ratio property of the Lamperti law, and the fact that for $\alpha < 1$,

$$S_\rho^\rho S_\alpha \stackrel{\mathcal{L}}{=} S_{\alpha\rho}^\rho$$

(Zolotarev, 1986, p. 194), which is easily shown using Mellin transforms.

A further corollary of calculations with the Mellin transforms is that, still for $\alpha < 1$,

$$(S_{\alpha,\rho})_+ \stackrel{\mathcal{L}}{=} \left(\frac{S_{\alpha\rho}}{S_\rho}\right)^\rho.$$

Because $\mathbb{P}\{S_{\alpha,\rho} > 0\} = \rho$, we have

$$S_{\alpha,\rho} \stackrel{\mathcal{L}}{=} \begin{cases} \left(\frac{S_{\alpha\rho}}{S_\rho}\right)^\rho & \text{with probability } \rho, \\ -\left(\frac{S_{\alpha(1-\rho)}}{S_{1-\rho}}\right)^{1-\rho} & \text{with probability } 1-\rho, \end{cases}$$

and thus

$$S_{\alpha,1/2} \stackrel{\mathcal{L}}{=} S\sqrt{\frac{S_{\alpha/2}}{S_{1/2}}},$$

where S is a random sign. Since, as we will see later, $S_{1/2} \stackrel{\mathcal{L}}{=} 1/(2N^2)$, we recover the well-known relationship

$$S_{\alpha,1/2} \stackrel{\mathcal{L}}{=} N\sqrt{2S_{\alpha/2}}.$$

In other words, the symmetric stable random variables are normal scale mixtures.

Recall that one can write all random variables $S_{\alpha,\rho}$ for $\alpha < 1$ as $pS_\alpha - qS'_\alpha$ for appropriate p and q. Using a formula shown earlier, we have, for example,

$$S_{\alpha,1/2} \stackrel{\mathcal{L}}{=} \frac{S_\alpha - S'_\alpha}{(2\cos(\pi\alpha/2))^{\frac{1}{\alpha}}}.$$

As $\alpha \uparrow 1$, the right hand side behaves as $0/0$. One can show that the limit law is indeed $S_{1,1/2} \stackrel{\mathcal{L}}{=} C$.

In all of the representations above, the unilateral strictly stable law looms large, and this is why we need to have good random variate generators for it.

The unilateral strictly stable law.

We recall that $S_\alpha \geq 0$:

$$(S_{\alpha,1})_+ \stackrel{\mathcal{L}}{=} S_{\alpha,1} \stackrel{\mathcal{L}}{=} S_\alpha, \alpha \in (0,1).$$

A simple random variate generator for S_α has been suggested by Kanter (1975), who used an integral representation of Ibragimov and Chernin (1959) and Zolotarev (1966) (see Zolotarev (1986, p. 74)), which states that the distribution function of $S_\alpha^{\alpha/(1-\alpha)}$ is given by

$$\frac{1}{\pi} \int_0^\pi e^{-\frac{A(u)}{x}} \, du,$$

where A is <u>Zolotarev's function</u>:

$$A(u) \stackrel{\text{def}}{=} \left\{ \frac{(\sin(\alpha u))^\alpha (\sin((1-\alpha)u))^{1-\alpha}}{\sin u} \right\}^{\frac{1}{1-\alpha}}.$$

By taking limits, we note that $S_1 = 1$, so that the family is properly defined for all $\alpha \in (0,1]$. We say that $K_\alpha \stackrel{\text{def}}{=} (A(\pi U))^{1-\alpha}$ is a <u>Kanter random variable</u> of parameter $\alpha \in (0,1)$, where U is uniform on $[0,1]$. Zolotarev's integral representation implies that

$$S_\alpha^\alpha \stackrel{\mathcal{L}}{=} \left(\frac{A(\pi U)}{E} \right)^{1-\alpha} \stackrel{\mathcal{L}}{=} \frac{K_a}{E^{1-\alpha}} \tag{2}$$

where E is exponential. This is Kanter's method.

Combined with the remark of the previous section, this yields a simple method requiring three independent uniform random variables for the generation of $S_{\alpha,\beta}$ when $\alpha \leq 1$: one uniform is needed to obtain E, one for computing $A(\pi U)$, and a third one for generating the shifted Cauchy random variate.

For later reference, it is important to note that the S_α has a simple Laplace transform:

$$\mathbb{E}\left\{ e^{-\lambda S_\alpha} \right\} = e^{-\lambda^\alpha}, \lambda \geq 0.$$

Also, from the ratio property for Lamperti random variables,

$$L_\rho \stackrel{\mathcal{L}}{=} \frac{K_\rho E^{1-\rho}}{K'_\rho E'^{1-\rho}} \stackrel{\mathcal{L}}{=} \left(\frac{1-U}{U} \right)^{1-\rho} \frac{K_\rho}{K'_\rho},$$

where K_ρ, K'_ρ are Kanter (ρ), U is uniform $[0,1]$, and E, E' are exponential, and all five random variables are independent.

It is convenient to rewrite K_α is

$$K_\alpha = \left(\frac{\sin(\alpha \pi U)}{\sin(\pi U)} \right)^\alpha \times \left(\frac{\sin((1-\alpha)\pi U)}{\sin(\pi U)} \right)^{1-\alpha}.$$

In particular, for $\alpha = 1/2$, this yields

$$K_{1/2} = \frac{\sin(\pi U/2)}{\sin(\pi U)} = \frac{1}{2\cos(\pi U/2)}.$$

By the well-known Box-Müller formula (1958), reproved below by elementary means, using $\cos^2(\pi U/2) \stackrel{\mathcal{L}}{=} \cos^2(\pi U)$,

$$N \stackrel{\mathcal{L}}{=} \sqrt{2E}\cos(\pi U) \stackrel{\mathcal{L}}{=} S\sqrt{2E}\cos(\pi U/2),$$

where N denotes a normal random variable and S denotes a random independent sign. Therefore,

$$S_{1/2} \stackrel{\mathcal{L}}{=} \frac{1}{4E\cos^2(\pi U/2)} \stackrel{\mathcal{L}}{=} \frac{1}{2N^2},$$

The strictly stable law: $\alpha > 1$.

Within this range, we have $\alpha\rho \le 1$ and $\alpha(1-\rho) \le 1$. Zolotarev (1981, 1986) realized that $(S_{\alpha,\rho})_+$ plays a key role. The Mellin transform of $S_{\alpha,\rho}$ is ρ times the Mellin transform of $(S_{\alpha,\rho})_+$, and from that, one can deduce, see Zolotarev (1981), pages 188–191, the following, in (α,ρ) notation:

$$\begin{aligned} S_{\alpha,1/2} &\stackrel{\mathcal{L}}{=} S|S_{\alpha,1/2}|, \\ \frac{(S_{\alpha,\rho_1})_+}{(S_{\alpha,\rho_2})_+} &\stackrel{\mathcal{L}}{=} \frac{(S_{\alpha,\rho_2})_+}{(S_{\alpha,\rho_1})_+}, \\ (S_{\alpha,\rho})_+ &\stackrel{\mathcal{L}}{=} \frac{1}{(S_{1/\alpha,\alpha\rho})_+^{\frac{1}{\alpha}}}. \end{aligned}$$

Especially the last identity is of interest because it relates $(S_{\alpha,\rho})_+$ for $\alpha > 1$ to strictly stable random variables whose first parameter is less than one. This matters because we clearly have

$$S_{\alpha,\rho} \stackrel{\mathcal{L}}{=} -S_{\alpha,1-\rho},$$

and $\mathrm{P}\{S_{\alpha,\rho} > 0\} = \rho$, so that

$$S_{\alpha,\rho} \stackrel{\mathcal{L}}{=} \begin{cases} (S_{\alpha,\rho})_+ & \text{with probability } \rho, \\ -(S_{\alpha,1-\rho})_+ & \text{with probability } 1-\rho. \end{cases}$$

Therefore, when $\alpha > 1$, we have, using shifted Cauchy random variables, using accents to denote independent random variables,

$$\begin{aligned} S_{\alpha,\rho} &\stackrel{\mathcal{L}}{=} -S_{\alpha,1-\rho} \\ &\stackrel{\mathcal{L}}{=} \begin{cases} \frac{1}{(S_{1/\alpha,\alpha\rho})_+^{\frac{1}{\alpha}}} & \text{with probability } \rho, \\ -\frac{1}{(S_{1/\alpha,\alpha(1-\rho)})_+^{\frac{1}{\alpha}}} & \text{with probability } 1-\rho. \end{cases} \\ &\stackrel{\mathcal{L}}{=} \frac{B_\rho}{(S_{1/\alpha,\alpha\rho})_+^{\frac{1}{\alpha}}} - \frac{1-B_\rho}{(S_{1/\alpha,\alpha(1-\rho)})_+^{\frac{1}{\alpha}}} \\ &\stackrel{\mathcal{L}}{=} \frac{1}{S_{1/\alpha}^{1/\alpha}} \times \left(B_\rho L_{\alpha\rho}^{-\frac{1}{\alpha}} - (1-B_\rho) L_{\alpha(1-\rho)}^{-\frac{1}{\alpha}} \right) \\ &\stackrel{\mathcal{L}}{=} \frac{1}{S_{1/\alpha}^{1/\alpha}} \times \left(B_\rho L_{\alpha\rho}^{\frac{1}{\alpha}} - (1-B_\rho) L_{\alpha(1-\rho)}^{\frac{1}{\alpha}} \right) \end{aligned}$$

$$
\begin{aligned}
&\stackrel{\mathcal{L}}{=} \frac{1}{S_{1/\alpha}^{1/\alpha}} \times \left(B_\rho \frac{S_{\alpha\rho}^\rho}{S'^{\rho}_{\alpha\rho}} - (1-B_\rho) \frac{S_{\alpha(1-\rho)}^{1-\rho}}{S'^{1-\rho}_{\alpha(1-\rho)}} \right) \\
&\stackrel{\mathcal{L}}{=} \begin{cases} \frac{S_{\alpha\rho}^\rho}{S_{1/\alpha}^{1/\alpha} S'^{\rho}_{\alpha\rho}} & \text{with probability } \rho, \\ -\frac{S_{\alpha(1-\rho)}^{1-\rho}}{S_{1/\alpha}^{1/\alpha} S'^{1-\rho}_{\alpha(1-\rho)}} & \text{with probability } 1-\rho \end{cases} \\
&\stackrel{\mathcal{L}}{=} \begin{cases} \left(\frac{L_{\alpha\rho}}{S_{1/\alpha}}\right)^{1/\alpha} & \text{with probability } \rho, \\ -\left(\frac{L_{\alpha(1-\rho)}}{S_{1/\alpha}}\right)^{1/\alpha} & \text{with probability } 1-\rho. \end{cases}
\end{aligned}
$$

These expressions provide a multitude of ways of generating strictly stable random variates with $\alpha > 1$. Essentially, we either need one shifted Cauchy and one unilateral stable random variate, or three unilateral stable random variates. However, unlike the case $\alpha < 1$, the parameters of the shifted Cauchy and unliteral stable are no longer decoupled. In any case, we highly recommend this shifted Cauchy and unilateral stable product in practice since it is less error-prone than other methods published in the literature, and one can concentrate on optimizing random variate generation for both component random variates separately.

It helps to check some special cases. Taking $\rho = 1/2$, the symmetric case, and using S to denote a random sign, we see that

$$
S_{\alpha,1/2} \stackrel{\mathcal{L}}{=} S \left(\frac{L_{\alpha/2}}{S_{1/\alpha}}\right)^{1/\alpha}.
$$

In particular, setting $\alpha = 2$, we have

$$
S_{2,1/2} \stackrel{\mathcal{L}}{=} \frac{S}{\sqrt{S_{1/2}}}.
$$

This is just another way of stating that the Lévy law (i.e., that of $S_{1/2}$) satisfies

$$
S_{1/2} \stackrel{\mathcal{L}}{=} \frac{1}{2N^2}.
$$

The extreme values for ρ are $1/\alpha$ and $1-1/\alpha$. For this case, one side of the strictly stable law simplifies dramatically. We have, making things above more specific:

$$
\begin{aligned}
S_{\alpha,1/\alpha} &\stackrel{\mathcal{L}}{=} -S_{\alpha,1-1/\alpha} \\
&\stackrel{\mathcal{L}}{=} \begin{cases} \frac{1}{S_{1/\alpha}^{\frac{1}{\alpha}}} & \text{with probability } 1/\alpha, \\ -\frac{1}{(S_{1/\alpha,\alpha-1})_+^{\frac{1}{\alpha}}} & \text{with probability } 1-1/\alpha. \end{cases} \\
&\stackrel{\mathcal{L}}{=} \frac{B_{1/\alpha}}{S_{1/\alpha}^{\frac{1}{\alpha}}} - \frac{1-B_{1/\alpha}}{(S_{1/\alpha,\alpha-1})_+^{\frac{1}{\alpha}}} \\
&\stackrel{\mathcal{L}}{=} \frac{1}{S_{1/\alpha}^{1/\alpha}} \times \left(B_{1/\alpha} - (1-B_{1/\alpha}) L_{\alpha-1}^{-\frac{1}{\alpha}} \right)
\end{aligned}
$$

$$\stackrel{\mathcal{L}}{=} \frac{1}{S_{1/\alpha}^{1/\alpha}} \times \left(B_{1/\alpha} - (1 - B_{1/\alpha}) L_{\alpha-1}^{\frac{1}{\alpha}} \right)$$

$$\stackrel{\mathcal{L}}{=} \frac{1}{S_{1/\alpha}^{1/\alpha}} \times \left(B_{1/\alpha} - (1 - B_{1/\alpha}) \frac{S_{\alpha-1}^{1-1/\alpha}}{S'^{1-1/\alpha}_{\alpha-1}} \right)$$

$$\stackrel{\mathcal{L}}{=} \begin{cases} \frac{1}{S_{1/\alpha}^{1/\alpha}} & \text{with probability } 1/\alpha, \\ -\frac{S_{\alpha-1}^{1-1/\alpha}}{S_{1/\alpha}^{1/\alpha} S'^{1-1/\alpha}_{\alpha-1}} & \text{with probability } 1 - 1/\alpha \end{cases}$$

$$\stackrel{\mathcal{L}}{=} \begin{cases} \left(\frac{1}{S_{1/\alpha}} \right)^{1/\alpha} & \text{with probability } 1/\alpha, \\ -\left(\frac{L_{\alpha-1}}{S_{1/\alpha}} \right)^{1/\alpha} & \text{with probability } 1 - 1/\alpha. \end{cases}$$

The difference method alluded to above permits one to write all $S_{\alpha,\rho}$ for $\alpha > 1$ as $pS_{\alpha,1/\alpha} - qS'_{\alpha,1/\alpha}$ for appropriate p and q. Using a formula shown earlier, we have, for example,

$$S_{\alpha,1/2} \stackrel{\mathcal{L}}{=} \frac{S_{\alpha,1/\alpha} - S'_{\alpha,1/\alpha}}{(2\cos(\pi\alpha/2))^{\frac{1}{\alpha}}}.$$

Chambers, Mallows and Stuck (1976) (see also Weron, 1996) proposed a general method for strictly stable random variables. However, using the relationships above, their formula is not computationally much more advantageous than the ones suggested above. Using an integral representation of Zolotarev (1966), they showed that

$$S_{\alpha,\theta} \stackrel{\mathcal{L}}{=} E^{1-1/\alpha} \times \frac{\sin((\pi/2)\alpha(U - 1/2 + \theta))}{\sin^{1/\alpha}((\pi/2)(U - 1/2))\cos^{1-1/\alpha}((\pi/2)(U - 1/2 - \alpha(U - 1/2 + \theta)))}.$$

However, their formula provides little insight into its genesis and into the structural properties of stable laws.

Other random variate generators

For symmetric strictly stable random variables, having characteristic function $e^{-|t|^\alpha}$, $0 < \alpha < 1$, a specially simple method exists that is based on the fact that the characteristic function is of the Polya type, i.e., it is convex on the positive halfline and symmetric. Devroye (1984) showed that

$$S_{\alpha,1/2} \stackrel{\mathcal{L}}{=} \frac{V}{\left(E + E' 1_{[U<\alpha]}\right)^{\frac{1}{\alpha}}},$$

where E and E' are independent exponential random variables, and V has the so-called de la Vallée-Poussin density

$$\frac{1}{\pi} \left(\frac{\sin x}{x} \right)^2.$$

A particularly efficient method for V consists in generating independent uniform $[-1, 1]$ random pairs (U, X), and setting $X \leftarrow 1/X$ if $U < 0$, until $|U| \min(1, X^2) \le \sin^2(X)$. Upon exit, return $2X$.

It would be nice if simple extensions of this property can be found for the entire strictly stable family, and especially the unilateral strictly stable laws.

There is a convergent series expansion for the density of all stable laws due to Bergstrom (1952) and Feller (1971). Based on that series and estimates related to it due to Bartels (1981), Devroye (1986) developed a general rejection method for the strictly stable family.

The weakly stable distribution

In this section, we look at the random variable W_β, $\beta \in [-1,1]$, which is the case $\alpha = 1$ for Zolotarev's form (B). The characteristic function φ is defined by

$$\log \varphi(t) = -|t|\frac{\pi}{2} - i\beta t \log |t|.$$

For $\beta \in (0,1]$, Zolotarev (1986, p. 74ff) showed that the distribution function of W_β is given by

$$\frac{1}{2}\int_{-1}^{1} \exp\left(-e^{-x/\beta} f(z,\beta)\right)\, dz,$$

where

$$f(z,\beta) = \frac{\pi}{2}\frac{1+\beta z}{\cos(\pi z/2)} \exp\left(\frac{\pi}{2}\left(z+\frac{1}{\beta}\right)\tan(\pi z/2)\right).$$

If V denotes a uniform $[-1,1]$ random variable, then that distribution function can also be written as

$$\mathbb{E}\left\{\exp\left(-e^{-x/\beta} f(V,\beta)\right)\right\}.$$

If E is an exponential random variable, and $\delta > 0$ is a constant, then

$$-\beta(\log E + \log \delta)$$

has distribution function

$$\exp\left(-\exp\left(-\frac{x}{\beta} + \log\delta\right)\right) = \exp\left(-\delta \exp\left(-\frac{x}{\beta}\right)\right).$$

This is the Gumbel distribution with translation and scale parameter. As a corollary, for $\beta > 0$,

$$W_\beta \stackrel{\mathcal{L}}{=} -\beta \log E - \beta \log f(V,\beta).$$

Similarly, as $W_{-\beta} \stackrel{\mathcal{L}}{=} -W_\beta$, we conclude that for all $\beta \in (0,1]$,

$$\begin{aligned} W_\beta &\stackrel{\mathcal{L}}{=} -\beta \log E - \beta \log f(V,|\beta|) \\ &\stackrel{\mathcal{L}}{=} -\beta \log E - \beta \log\left(\frac{\pi}{2}\frac{1+|\beta| V}{\cos(\pi V/2)}\right) - \frac{\pi}{2}\left(\beta V + \frac{\beta}{|\beta|}\right)\tan(\pi V/2) \\ &\stackrel{\mathcal{L}}{=} -\beta \log E - \beta \log\left(\frac{\pi}{2}\frac{1+\beta V}{\cos(\pi V/2)}\right) - \frac{\pi}{2}(\beta V + 1)\tan(\pi V/2) \quad \text{(replace } V \text{ by } V \operatorname{sign}\beta). \end{aligned}$$

The last representation shows that we can extend the range for β to the closed interval $[0,1]$. In particular, we rediscover that

$$W_0 \stackrel{\mathcal{L}}{=} -\frac{\pi}{2}\tan(\pi V/2) \stackrel{\mathcal{L}}{=} \frac{\pi}{2} C,$$

something that follows immediately from the definition of W_0 as well.

A special role is played by W_1:

$$W_1 \overset{\mathcal{L}}{=} -\log E - \log\left(\frac{\pi(1+V)}{2\cos(\pi V/2)}\right) - \frac{\pi(1+V)\tan(\pi V/2)}{2}.$$

Replacing $(V+1)/2$ by U, a uniform $[0,1]$ random variable, and using standard trigonometric operations, we see that

$$W_1 \overset{\mathcal{L}}{=} -\log E - \log\left(\frac{\pi U}{\sin(\pi U)}\right) + \frac{\pi U}{\tan \pi U}.$$

The importance of W_1 follows from the following observation: If $\lambda, \mu \in \mathbf{R}$, and W_1', W_1 are i.i.d., then

$$\lambda W_1 + \mu W_1'$$

has characteristic function φ defined by

$$\log \varphi(t) = -\frac{\pi}{2}|t|(|\lambda| + |\mu|) - (\lambda+\mu) it \log|t| - it(\lambda \log|\lambda| + \mu \log|\mu|).$$

So, if we set $\lambda = (1+\beta)/2$, $\mu = (\beta-1)/2$, $\beta \in [-1,1]$, then

$$\log \varphi(t) = -\frac{\pi}{2}|t| - i\beta t \log|t| - it\gamma,$$

where

$$\gamma \overset{\text{def}}{=} \frac{1+\beta}{2} \log\left(\frac{1+\beta}{2}\right) - \frac{1-\beta}{2} \log\left(\frac{1-\beta}{2}\right).$$

Therefore,

$$W_\beta - \gamma \overset{\mathcal{L}}{=} \frac{1+\beta}{2} W_1 - \frac{1-\beta}{2} W_1'.$$

Put differently,

$$W_\beta \overset{\mathcal{L}}{=} \frac{1+\beta}{2}\left(W_1 + \log\left(\frac{1+\beta}{2}\right)\right) - \frac{1-\beta}{2}\left(W_1' + \log\left(\frac{1-\beta}{2}\right)\right).$$

In particular, $W_0 \overset{\mathcal{L}}{=} (W_1 - W_1')/2$.

The stable distribution

The general stable law, or simply the stable law, is described by its characteristic function in one of several forms. The choice of one over another often turns to the issue of continuity with respect to the parameters. In this section, we mention a few of these forms. When the main parameter α is not one, these correspond to affine transformations of strictly stable laws. Only the threshold case $\alpha = 1$ adds a new element. The present section is not essential to the remainder of the survey, and may be skipped without harm.

Assume first $\alpha \neq 1$, and let us use the parameters of the strictly stable law. The polar form, or Zolotarev's form (B) (Zolotarev, 1986, p. 12), with a scale parameter $\lambda > 0$ and translation parameter $\gamma \in \mathbf{R}$ added, is:

$$\log \varphi(t) = \lambda\left(it\gamma - |t|^\alpha e^{-i(\pi/2)\theta\alpha\,\mathrm{sign}(t)}\right).$$

Recall that $\theta = \beta$ when $\alpha < 1$ and $\theta = \beta(\alpha-2)/\alpha$ otherwise. By comparison with the strictly stable law, it is immediate that this is the characteristic function of

$$\lambda\gamma + \lambda^{1/\alpha} S_{\alpha,\beta}.$$

Zolotarev's form (A) (Zolotarev, p. 9; see also Weron (2004) or Samorodnitsky and Taqqu (1994)) is

$$\log \varphi(t) = \lambda \left(it\gamma - |t|^\alpha + i|t|^\alpha \beta \tan(\pi\alpha/2) \operatorname{sign}(t) \right),$$

where $\alpha \in (0,2)$ and $\beta \in [-1,1]$ play the same role as before. This is the distribution of

$$\lambda\gamma + \left(\frac{\lambda}{\cos(\theta\alpha\pi/2)} \right)^{\frac{1}{\alpha}} S_{\alpha,\beta'}$$

where

$$\beta' = \begin{cases} \frac{2}{\alpha\pi} \arctan(\beta \tan(\alpha\pi/2)) & \text{if } \alpha < 1, \\ \frac{2}{(\alpha-2)\pi} \arctan(\beta \tan(\alpha\pi/2)) & \text{if } \alpha > 1. \end{cases}$$

There are other popular forms as well, such as Zolotarev's (M), which was already used by Chambers, Mallows and Stuck (1976), and has been further discussed by Cheng and Liu (1997) and Nolan (1997). These are of course further linear transformations, often motivated by continuity with respect to the parameters, especially when $\alpha \to 1$, and the limit corresponds to the stable law with $\alpha = 1$ and the same value of β. For random variate generation, this is inconsequential, however.

The BFRY law

Named after Bertoin, Fujita, Roynette and Yor (2006), we define the BFRY random variable with parameter $\alpha \in (0,1)$ by its density

$$f_\alpha(x) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{1-e^{-x}}{x^{1+\alpha}}, x > 0.$$

This random variable occurs in the study of the excursion duration of Bessel processes. It is infinitely divisible and poses no challenge for random variate generation, as pointed out by Bertoin et al: if X_α denotes a BFRY (α) random variable, U is uniform $[0,1]$ and G_a denotes a gamma (a) random variable with shape parameter a and scale parameter one, then

$$X_\alpha \stackrel{\mathcal{L}}{=} \frac{G_{1-\alpha}}{U^{1/\alpha}}.$$

Its Laplace transform is

$$\mathbb{E}\left\{ e^{-\lambda X_\alpha} \right\} = (1+\lambda)^\alpha - \lambda^\alpha, \lambda \ge 0.$$

It is self-decomposable and generates a perpetuity by this result of Jurek (1999) and Bertoin et al (2006):

$$X_\alpha \stackrel{\mathcal{L}}{=} U^{\frac{1}{1-\alpha}} (X_\alpha + \mathcal{K}_\alpha)$$

where U, X_α and $\mathcal{K}_\alpha$ on the right-hand side are independent, and $\mathcal{K}_\alpha \stackrel{\mathcal{L}}{=} E/\mathcal{G}_\alpha$, and $\mathcal{G}_\alpha$ has density

$$f(x) = \frac{\alpha \sin(\pi\alpha)}{(1-\alpha)\pi} \frac{x^{\alpha-1}(1-x)^{\alpha-1}}{(1-x)^{2\alpha} - 2(1-x)^\alpha x^\alpha \cos(\pi\alpha) + x^{2\alpha}}, 0 \le x \le 1.$$

With $\mathcal{K}_\alpha(i), i \ge 0$, denoting i.i.d. copies of $\mathcal{K}_\alpha$, and $U(i)$, $i \ge 0$ denoting i.i.d. copies of U, we thus have the perpetuity (Jurek, 1999)

$$X_\alpha \stackrel{\mathcal{L}}{=} \sum_{j=0}^{\infty} \prod_{i=0}^{j} \mathcal{K}_\alpha(i) U(i)^{\frac{1}{1-\alpha}}.$$

The second BFRY law.

$\mathcal{G}_\alpha$ will be called the second law of Bertoin, Fujita, Roynette and Yor (2006), who showed the following properties:

(i) $E\mathcal{G}_\alpha$ has Laplace transform

$$\mathbb{E}\left\{e^{-\lambda E\mathcal{G}_\alpha}\right\} = \frac{\alpha}{1-\alpha}\frac{1-(1+\lambda)^{\alpha-1}}{(1+\lambda)^\alpha-1}, \lambda \ge 0.$$

(ii) $\mathcal{G}_{1/2}$ is beta $(1/2,1/2)$, the arc sine law, having density

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}}, 0 < x < 1.$$

(iii) $\mathcal{G}_\alpha \stackrel{\mathcal{L}}{=} 1 - \mathcal{G}_\alpha$.

(iv) $\mathcal{G}_a \stackrel{\mathcal{L}}{\to} U$, the uniform $[0,1]$ random variable, as $\alpha \uparrow 1$. We can thus define, by continuity, $\mathcal{G}_1 = U$.

(v) As $\alpha \downarrow 0$, $\mathcal{G}_\alpha \stackrel{\mathcal{L}}{\to} \mathcal{G}_0 \stackrel{\text{def}}{=} 1/(1+\exp(\pi C))$, with C Cauchy. Its density is

$$f(x) = \frac{1}{x(1-x)} \times \frac{1}{\pi^2 + (\log(1-x)-\log(x))^2}, 0 < x < 1.$$

Random variate generation for the law $\mathcal{G}_\alpha$, is facilitated by two relationships that were pointed out by Bertoin et al (2006). It is a simple exercise to show that

$$\mathcal{G}_\alpha \stackrel{\mathcal{L}}{=} \frac{1}{1+L_{1-\alpha}^{1/\alpha}} \stackrel{\mathcal{L}}{=} \frac{L_{1-\alpha}^{1/\alpha}}{1+L_{1-\alpha}^{1/\alpha}} \stackrel{\mathcal{L}}{=} \frac{S_{1-\alpha}^{\frac{1-\alpha}{\alpha}}}{S_{1-\alpha}^{\frac{1-\alpha}{\alpha}} + S'^{\frac{1-\alpha}{\alpha}}_{1-\alpha}}.$$

Mittag-Leffler distribution

Further laws related to $\mathcal{G}_\alpha$ include the Mittag-Leffler law. A Mittag-Leffler random variable of parameter $\alpha \in (0,1)$, written M_α, is defined by

$$M_\alpha \stackrel{\text{def}}{=} \frac{1}{S_\alpha^\alpha}$$

(Chaumont and Yor, 2003, p. 114). M_α, unlike S_α, has short tails, and all its positive moments exist. For example,

$$\mathbb{E}\left\{M_\alpha^r\right\} = \frac{\Gamma(r+1)}{\Gamma(\alpha r+1)}, r > -1,$$

and

$$\mathbb{E}\left\{e^{\lambda M_\alpha}\right\} = \sum_{n=0}^\infty \frac{\lambda^n}{\Gamma(\alpha n+1)}, \lambda \in \mathbb{R}.$$

The Mittag-Leffler law is related to Lamperti's via the Laplace transform:

$$\mathbb{E}\left\{e^{-\lambda L_\alpha^{1/\alpha}}\right\} = \mathbb{E}\left\{e^{-\lambda S_\alpha/S'_\alpha}\right\} = \mathbb{E}\left\{e^{-\lambda^\alpha/S'^\alpha_\alpha}\right\} = \mathbb{E}\left\{e^{-\lambda^\alpha M_\alpha}\right\}.$$

It is worth noting that M_α is related to the almost sure limit of the number of partition blocks induced by a Pitman-Yor process (see, e.g., Theorem 3.8 of Pitman, 2006), which in turn is relevant for the study of clustering in Bayesian statistics and machine learning.

The third BFRY law

The third law of Bertoin et al (2006), which generalizes the second, has two parameters, a, b such that $a, b \in (0,1)$. It too is supported on $\mathbb{R}^+$, and is best described by its Stieltjes transform. The Stieltjes transform of a random variable X in general is given by

$$\frac{1}{\lambda}\mathbb{E}\left\{e^{-EX/\lambda}\right\} = \mathbb{E}\left\{\frac{1}{\lambda+X}\right\}, \lambda > 0.$$

Note that the Laplace transorm $\mathbb{E}\{e^{-\lambda EX}\}$ of EX thus is $(1/\lambda)S_{1/\lambda,1/2}$, where $S_{\lambda,1/2}$ is the Stieltjes transform of X. The Stieltjes transform of a $\mathcal{G}_{a,b}$ random variable, which follows the third law of Bertoin et al, is

$$\frac{a}{1-b}\,\frac{\left(\lambda^{b-1}-(1+\lambda)^{b-1}\right)\lambda^{a-b}}{(1+\lambda)^a-\lambda^a}, \lambda \ge 0.$$

Its density for $x \in (0,1)$ is given by

$$\frac{a}{\pi(1-b)}\,\frac{(1-x)^a x^{a-1}\sin(\pi a) + x^{2a-b}(1-x)^{b-1}\sin(\pi b) + (1-x)^{a+b-1}x^{a-b}\sin(\pi(a-b))}{(1-x)^{2a} - 2(1-x)^a x^a \cos(\pi a) + x^{2a}}.$$

We note that $\mathcal{G}_{a,a} \stackrel{\mathcal{L}}{=} \mathcal{G}_a$, and that $\mathcal{G}_{a,1-a} \stackrel{\mathcal{L}}{=} B_{a,1-a}$, where B denotes a beta random variable. Also, $E\mathcal{G}_{1,b}$ has Laplace transform

$$\frac{1-(1+\lambda)^{b-1}}{(1-b)\lambda}.$$

The fourth BFRY law

Bertoin et al (2006) also showed the existence of a random variable $X_{a,b} \ge 0$ with Laplace transform

$$\frac{b}{a}\,\frac{(1+\lambda)^a-1}{(1+\lambda)^b-1}, \lambda \ge 0.$$

Here the parameters are restricted as follows: $0 < a \le b \le 1$. This family of distributions is infinitely divisible and has the remarkable property that for $0 < a < b < c < 1$,

$$X_{a,c} \stackrel{\mathcal{L}}{=} X_{a,b} + X_{b,c},$$

where the two random variables on the right hand side are independent. Quite interestingly, $\mathbb{E}\{X_{a,b}\} = (b-a)/2$. By taking limits, we can define $X_{a,1}$ as the law with Laplace transform

$$\frac{(1+\lambda)^a-1}{a\lambda}, \lambda > 0.$$

Similarly, we define $X_{0,b}$ as the law with Laplace transform

$$\frac{b\log(1+\lambda)}{(1+\lambda)^b-1}, \lambda>0.$$

Using the fact that G_a has Laplace transform $1/(1+\lambda)^a$, it is a trivial exercise to show that

$$X_{a,1} \stackrel{\mathcal{L}}{=} UG_{1-a},$$

where U is uniform $[0,1]$. In particular, $X_{0,1}$ has the law with Laplace transform

$$\frac{\log(1+\lambda)}{\lambda},$$

which is easily seen to be the Laplace transform of EU, with E exponential and U uniform on $[0,1]$.

The random variables $X_{a,b}$ and $\mathcal{G}_a$ are intimately related in a calculus, developed by Bertoin et al (2006). Most properties follow from the Laplace transforms. For example, for $1-a \le a$,

$$G_{1-a} + X_{1-a,a} \stackrel{\mathcal{L}}{=} E\mathcal{G}_a,$$

where E is exponential and G_{1-a} is gamma. This follows after multiplying the Laplace transforms.

Lamperti's second law

Barlow, Pitman and Yor (1989) (see also Watanabe, 1995) found the law of the occupation time of one side of a skew Bessel process of dimension in $(0,2)$ with skewness parameter $p \in (0,1)$. It too was originally studied by Lamperti (1958) and is characterized by p and a shape parameter $\rho \in (0,1)$. We write a random variable as $L_{\rho,p}$. The Stieltjes transform is

$$\mathbb{E}\left\{\frac{1}{\lambda+L_{\rho,p}}\right\} = \frac{p(1+\lambda)^{\rho-1}+(1-p)\lambda^{\rho-1}}{p(1+\lambda)^{\rho}+(1-p)\lambda^{\rho}}, \lambda>0.$$

and the density is

$$f(x) = \frac{\sin(\rho\pi)}{\pi}\;\frac{p(1-p)x^{\rho-1}(1-x)^{\rho-1}}{p^2(1-x)^{2\rho}+(1-p)^2x^{2\rho}+2p(1-p)x^{\rho}(1-x)^{\rho}\cos(\rho\pi)}, 0<x<1.$$

For $p=1/2$, $\rho=1/2$, we obtain Lévy's arc sine law. Random variate generation for $L_{\rho,p}$ can be done by inversion of the distribution function, something first remarked by James (2010b). The distribution function is

$$F(x) = 1-\frac{1}{\pi\rho}\arctan\left(\frac{\sin(\pi\rho)}{\cos(\pi\rho)+\frac{(1-p)x^{\rho}}{p(1-x)^{\rho}}}\right).$$

Inversion shows that, with U uniform $[0,1]$, L_ρ Lamperti of parameter ρ, and $q=p/(1-p)$,

$$L_{\rho,p} \stackrel{\mathcal{L}}{=} \frac{(qL_\rho)^{\frac{1}{\rho}}}{1+(qL_\rho)^{\frac{1}{\rho}}}.$$

We recall that

$$L_\rho \stackrel{\mathcal{L}}{=} \frac{\sin(U\pi\rho)}{\sin((1-U)\pi\rho)}.$$

By the property of Lamperti random variables, we also see that

$$L_{\rho,p} \stackrel{\mathcal{L}}{=} \frac{p^{1/\rho} S_\rho}{p^{1/\rho} S_\rho + (1-p)^{1/\rho} S'_\rho}$$

where S_ρ, S'_ρ are independent unilateral stable random variables.

Occupation times of Bessel bridges and some other processes lead to Poisson-Dirichlet means (see Barlow, Pitman and Yor, 1989). Exact random variate generation for these distributions was dealt with at length in Devroye and James (2011) using an extension of the coupling-from-the-past method coined Double CFTP. Poisson-Dirichlet means are important in Bayesian and nonparametric statistics. Further references to linear functionals of the Poisson-Dirichlet process—popularized today as the Pitman-Yor process— include Yano and Yano (2008) and James, Lijoi and Prünster (2008). For Drichlet means, see, e.g., Cifarelli and Regazzini (1990).

Linnik and generalized Linnik distributions

The generalized Linnik distribution (Devroye, 1990, 1996) has Laplace transform

$$\frac{1}{(1+\lambda^\alpha)^\beta},$$

where $\alpha \in (0,1)$ and $\beta > 0$. The standard Linnik has $\beta = 1$. A Linnik random variate, denoted by $\Delta_{\alpha,\beta}$, can be generated, as indicated by Devroye (1990, 1996), as

$$G_\beta^{1/\alpha} S_\alpha.$$

See Huillet (2000) and Lin (2001) for additional properties. However, there are many other relationships worth noting. For example,

$$\Delta_{\alpha,1} \stackrel{\mathcal{L}}{=} E L_\alpha^{\frac{1}{\alpha}} \stackrel{\mathcal{L}}{=} E^{\frac{1}{\alpha}} S_\alpha,$$

and for $\beta \in (0,1)$,

$$\Delta_{\alpha,\beta} \stackrel{\mathcal{L}}{=} B_{\beta,1-\beta}^{\frac{1}{\alpha}} E L_\alpha^{\frac{1}{\alpha}}.$$

For proofs of this, see below.

Mellin transforms

Let us introduce a few classical distributions. The Pearson VI distribution with parameters $a, b > 0$, denoted $P_{a,b}$, is the distribution of G_a/G_b. Clearly, $1/P_{a,b} \stackrel{\mathcal{L}}{=} P_{b,a}$. The characteristic function of $\log(P_{a,b})$ is the product of those for $\log(G_a)$ and $\log(G_b^{-1})$. The special case $P_{1,b} = \log(E/G_b)$ is important because $P_{1,b}$ has density $1/(1+x)^{1+b}, x > 0$, and thus,

$$P_{1,b} \stackrel{\mathcal{L}}{=} \frac{1}{U^{\frac{1}{b}}} - 1.$$

For historic reasons, statisticians are more accustomed to the F-distribution (also called the Fisher-Snedecor or Snedecor's F-) distribution:

$$F_{a,b} \stackrel{\text{def}}{=} \frac{b}{a} \frac{Ga/2}{G_{b/2}} = \frac{b}{a} P_{a/2,b/2}.$$

However, we should all switch to a common metric system and just use $P_{a,b}$. The Student t distribution of parameter a is the distribution of

$$T_a \stackrel{\text{def}}{=} \frac{N}{\sqrt{\frac{G_{a/2}}{a/2}}}.$$

The best known special case is the Cauchy law (written C), which is just T_1. The final random variable, $T_{a,b}$ is a tilted stable that will be defined below. The table below gives the Mellin transform $M(it)$ for most of the distributions discussed in this survey.

Random variable	Characteristic function	Range for the parameter(s)
$\log(G_a)$	$\frac{\Gamma(a+it)}{\Gamma(a)}$	$(a > 0)$
$\log((S_{a,\rho})_+)$	$\frac{\sin(\pi\rho it)}{\rho\sin(\pi it)} \frac{\Gamma(1-it/a)}{\Gamma(1-it)}$	$(0 < a \le 2, \max(0, 1-1/\alpha) \le \rho \le \min(1, 1/\alpha))$
$\log(S_a)$	$\frac{\Gamma(1-it/a)}{\Gamma(1-it)}$	$(0 < a \le 1)$
$\log(M_a)$	$\frac{\Gamma(1+it)}{\Gamma(1+ita)}$	$(0 < a \le 1)$
$\log(E^a)$	$\Gamma(1+ita)$	$(a \ge 0)$
$\log(B_{a,b})$	$\frac{\Gamma(a+it)\Gamma(a+b)}{\Gamma(a)\Gamma(a+b+it)}$	$(a, b > 0)$
$\log(U^a)$	$\frac{1}{1+ita}$	$(a \in \mathbb{R})$
$\log(P_{a,b})$	$\frac{\Gamma(a+it)\Gamma(b-it)}{\Gamma(a)\Gamma(b)}$	$(a, b > 0)$
$\log(T_a^2/a)$	$\frac{\Gamma(1/2+it)\Gamma(a/2-it)}{\Gamma(1/2)\Gamma(a/2)}$	$(a > 0)$
$\log(C^{2a})$	$\frac{\Gamma(1/2+ita)\Gamma(1/2-ita)}{\pi}$	$(a \ge 0)$
$\log(K_a)$	$\frac{\Gamma(1-it)}{\Gamma(1-ita)\Gamma(1-it(1-a))}$	$(0 \le a \le 1)$
$\log(L_a)$	$\frac{\Gamma(1-it)\Gamma(1+it)}{\Gamma(1-ita)\Gamma(1+ita)}$	$(0 \le a \le 1)$
$\log(\Delta_{a,b})$	$\frac{\Gamma(b+it/a)\Gamma(1-ita)}{\Gamma(b)\Gamma(1-it)}$	$(0 < a \le 1, b > 0)$
$\log(X_a)$	$\frac{\Gamma(1-a+it)}{\Gamma(1-a)(1-it/a)}$	$(0 < a < 1)$
$\log(T_{a,b})$	$\frac{\Gamma(1+b/a-it/a)\Gamma(1+b)}{\Gamma(1+b/a)\Gamma(1+b-it)}$	$(0 < a < 1, b \ge 0)$

Polynomial tilting

For a random variable $X \geq 0$, we can define its polynomially tilted (or: b-tilted) version X_b with parameter $b \in \mathrm{R}$ as the random variable in which the probability mass of X is multiplicatively altered as x^{-b}. Observe that if $\log X$ has characteristic function φ, then $\log X_b$ has characteristic function

$$\frac{\varphi(t+ib)}{\varphi(ib)}.$$

For example, the b-tilted version of S_a, denoted by $T_{a,b}$ (see Perman, Pitman and Yor (1992) or James (2006b, 2010a)) is such that $\log T_{a,b}$ has characteristic function

$$\frac{\Gamma(1+(b-it)/a)\Gamma(1+b)}{\Gamma(1+b/a)\Gamma(1+b-it)}.$$

In another example, the b-tilted G_a, with $0 \leq b < a$, has a logarithm with characteristic function

$$\frac{\Gamma(a-b+it)}{\Gamma(a-b)},$$

which is G_{a-b}. In a third example, it is easy to see from the table of the previous section that the b-tilted version of $L_a^{1/a}$ is distributed as $S_a/T_{a,b}$ (James, 2010a, 2010b).

In the reaminder of this section, we deal exclusively with $T_{a,b}$, a family of distributions that was studied in depth by Perman, Pitman and Yor (1992), James and Yor (2007) and James (2006a, 2006b, 2010a, 2010b). The following identity of Perman, Pitman and Yor (1992) follows from the table given above:

$$T_{a,b} \stackrel{\mathcal{L}}{=} \frac{T_{a,a+b}}{B_{a+b,1-a}}.$$

This remains valid even for $b < 0$ as long as $a + b > 0$. As a special case, we have

$$T_{a,1-a} \stackrel{\mathcal{L}}{=} \frac{T_{a,1}}{B_{1,1-a}} \stackrel{\mathcal{L}}{=} \frac{T_{a,1}}{1-U^{1/(1-a)}}.$$

James (2006a, 2006b, 2010a, 2010b) showed much more than this. In fact, as can be verified by the table of Mellin transforms,

$$T_{a,b}^a \stackrel{\mathcal{L}}{=} \frac{T_{a,\delta}^a}{B_{\frac{a+b}{a},\frac{\delta-(a+b)}{a}} B_{\delta,1+b-\delta}^a}$$

with $a + b \leq \delta \leq 1 + b$. Therefore, all $T_{a,b}$ random variates can be obtained from beta variates and a $T_{a,1}$ variate, thus shining a focused light on $T_{a,1}$. Furthermore, James (2010a) shows the following:

$$G_{1+b/a}^{1/a} \stackrel{\mathcal{L}}{=} \frac{G_{a+b}}{T_{a,a+b}} \stackrel{\mathcal{L}}{=} \frac{G_{b+1}}{T_{a,b}}.$$

The special case $b = 1 - a$ yields

$$G_{1/a}^{1/a} \stackrel{\mathcal{L}}{=} \frac{E}{T_{a,1}} \stackrel{\mathcal{L}}{=} \frac{G_{2-a}}{T_{a,1-a}}.$$

This matters a lot, because it implies that a gamma random variate $G_{1/a}$ can be generated from an exponential random variate and $T_{a,1}$. Other corollaries are that $E^{1/a} \stackrel{\mathcal{L}}{=} G_a/T_{a,a}$ and that $G_2^{1/a} \stackrel{\mathcal{L}}{=} G_{a+1}/T_{a,a}$.

It is worth noting here that $T_{a,b}$ can be generated in expected time uniformly bounded over all $a \in (0, 1]$, $b \geq 0$ (Devroye, 2009), using the rejection method. Unlike for the unilateral stable distribution, no simple one-line method in the spirit of Kanter's exists for $T_{a,b}$ or even just $T_{a,1}$. Devroye's method is based upon the following identity (Devroye, 2009; James, 2010a):

$$T_{a,b} \stackrel{\mathcal{L}}{=} \frac{K_{a,b}}{G_{1+b\frac{1-a}{a}}^{\frac{1-a}{a}}},$$

where $K_{a,b}$ is a random variate that generalizes K_a. Instead of dealing with $T_{a,b}$ directly, the paper, instead, provides a simple random variate generator for $K_{a,b}$. From the table of Mellin transforms, one can verify that, with a new parametrization that uses $\theta \geq 0$ and $a \in (0,1]$, $\log K_{a,a\theta}^a$ has characteristic function

$$\frac{\Gamma(1+\theta-it)}{\Gamma(1+\theta)} \frac{\Gamma(1+a\theta)}{\Gamma(1+a\theta-ita)} \frac{\Gamma(1+(1-a)\theta)}{\Gamma(1+(1-a)\theta-it(1-a))}$$

which remains the same upon replacement of a by $1-a$. Also, observe that $K_{a,0} \stackrel{\mathcal{L}}{=} K_a$. Numerous other identities flow from this representation, such as:

$$G_{1+\theta} \stackrel{\mathcal{L}}{=} \frac{G_{1+a\theta}^a G_{1+(1-a)\theta}^{1-a}}{K_{a,a\theta}^a},$$

and, with $\theta = 0$,

$$E \stackrel{\mathcal{L}}{=} \frac{E^a {E'}^{1-a}}{K_a^a}.$$

Several classical families of distributions

We briefly survey a host of distributional identities that follow immediately from the Mellin transforms. Let us agree that all random variables mentioned below in expressions are independent, and accented random variables are distributed as, but independent of, unaccented ones.

Beta-gamma calculus. The well-known beta-gamma calculus (see, e.g., Dufresne, 1990) is based on

$$G_a \stackrel{\mathcal{L}}{=} G_{a+b} B_{a,b},$$

which follows without work from above. It falls short, though, of the more useful property that

$$(G_a, G_b) \stackrel{\mathcal{L}}{=} G_{a+b}(B_{a,b}, 1-B_{a,b}).$$

That

$$\frac{G_a}{G_a+G_b} \stackrel{\mathcal{L}}{=} B_{a,b}$$

also falls outside the scope of this methodology. One particularly useful observation is that

$$G_a \stackrel{\mathcal{L}}{=} G_{a+1} B_{a,1} \stackrel{\mathcal{L}}{=} G_{a+1} U^{\frac{1}{a}}.$$

This implies that for gamma random variate generation, the case $a < 1$ can always be reduced to $a > 1$.

Beta calculus. For random variate generation, it is a good start to note that $B_{a,b} \stackrel{\mathcal{L}}{=} 1 - B_{b,a}$ and that

$$B_{1/2,a} \stackrel{\mathcal{L}}{=} (2B_{a,a} - 1)^2.$$

Furthermore, the easiest case is the following, an immediate consequence of the table above:

$$B_{a,1} \stackrel{\mathcal{L}}{=} U^{\frac{1}{a}} \stackrel{\mathcal{L}}{=} 1 - B_{1,a}.$$

We know that $B_{1/2,1/2}$ is an arc sine random variable, thusly named because it is distributed as $\sin^2(\pi U)$ or as $\cos^2(\pi U)$. Its density is $1/(\pi\sqrt{x(1-x)})$ on $(0,1)$. Since $\log(B_{1/2,1/2})$ has characteristic function

$$\frac{\Gamma(1/2 + it)}{\Gamma(1/2)\Gamma(1 + it)},$$

we have $B_{1/2,1/2}E \stackrel{\mathcal{L}}{=} G_{1/2}$. Multiplying by two and taking a square root yields the polar method for a normal random variable (Box and Müller, 1958):

$$|N| = \sqrt{2G_{1/2}} \stackrel{\mathcal{L}}{=} \sqrt{2EB_{1/2,1/2}} \stackrel{\mathcal{L}}{=} \sqrt{2E}\,\sin(\pi U).$$

Taking care of the random sign, we obtain the more classical Box-Müller formula, $N \stackrel{\mathcal{L}}{=} \sqrt{2E}\,\sin(2\pi U)$. Finally, we can easily deduce the chain rule for betas from our characteristic function: for any parameters $a_i > 0$,

$$B_{a_1,a_2} B_{a_1+a_2,a_3} \cdots B_{a_1+\cdots+a_{n-1},a_n} \stackrel{\mathcal{L}}{=} B_{a_1,a_2+\cdots+a_n}.$$

For small integer values of the parameters this leads to identities that can be used for random variate generation. Take for example $B_{2,5}$. By repeated use of the chain rule, we have

$$B_{2,5} \stackrel{\mathcal{L}}{=} B_{2,2}B_{4,3} \stackrel{\mathcal{L}}{=} B_{2,2}\left(1 - B_{3,4}\right) \stackrel{\mathcal{L}}{=} B_{2,2}\left(1 - B_{3,3}B_{6,1}\right) \stackrel{\mathcal{L}}{=} B_{2,2}\left(1 - B_{3,3}U^{\frac{1}{6}}\right).$$

This is a function of five independent uniform random variates. The beta distribution also plays a key role with Pearson VI variates, as

$$P_{a,b} \stackrel{\mathcal{L}}{=} B_{a,c}P_{a+c,b}$$

for all $a, b, c > 0$.

Gamma-stable calculus. Our calculus shows straightforwardly that

$$E^a M_a \stackrel{\mathcal{L}}{=} \left(\frac{E}{S_a}\right)^a \stackrel{\mathcal{L}}{=} E \stackrel{\mathcal{L}}{=} \left(\frac{G_a}{T_{a,a}}\right)^a,$$

a distributional identity that can be found in Chaumont and Yor (2003).

Stable calculus. We rediscover the remarkable identity that was at the basis of Kanter's method for generating unilateral stable random variates, namely

$$S_a^a \stackrel{\mathcal{L}}{=} K_a E^{-(1-a)}, 0 < a < 1.$$

Just as the beta distribution, the stable law has a chain rule that is obvious from the table above: for $a_i \in (0,1)$:

$$S_{a_1 \cdots a_n}^{a_1 \cdots a_n} \stackrel{\mathcal{L}}{=} S_{a_1}^{a_1} \cdot S_{a_2}^{a_1 a_2} \cdot S_{a_3}^{a_1 a_2 a_3} \cdots S_{a_n}^{a_1 \cdots a_n}.$$

Take all a_i's equal to a. Then $S_{a^n}^{a^n}$ can be written as a product of n powers of independent S_a-distributed random variables. this is a remarkable multiplicative property that can be added to the well-known additive properties of the stable law, like, e.g.,

$$S_a \stackrel{\mathcal{L}}{=} n^{-a} \sum_{j=1}^{n} S_a(j)$$

where the $S_a(j)$'s are i.i.d. copies of S_a (see, e.g., Zolotarev, 1985). Since $S_{1/2}$ is inverse gaussian and easy to generate, the chain formula gives quick-and-dirty ways of generating unilateral stable variates when the parameter is an integer power of 1/2.

Properties of the first bfry law. The first bfry law plays a special role because of its many connections with other distributions. Using the well-known identity

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

for z complex, we see that $\log X_a$ has characteristic function

$$\frac{a\Gamma(1-a+it)}{\Gamma(1-a)(a-it)} = \frac{\sin(\pi a)}{\sin(\pi(a-it))} \frac{\Gamma(1+a)}{\Gamma(1+a-it)}.$$

Thus, $\log(X_a G_{1+a})$ has characteristic function

$$\frac{\sin(\pi a)}{\sin(\pi(a-it))}.$$

We conclude the following;

$$X_a G_{1+a} \stackrel{\mathcal{L}}{=} \frac{G_{1-a} G_{1+a}}{U^{1/a}} \stackrel{\mathcal{L}}{=} \frac{G_{1-a}}{G_a} \stackrel{\mathcal{L}}{=} P_{1-a,a} \stackrel{\mathcal{L}}{=} \frac{1}{P_{a,1-a}}.$$

The logistic and Cauchy distributions. The logistic distribution on $\mathbf{R}$ is the distribution of $\log P_{1,1}$. We have the well-known distributional identities:

$$\log P_{1,1} \stackrel{\mathcal{L}}{=} \log\left(\frac{E}{E'}\right) \stackrel{\mathcal{L}}{=} \log\left(\frac{1-U}{U}\right).$$

Its characteristic function is

$$\Gamma(1-it)\Gamma(1+it) = \frac{\pi it}{\sin \pi it} = \frac{2\pi t}{e^{\pi t} - e^{-\pi t}} = \frac{\pi t}{\sinh(\pi t)}.$$

Its distribution function is $1/(1+e^{-x})$. A Cauchy random variable C, in contrast, has density $1/(\pi(1+x^2))$ on the real line. Using either the definition of the Student t distribution or the probability integral transform, we have the identities, using N to denote a generic normal random variable,

$$C \stackrel{\mathcal{L}}{=} T_1 \stackrel{\mathcal{L}}{=} \frac{N}{N'} \stackrel{\mathcal{L}}{=} \tan(\pi U).$$

Thus,

$$C^2 \stackrel{\mathcal{L}}{=} T_1^2 \stackrel{\mathcal{L}}{=} P_{1/2,1/2} \stackrel{\mathcal{L}}{=} \tan^2(\pi U) \stackrel{\mathcal{L}}{=} P_{1/2,1/2} \stackrel{\mathcal{L}}{=} \sin(\pi U')\,\frac{1-U^2}{U^2}.$$

From this, $\log(C^2)$ is easily seen to have characteristic function

$$\frac{\Gamma(1/2+it)\Gamma(1/2-it)}{\Gamma^2(1/2)} = \frac{1}{\sin(\pi(1/2+it))} = \frac{2}{e^{\pi t}+e^{-\pi t}} = \frac{1}{\cosh(\pi t)},$$

which is related to that of the logistic.

Lamperti calculus. By definition, but also from the table above, we have $L_a \stackrel{\mathcal{L}}{=} (S_a/S'_a)^a$. It is easy to verify an identity that can be found, e.g., in James (2010b), Devroye (1996), Jayakumar and Pillai (1996) and Lin (2001):

$$\Delta_{a,a} \stackrel{\mathcal{L}}{=} EL_a^{\frac{1}{a}} \stackrel{\mathcal{L}}{=} E^{\frac{1}{a}} S_a.$$

By the nature of the Lamperti law, the chain rule for the stable distribution remains valid, mutatis mutandis, for the Lamperti law: for any $a_i \in (0,1)$, we have

$$L_{a_1\cdots a_n} \stackrel{\mathcal{L}}{=} L_{a_1} \cdot L_{a_2}^{a_1} \cdot L_{a_3}^{a_1 a_2} \cdots L_{a_n}^{a_1\cdots a_{n-1}}.$$

For example,

$$L_{a^2} \stackrel{\mathcal{L}}{=} L_a {L'}_a^{a}.$$

The Student t-Pearson VI calculus. Since we work with positive random variables, it is convenient to consider the square of the Student t distribution of parameter a. We note, from the definition,

$$\frac{T_a^2}{a} = \frac{(1/2)N^2}{G_{a/2}} \stackrel{\mathcal{L}}{=} \frac{G_{1/2}}{G_{a/2}} \stackrel{\mathcal{L}}{=} P_{1/2,a/2} \stackrel{\mathcal{L}}{=} B_{1/2,1/2}P_{1,a/2} \stackrel{\mathcal{L}}{=} \sin^2(\pi U')\left(\frac{1}{U^{\frac{2}{a}}}-1\right),$$

which gives a one-liner for the Student t distribution due to Bailey (1994):

$$T_a \stackrel{\mathcal{L}}{=} \sqrt{a}\sin(2\pi U)\sqrt{\frac{1}{U^{\frac{2}{a}}}-1}.$$

Since T_a^2 is Pearson VI after rescaling, we recall from the definition of the Pearson VI and the gamma-beta calculus that

$$P_{a,b} \stackrel{\mathcal{L}}{=} \frac{G_a}{G_b} \stackrel{\mathcal{L}}{=} \frac{G_{a+b}B_{a,b}}{G_{a+b}(1-B_{a,b})} = \frac{B_{a,b}}{1-B_{a,b}}.$$

This, in turn, implies that

$$B_{a,b} \stackrel{\mathcal{L}}{=} \frac{P_{a,b}}{1+P_{a,b}}.$$

Returning to the Student distribution, we have

$$\frac{T_a^2}{a} \stackrel{\mathcal{L}}{=} P_{1/2,a/2} \stackrel{\mathcal{L}}{=} \frac{B_{1/2,a/2}}{1-B_{1/2,a/2}},$$

and

$$B_{1/2,a/2} \stackrel{\mathcal{L}}{=} \frac{(T_a^2/a)}{1+(T_a^2/a)}.$$

Furthermore,

$$B_{a,a} \stackrel{\mathcal{L}}{=} \frac{1+S\sqrt{\frac{(T_a^2/a)}{1+(T_a^2/a)}}}{2}.$$

The interest of the last two identities is that the beta random variables on the left can be written, via Bailey's formula for the Student t variate, as functions of two independent uniform random variates, and this formula, unlike the one derived earlier which could only be used for $a > 1/2$, is valid for all values $a > 0$. For other one-liners for the symmetric beta distribution that use two independent uniform random variates: see Ulrich (1994) and Devroye (1984, 1996, 2006). The asymmetric beta with both parameters different from 1 and 1/2 is still difficult to write as a simple function of a few independent uniform random variates. Finally, just as for the gamma, we have a reduction identity:

$$P_{a,b} \stackrel{\mathcal{L}}{=} P_{a+1,b} U^{\frac{1}{a}}.$$

Logistic distribution. The generalized extreme value distribution with parameter $a > 0$ is that of $\log G_a$: it has density

$$\frac{e^{-(a-1)x-e^{-x}}}{\Gamma(a)}, x \in \mathbb{R}.$$

The standard extreme value distribution, or Gumbel distribution, has parameter $a = 1$. The characteristic function of $\log G_a$ is $\Gamma(a+it)/\Gamma(a)$. Using the table above, an entire calculus for extreme value distributions can be obtained, parallelling that for the gamma distribution.

The Riesz-Bessel distribution.

Exponentially tilted, or Esscher transformed (Sato, 1999), random variables have many applications. If $X \geq 0$ is a given random variable with Laplace transform $\mathcal{L}(\lambda)$, then we say that X_μ^* is the exponentially tilted version of X with parameter $\mu \geq 0$ ($\mu < 0$ can be considered as well for small-tailed X) if the ratio of probability measures of dx under X_μ^* and X is equal to $e^{-\mu x}$. Equivalently, X_μ has Laplace transform

$$\frac{\mathcal{L}(\lambda+\mu)}{\mathcal{L}(\mu)}.$$

Random variate generation is classically done by rejection: keep generating pairs (X, U) with U uniform $[0,1]$, until for the first time $U < \exp(-\mu X)$, and return X. However, the expected number of iterations before halting is

$$\frac{1}{\mathbb{P}\{U < \exp(-\mu X)\}} = \frac{1}{\mathbb{E}\{\exp(-\mu X)\}} = \frac{1}{\mathcal{L}(\mu)}.$$

This can be quite inefficient. One can do much better by using special designs. For example, ${(S_a)^*}_\mu$, the exponentially tilted unilateral stable, has Laplace transform

$$\exp\left(\mu^a - (\lambda+\mu)^a\right), \lambda \geq 0.$$

Using the so-called double rejection method, Devroye (2009) describes a random variate generator that is uniformly fast over all values of $\mu \geq 0$ and $\alpha \in (0,1]$.

When describing the Riesz-Bessel-Lévy subordinator, Anh and McVinish (2004) introduce the Riesz-Bessel distribution through its Laplace transform. We say that $R_{\alpha,\gamma,t} \geq 0$ is a Riesz-Bessel random variable if its Laplace transform is

$$\mathbb{E}\left\{e^{-\lambda R_{\alpha,\gamma,t}}\right\} = e^{-t\lambda^\alpha (1+\lambda)^\gamma}, \lambda \geq 0.$$

Here the parameters are $t \geq 0$, $\alpha \in (0,1]$ and γ such that $\alpha + \gamma \in [0,1]$. By checking an identity via Laplace transforms, we see that for $\beta \in (0,1]$,

$$R_{\alpha,\gamma,t^{1/\beta} S_\beta} \stackrel{\mathcal{L}}{=} R_{\alpha\beta,\gamma\beta,t}.$$

Or, reparametrized such $0 < \alpha \leq \beta \leq 1$, $0 \leq \alpha + \gamma \leq \beta \leq 1$,

$$R_{\alpha/\beta,\gamma/\beta,t^{1/\beta} S_\beta} \stackrel{\mathcal{L}}{=} R_{\alpha,\gamma,t}.$$

Anh, McVinish and Pesee (2005) propose various random variate generators for $R_{\alpha,\gamma,t}$. The last identity shows that only the cases $\alpha = 1$ or $\alpha + \gamma = 1$ matter:

(i) When $\gamma \leq 0$, take $\beta = \alpha$, and note that

$$R_{\alpha,\gamma,t} \stackrel{\mathcal{L}}{=} R_{1,\gamma/\alpha,t^{1/\alpha} S_\alpha}.$$

Then sample from the latter law. See below for a uniformly fast generator for $R(1,\gamma,t)$, $\gamma \in (-1,0]$.

(ii) When $\gamma > 0$, take $\beta = \alpha + \gamma$, and note that

$$R_{\alpha,\gamma,t} \stackrel{\mathcal{L}}{=} R_{\alpha/(\alpha+\gamma),\gamma/(\alpha+\gamma),t^{1/(\alpha+\gamma)} S_{\alpha+\gamma}}.$$

Anh, McVinish and Pesee (2005) give an exact generator for $R_{a,1-a,t}$, $a \in (0,1)$, which permits one, by the identity given above, to cover $R_{\alpha,\gamma,t}$ for all $\gamma > 0$. However, their algorithm is cumbersome and not uniformly efficient with respect to the parameter range. The development of a uniformly fast generator for $R_{a,1-a,t}$ remains open.

With infinitely divisible distributions, compound Poisson distributions play an important role. Assume that we have a family of distributions (of X_θ) with characteristic function $(\varphi(t))^\theta$, where $\theta > 0$ is a parameter that makes the infinite divisibility obvious and explicit, and φ is a basic charcateristic function of a random variable $X = X_1$, then the compound Poisson trick consists of replacing θ by a Poisson (b) random variable P_b times a scale factor a. The characteristic function of X_{aP_b} is

$$\mathbb{E}\left\{e^{itX_{aP_b}}\right\} = \sum_{j=0}^{\infty} \frac{b^j e^{-b}}{j!}\,\mathbb{E}\left\{e^{itX_{aj}}\right\} = \sum_{j=0}^{\infty} \frac{(b(\varphi(t))^a)^j e^{-b}}{j!} = e^{b((\varphi(t))^a - 1)}.$$

Similarly, if $X_\theta \geq 0$ has Laplace transform $\mathcal{L}(\lambda)$, then the Laplace transform of X_{aP_b} is

$$\mathrm{E}\left\{e^{-\lambda X_{aP_b}}\right\} = \sum_{j=0}^{\infty} \frac{b^j e^{-b}}{j!} \mathrm{E}\left\{e^{-\lambda X_{aj}}\right\} = \sum_{j=0}^{\infty} \frac{(b(\mathcal{L}(\lambda))^a)^j e^{-b}}{j!} = e^{t((\mathcal{L}(\lambda))^a - 1)}.$$

For example, G_{aP_b} has Laplace transform

$$\exp\left(b\left((1+\lambda)^{-a} - 1\right)\right).$$

This happens to be Laplace transform of $R(1,-1,b)$ when $a = 1$:

$$R_{1,-1,t} \stackrel{\mathcal{L}}{=} G_{P_t}, t \geq 0.$$

The ideas of compound Poisson and exponential tilting can be combined to yield a generator for $R_{1,\gamma,t}$, $\gamma \in (-1, 0]$, as noted by Anh, McVinish and Pesee (2005):

$$R_{1,\gamma,t} \stackrel{\mathcal{L}}{=} G_{-\gamma P_t} + t^{\frac{1}{1+\gamma}} \left(S_{1+\gamma}\right)^*_{t^{\frac{1}{1+\gamma}}}.$$

Indeed, the Laplace transform of the right-hand-side is

$$\begin{aligned} e^{t((1+\lambda)^\gamma - 1)} \times e^{t\left(1-(\lambda+1)^{1+\gamma}\right)} &= e^{t\left((1+\lambda)^\gamma - (\lambda+1)^{1+\gamma}\right)} \\ &= e^{-t\lambda(1+\lambda)^\gamma}. \end{aligned}$$

The more general random variable

$$G_{bP_t} + t^{\frac{1}{a}} \left(S_a\right)^*_{t^{\frac{1}{a}}}$$

has Laplace transform

$$\exp\left(-t\left(\frac{1}{(1+\lambda)^b} - (1+\lambda)^a\right)\right).$$

Open problems

Surely, more properties of the stable laws that are beneficial for its understanding and for simulation lurk just around the corner. Multivariate stable laws need to be looked at in depth as well. Part of the motivation of the paper stems from our quest to find a one-liner for the general gamma and beta distributions. Thus far, we have been unable to find one.

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