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A NOTE ON LINNIK'S DISTRIBUTION

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Abstract: We provide a short proof of the validity and unimodality of Linnik's characteristic function $1/(1+|t|^{\alpha})$, $0<\alpha\leqslant 2$, by noting that it corresponds to the distribution of $S_{\alpha}V^{1/\alpha}$, where S_{α} is a symmetric stable random variable with parameter α, and V is an independent exponential random variable.

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In 1953, Linnik showed that the function

$$\phi(t)=1/(1+|t|^{\alpha})$$

is a valid characteristic function for $\alpha\in(0,2]$. Later, Laha (1961) (see Also Lukacs, 1970, pp. 96–97) showed that the distribution is in fact unimodal. Using the fact that for $\alpha\leqslant 1$, ϕ is a Pólya characteristic function (convex on the positive halfline), Devroye (1986) showed how random variates from this distribution can easily be generated. Recently, Dale Anderson (UC Riverside) asked me how one could quickly generate random variates for all values of α. This led to the following simple observation.

Property. *Let $\alpha\in(0,2]$ and $\beta>0$ be given constants. Let S_{α} be a symmetric stable random variable with characteristic function $\mathrm{e}^{-|t|^{\alpha}}$, and let V_{β} be an independent random variable with density*

$$\exp(-v^{\beta})/\Gamma(1+1/\beta),\quad v>0.$$

Then $X=S_{\alpha}V_{\beta}^{\beta/\alpha}$ has characteristic function

$$\phi(t)=1/(1+|t|^{\alpha})^{1/\beta}.$$

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Proof. Note that

$$\begin{aligned}E(\mathrm{e}^{\mathrm{i}tX})&=E\left(\mathrm{e}^{-|t|^{\alpha}V_{\beta}^{\beta}}\right)\\&=\int_0^{\infty}\frac{\mathrm{e}^{-v^{\beta}-|t|^{\alpha}v^{\beta}}}{\Gamma(1+1/\beta)}\,\mathrm{d}v\\&=1/(1+|t|^{\alpha})^{1/\beta}.\qquad\square\end{aligned}$$

Linnik's distribution. $\beta=1$. The characteristic function identified by Linnik is that of $S_{\alpha}V_1^{1/\alpha}$, where V_1 is exponentially distributed. This property provides a short proof of the validity of Linnik's characteristic function. Since S_{α} is unimodal (Ibragimov and Chernin, 1959; Kanter, 1975), it follows that X is unimodal as well. In general, there exists an infinite peak at the origin if and only if $\alpha\leqslant\beta$. From the representation, we also deduce that the first absolute moment is finite if and only if $\alpha>1$.

Random variate generation. S_{α} can be generated by the method proposed by Chambers, Mallows and Stuck (1976) as

$$\frac{\sin(\alpha U)}{\cos^{1/\alpha}U}\left(\frac{\cos((1-\alpha)U)}{Y}\right)^{(1-\alpha)/\alpha},$$

where U is uniformly distributed on [0, 1] and Y is exponentially distributed and independent of U. Furthermore, V_β has the exponential power distribution, for which several good generators are available (Johnson, 1979; Tadikamalla, 1980). Several other generators are given in Devroye (1986). In any case, V_β is distributed as $UY^{1/\beta}$, where U is uniform on [0, 1] and Y is a gamma random variable with parameter $1 + 1/\beta$. Gamma random variates in turn can be obtained in average time uniformly bounded over the parameters.

Special cases. (A) If Y is Cauchy (characteristic function $e^{-|t|}$) and Z is exponential, then YZ is Linnik with parameters $\alpha = 1$ and $\beta = 1$.

(B) If N is standard normal, and Z is exponential, then $N\sqrt{\frac{1}{2}Z}$ is Linnik with parameters $\alpha = 2$ and $\beta = 1$, which in turn is easily seen to be Laplace.

(C) If N is standard normal, then $S_\alpha(|N|/\sqrt{2})^{2/\alpha}$ has characteristic function $1/\sqrt{1 + |t|^\alpha}$.

(D) It is equally simple to verify that if V is exponential, α_j forms a sequence of numbers from (0, 2], $\gamma_j \geqslant 0$, and S_{α_j} is a sequence of independent symmetric stable random variables with the given parameters, then $\sum_{j=1}^{n} S_{\alpha_j}(\gamma_j V)^{1/\alpha_j}$ has characteristic function

$$1/\left(1 + \sum_{j=1}^{n} \gamma_j |t|^{\alpha_j}\right).$$

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