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Strong Laws for the Maximal k-Spacing When $k \leqq c \log n$

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Summary. We consider the maximal k-spacing $M_n = \max_{0 \leqq i \leqq n+1-k} (U_{ni+k} - U_{ni})$ where $U_{n1} \leqq \ldots \leqq U_{nn}$ are the order statistics of an i.i.d. sample of size n from the uniform distribution on $[0,1]$, and $U_{n0}=0$, $U_{nn+1}=1$. The integer k is allowed to vary with n at a rate not exceeding $\log n$. We obtain laws of the iterated logarithm for all the k's in the given range. For small k, the methods used in the proofs are borrowed from extreme value theory. For larger k, the techniques are reminiscent of those used in the proof of the Erdös-Rényi theorem.

1. Introduction and Results

Let $U_1, \ldots, U_n$ be i.i.d. uniform $[0,1]$ random variables with order statistics $U_{n1} \leqq U_{n2} \leqq \cdots \leqq U_{nn}$, and set $U_{n0}=0$, $U_{nn+1}=1$. We define the *maximal k-spacing* by

$$M_n = \max_{0 \leqq i \leqq n+1-k} (U_{ni+k} - U_{ni})$$

where k is an integer between 1 and n. We will allow integers $k=k_n$ that vary with n, but in any case, k_n is assumed to be nondecreasing. The following strong law of large numbers is known for M_n (see e.g. Mason, 1984):

Theorem 1. A) *If* $k_n = o(\log n)$, *then* $n M_n/\log n \to 1$ *a.s.*

B) *If* $k_n \sim c \log n$ $(0<c<\infty)$, *then* $n M_n/\log n \to c\alpha_c^+$ *a.s. where* α_c^+ *is the unique root greater than* 1 *of*

$$f(\lambda) = \lambda + \log\left(\frac{1}{e\lambda}\right) = \frac{1}{c}.$$

C) *If* $k_n/n \downarrow 0$, $k_n/\log n \to \infty$, $\log\left(\frac{n}{k_n}\right) \Big/ \log\log n \to \infty$ (*these are called the Csörgő-Révész-Stute conditions*), *then*

$$(n M_n - k_n) \Big/ \sqrt{2 k_n \log\left(\frac{n}{k_n}\right)} \to 1 \quad a.s.$$

Theorems of this type are important in the study of the oscillation behavior of the uniform empirical quantile process (Mason, 1984) and the strong uniform consistency of density estimators (Hall, 1981; Stute, 1982a, 1982b; Révész, 1982). Also, in the context of goodness-of-fit tests, it seems better to take either a large constant k or a slowly growing k (del Pino, 1979).

In a series of papers (Devroye (1981, 1982), Deheuvels (1982)), the authors were able to obtain upper and lower class sequences for M_n when $k=1$. For example, limiting ourselves to the first term, we have

$$\limsup_{n\to\infty} (n M_n - \log n)/2 \log\log n = 1 \text{ a.s.} \tag{1}$$

and

$$\liminf_{n\to\infty} (n M_n - \log n)/\log\log\log n = -1 \text{ a.s.} \tag{2}$$

In this paper, we obtain similar results for all k growing at a rate of $\log n$ or slower. In the spirit of the standard law of the iterated logarithm for sums of i.i.d. random variables, we will limit ourselves in most cases to one or two additional asymptotic terms as in (1) or (2). Occasionally, the arguments yield more terms, and we will indicate so in small remarks further on. Our main theorem is

Theorem 2. *When $k\to\infty$, $k=o(\log n)$, then*

$$\frac{n M_n - \log n}{(k-1)\log\left(\frac{e\log n}{k}\right)} \to 1 \textit{ a.s.}$$

Theorem 2 shows us very clearly that $n M_n$ is of the order of $\log n$ plus a "correction term", $(k-1)\log\left(\frac{e\log n}{k}\right)$. Just how far $n M_n - \log n$ is from this correction term is indicated in Theorems 3, 4 and 5, and depends very heavily on the rate of increase of k.

Theorem 3. (Extremely small k.) *If $\log k = o(\log\log n)$, then*

$$\limsup_{n\to\infty} \frac{n M_n - \log n - (k-1)\log\left(\frac{e\log n}{k}\right)}{2\log\log n} = 1 \textit{ a.s.}$$

If $\log k = o(\log\log\log n)$, then

$$\liminf_{n\to\infty} \frac{n M_n - \log n - (k-1)\log\left(\frac{e\log n}{k}\right)}{\log\log\log n} = -1 \textit{ a.s.}$$

Theorem 3 includes the case of fixed k, and can thus be considered as an extension of (1) and (2). The lim sup part of Theorem 3 is valid for all k that vary as $(\log\log n)^p$ for any power $p>0$. The lim inf part is only valid for a more restricted class of k's.

Theorem 4. (Intermediate k.) *If* $k = o(\sqrt{\log n})$, *then*

$$\limsup_{n \to \infty} \frac{n M_n - \log n - (k-1) \log \left(\frac{e \log n}{k}\right)}{2 \log \log n} = 1 + c\, d \quad a.s.$$

where $c \in [-\frac{3}{2}, -\frac{1}{2}]$ *is a constant and* $d = \liminf_{n \to \infty} \frac{\log k}{2 \log \log n}$ (*which is a number between* 0 *and* 1/2). *Also,*

$$\liminf_{n \to \infty} \frac{n M_n - \log n - (k-1) \log \left(\frac{e \log n}{k}\right)}{\log k} = c - d' \quad a.s.$$

where c is as above, and $d' = \limsup_{n \to \infty} \frac{\log \log \log n}{\log k}$ (*which is a number between* 0 *and* ∞ *inclusive, but in the latter case* ($d' = \infty$), *we refer to Theorem* 3).

Theorem 5. (Large k.) *If* $k = o(\log n)$ *and* $k/\sqrt{\log n} \to \infty$, *then*

$$\frac{n M_n - \log n - (k-1) \log \left(\frac{e \log n}{k}\right)}{\left(\frac{k-1}{\log n}\right) (k-1) \log \left(\frac{e \log n}{k}\right)} \to 1 \quad a.s.$$

Theorem 6. (The limit case.) *If* $k = c \log n + o(\log \log n)$, *for some constant* $c > 0$, *then*

$$V_n = \frac{n M_n - (1+a)\, c \log n}{\log \log n}$$

satisfies

$$\limsup_{n \to \infty} V_n = c^* \frac{1+a}{a} \qquad \text{almost surely}$$

and

$$\liminf_{n \to \infty} V_n = -c' \frac{1+a}{a} \qquad \text{almost surely}$$

where a is the unique positive solution of the equation $\exp\left(-\frac{1}{c}\right) = (1+a)\, e^{-a}$, *and* c^* *and* c' *are constants taking values in* $[-\frac{1}{2}, \frac{3}{2}]$ *and* $[-\frac{3}{2}, -\frac{1}{2}]$ *respectively.*

The only case not covered by Theorems 3–6 is when $k \sim c\sqrt{\log n}$ for some constant c. However, what happens in this border case is not difficult to deduce from the various detailed lemmas given below. Theorem 6 is an Erdös-Rényi type theorem (after Erdös and Rényi, 1970) in the style of Deheuvels and Devroye (1983).

Examples. 1. If $k \operatorname{Log} k = o(\operatorname{Log}_2 n)$, where $\log_j$ is the j-times iterated logarithm.

We obtain

$$\limsup_{n\to\infty} \frac{n M_n - \log n - k \log_2 n}{\log_2 n} = 1 \qquad \text{almost surely,}$$

$$\liminf_{n\to\infty} \frac{n M_n - \log n - k \log_2 n}{\log_2 n} = -1 \qquad \text{almost surely.}$$

2. If $k=[(\log n)^\alpha]$, $0<\alpha<1$, from Theorem 2, we conclude that

$$\frac{n M_n - \log n}{(\log n)^\alpha \log_2 n} \to 1-\alpha \qquad \text{almost surely.}$$

3. If $k=[(\log_2 n)^\alpha]$, $\alpha>1$, we have

$$\frac{n M_n - \log n - (\log_2 n)^{1+\alpha}}{(\log_2 n)^\alpha \log_3 n} \to \alpha \qquad \text{almost surely.}$$

The other theorems yield of course finer expansions.

2. Top Bound. Outer Class Sequence

In this section, we will let u_n be a sequence of positive numbers. Our goal here is to show the following:

Theorem 7.

$$P(M_n > u_n \ i.o.) = 0$$

when $k_n = o(\log n)$ *and*

$$u_n = \frac{1}{n}\left(\log n + (k-1)\log\left(\frac{e\log n}{k}\right) + (2+\varepsilon)\log\log n - \tfrac{1}{2}\log(2\pi k) + (1+\delta)\frac{k-1}{\log n}\cdot(k-1)\log\left(\frac{e\log n}{k}\right)\right),$$

where ε, δ *are arbitrary positive numbers.*

From this, we obtain the outer class parts for the "lim sup" results in Theorems 2 through 5, in view of the relative sizes of the terms that appear in the expression for u_n.

Lemma 1. (About the relative sizes of functions of k and n.) *Let* $k=k_n$ *be nondecreasing, and let* $k=o(\log n)$. *Then:*

A) $(k-1)\log\left(\frac{e\log n}{k}\right) = o(\log n)$.

B) $\log\log n = o\left((k-1)\log\left(\frac{e\log n}{k}\right)\right)$ *whenever* $k\uparrow\infty$.

C) $\frac{k-1}{\log n}\cdot(k-1)\log\left(\frac{e\log n}{k}\right) = o\left(k\log\left(\frac{e\log n}{k}\right)\right)$, $=o(\log k) = o(\log\log n)$ (*whenever* $k=o(\sqrt{\log n})$).

D) $\log\log n$ *and* $\log k$ *are both* $o\left(\left(\frac{k-1}{\log n}\right)(k-1)\log\left(\frac{e\log n}{k}\right)\right)$ *when* $k/\sqrt{\log n} \to \infty$.

Proof of Lemma 1. We start with the elementary remark that the function $(u-1)\log\left(\frac{e\log n}{u}\right)$ is strictly increasing for $u \in [1, \log n]$.

Statement A follows from the fact that $\lim_{t\to\infty} \frac{1}{t}\log t = 0$ after taking $t = (e\log n)/k$.

To prove B, consider first any subsequence along which $\log\log n/k$ remains bounded. Then,

$$\frac{\log\log n}{k\log\left(\frac{\log n}{k}\right)} = \frac{\left(\frac{\log\log n}{k}\right)}{\log\left(\frac{\log n}{k}\right)} \to 0$$

in view of $\frac{\log n}{k} \to \infty$, still along this subsequence. If on the other hand $k/\log\log n \to 0$, then

$$\frac{\log\log n}{k\log\left(\frac{\log n}{k}\right)} \sim \frac{\log\log n}{k\log\log n} = \frac{1}{k} \to 0,$$

and we are done.

The first half of C is trivial for all $k \geqq 1$. The second half follows from the observation that

$$\frac{k^2\log\log n}{\log k\cdot\log n} = 2\left(\frac{k^2}{\log k^2}\right)\bigg/\left(\frac{\log n}{\log\log n}\right) \to 0$$

when $k^2/\log n \to 0$. Statement D follows from the fact that

$$\frac{\log\log n\cdot\log n}{k^2\log\left(\frac{\log n}{k}\right)} = 2\left(\frac{\log n}{k}\right)^2 \log^{-1}\left(\left(\frac{\log n}{k}\right)^2\right)\bigg/\left(\frac{\log n}{\log\log n}\right) \to 0$$

when $\left(\frac{\log n}{k}\right)^2 \bigg/ \log n = (\log n)/k^2 \to 0$.

Next, we need a few elementary lemmas:

Lemma 2. $((U_{n\,i+1} - U_{ni}),\ 0 \leqq i \leqq n)$ *are distributed as* $\left(\left(E_i \bigg/ \sum_{j=0}^{n} E_i\right),\ 0 \leqq i \leqq n\right)$ *where* $E_0, E_1, \ldots, E_n$ *are i.i.d. exponential random variables.*

Lemma 3. *If* $[A_n, B_n]$ *is the unique (a.s.) interval such that* $B_n - A_n = M_n$, *then the event* $[U_{n+1} \notin [A_n, B_n]]$ *implies* $[M_{n+1} \geqq M_n]$.

Lemma 4. *Let* u_n *be a nonincreasing sequence of positive numbers. Then* $P(M_n > u_n \text{ i.o.}) = 0$ *when*

(i) $P(M_n > u_n) \to 0$; *and*

(ii) $\sum_n u_n P(M_n > u_n) < \infty$.

Proof of Lemma 4. By a lemma of Barndorff-Nielsen (1961) (see also Devroye, 1981), (i) and (iii) below suffice for $P(M_n > u \text{ i.o.}) = 0$:

(iii) $\sum_n P(M_{n+1} \leqq u_{n+1}, M_n > u_n) < \infty$.

But the n-th term in (iii) does not exceed

$$P(U_{n+1} \in [A_n, A_n + 2u_n], M_n > u_n) \leqq 2u_n P(M_n > u_n).$$

In what follows, we will need the probability in the tail of the gamma density,

$$\psi_k(x) = \int_x^\infty \frac{u^{k-1} e^{-u}}{(k-1)!} du, \quad \text{integer } k.$$

Lemma 5. (Large deviation result for the gamma density.) *If y_n is a sequence of real numbers, and $k = k_n$ is a sequence of integers, and if $\lim_{n \to \infty} y_n/k = \infty$, then*

$$\psi_k(y_n) \sim \frac{y_n^{k-1} e^{-y_n}}{(k-1)!}.$$

Proof of Lemma 5. For $k < y_n$, the following inequality is valid:

$$1 \leqq \frac{\int_{y_n}^\infty u^{k-1} e^{-u} du}{y_n^{k-1} e^{-y_n}} \leqq \frac{1}{1 - \frac{k}{y_n}},$$

from which the lemma follows trivially.

Lemma 6. *Let $k = k_n$ be a sequence of integers with $k = o(\log n)$, let z_n be a sequence of real numbers with $|z_n| = o(\log n)$, and let u_n be defined by*

$$u_n = \frac{1}{n}\left(\log n + (k-1)\log\left(\frac{e \log n}{k}\right) + z_n\right).$$

Then

$$\log(n\psi_k(n u_n)) = 1 + o(1) - z_n - \tfrac{1}{2}\log(2\pi k)$$
$$+ (k-1)\log\left(1 + \left(z_n + (k-1)\log\left(\frac{e\log n}{k}\right)\right) \Big/ \log n\right), \quad \text{if } k \uparrow \infty,$$

and if $k \uparrow K < \infty$, we need only replace $1 - \frac{1}{2}\log(2\pi k)$ *by* $R(K) = (K-1)\log\left(\frac{K}{e}\right) - \log(K-1)!$.

For example, if $\varepsilon > 0$, $0 \leqq \eta_n = o\left(k \log\left(\frac{\log n}{k}\right)\right)$, and $k \uparrow \infty$, then the choice

$$z_n = \eta_n - \tfrac{1}{2}\log(2\pi k) + (1+\varepsilon)\frac{k-1}{\log n}(k-1)\log\left(\frac{e \log n}{k}\right)$$

gives

$$\log(n\psi_k(n u_n)) \leqq 2-\eta_n, \quad \text{all } n \text{ large enough.}$$

Proof of Lemma 6. By Lemma 5, $\log(n\psi_k(n u_n))$ is equal to $o(1)$ plus

$$\begin{aligned}
&\log n+(k-1)\log\left(\frac{e\log n}{k}\right)-(k-1)\log\left(\frac{k}{e}\right)\\
&\quad+(k-1)\log\left(1+\left(z_n+(k-1)\log\left(\frac{e\log n}{k}\right)\right)\Big/\log n\right)\\
&\quad-\log n-(k-1)\log\left(\frac{e\log n}{k}\right)-z_n-\log(k-1)!
\end{aligned}$$

which gives us our result if we can show that $R(k)=(k-1)\log\left(\frac{k}{e}\right)-\log(k-1)!$ $=1-\frac{1}{2}\log(2\pi k)+o(1)$ as $k\to\infty$, and this follows directly from Stirling's approximation and the fact that $(k-1)\log\left(\frac{k}{k-1}\right)=1+o(1)$.

The second part of Lemma 6 is obtained by simply replacing z_n in the asymptotic expression for $\log(n\psi_k(n u_n))$, which now becomes

$$\begin{aligned}
&1+o(1)-\eta_n+(k-1)\left[\log\left(1+\frac{z_n}{\log n}+\frac{k-1}{\log n}\log\left(\frac{e\log n}{k}\right)\right)\right.\\
&\quad\left.-(1+\varepsilon)\frac{k-1}{\log n}\log\left(\frac{e\log n}{k}\right)\right]\leqq 1+o(1)-\eta_n
\end{aligned}$$

for all n large enough. Here we used the fact that for $a, b\in R$, $\varepsilon>0$, $\log(1+a+b)$ $-(1+\varepsilon)b\leqq a-\varepsilon b$ and that $\eta_n=o\left(k\log\left(\frac{\log n}{k}\right)\right)$.

Lemma 7.

$$P(M_n>u_n)\leqq\exp\left(-\frac{\sqrt{n}}{2}\right)+n\psi_k(u_n(n-n^{3/4})), \quad \textit{any } u_n\geqq 0.$$

Proof of Lemma 7. Using the notation of Lemma 2, and the symbol G_i for a gamma (i) distributed random variable, we have

$$\begin{aligned}
P(M_n>u_n)&\leqq P\left(\sum_{i=0}^{n}E_i\leqq n-n^{3/4}\right)+P\left(\operatorname*{Max}_{0\leqq i\leqq n+1-k}\sum_{j=i}^{i+k-1}E_j>u_n(n-n^{3/4})\right)\\
&\leqq P\left(\frac{G_n-n}{n}\leqq -n^{-1/4}\right)+n\,P(G_k>u_n(n-n^{3/4}))\\
&\leqq\exp(-\tfrac{1}{2}\sqrt{n})+n\psi_k(u_n(n-n^{3/4})),
\end{aligned}$$

by an inequality for the tail of the gamma distribution (see e.g. Devroye (1981, Lemma 3.1)).

Proof of Theorem 7. By Lemmas 4 and 7, we must only show the following:

(i) $n\psi_k(u_n(n-n^{3/4}))\to 0$;

(ii) $\sum_n n u_n \psi_k(u_n(n-n^{3/4}))<\infty$.

If we write $u_n(\varepsilon)$ to make the dependence upon ε explicit, then it is easy to verify that $u_n(\varepsilon)(n-n^{3/4})\geqq u_n\left(\frac{\varepsilon}{2}\right)n$ for all n large enough. Indeed, the difference between left hand side and right hand side is $\frac{\varepsilon}{2}\log\log n-0(\log n/n^{1/4})$. Now, to bound $\psi_k\left(n u_n\left(\frac{\varepsilon}{2}\right)\right)$, we use Lemma 6 with the following formal replacement:

$$z_n=\left(2+\frac{\varepsilon}{2}\right)\log\log n-\tfrac{1}{2}\log(2\pi k)+(1+\delta)\frac{k-1}{\log n}(k-1)\log\left(\frac{e\log n}{k}\right).$$

In case $k\uparrow\infty$, we have for n large enough,

$$\psi_k\left(n u_n\left(\frac{\varepsilon}{2}\right)\right)\leqq\frac{1}{n}\exp\left(2-\left(2+\frac{\varepsilon}{2}\right)\log\log n\right).$$

Clearly (i) is satisfied. For (ii), we note that $nu_n\sim\log n$, so that we need only verify the summability of

$$\frac{1}{n}\log n\cdot\exp\left(\left(2+\frac{\varepsilon}{2}\right)\log\log n\right)=\frac{1}{n(\log n)^{1+\varepsilon/2}},$$

and we are done. If $k\uparrow K<\infty$, a similar argument implies (i) and (ii) also.

3. Bottom Bound. Outer Class Sequence

The main result of this section is:

Theorem 8.

$$P(M_n<u_n\ i.o.)=0$$

when $k=o(\log n)$ *and*

$$u_n=\frac{1}{n}\left(\log n+(k-1)\log\left(\frac{e\log n}{k}\right)+(1-\delta)\frac{k-1}{\log n}(k-1)\log\left(\frac{e\log n}{k}\right)\right.$$
$$\left.-\tfrac{1}{2}\log(2\pi k)-\log k-\log\log\log n-\varepsilon\right)$$

where $\delta>0$ *is an arbitrary positive number, and* ε *is a constant greater than* $\varepsilon_0=\log 2-\frac{2}{3}$. *If* $k\uparrow\infty$, *we can take* $\varepsilon_0=\log 2-1$.

From this, we obtain directly the outer class parts for the "lim inf" results in Theorems 2 through 5. (See also Lemma 1 for the relative sizes of the terms involved.) Assume for the time being that Theorem 8 is proved. In view of Lemma 1, we have thus obtained Theorems 2 and 5 as immediate corollaries of Theorems 7 and 8, since for these results inner class sequences are not needed.

To prove Theorem 8, we start with some fundamental inequalities for key probabilities:

Lemma 8. *Let $u_n \in R$. Then*

$$P(M_n < u_n) \leqq \exp\left(-\frac{1}{4}\sqrt{n}\right) + \exp\left(-\left[\frac{n}{k}\right] \psi_k(u_n(n+n^{3/4}))\right), \qquad n \geqq 16.$$

If also $u_n \geqq u_{n+1}$, then

$$\begin{aligned} &P(M_{n+1} < u_{n+1}, M_n \geqq u_n) \\ &\quad \leqq 2u_{n+1}\left[\exp\left(-\frac{1}{4}\sqrt{n}\right) + \left(1 + \left[\frac{n}{k}\right] \psi_k(u_n(n+n^{3/4}))\right) e^{-\left(\left[\frac{n}{k}\right]-1\right)\psi_k(u_n(n+n^{3/4}))}\right], \end{aligned}$$

for $n \geqq 16$, where $[u]$ is the largest integer not exceeding u.

Proof of Lemma 8. In the notation of Lemma 2,

$$P(M_n < u_n) \leqq P\left(\sum_{i=0}^{n} E_i \geqq n + n^{3/4}\right) + P\left(\max_{0 \leqq i \leqq [n/k]-1} G_i < u_n(n+n^{3/4})\right),$$

where $G_0, \ldots, G_i, \ldots$ are i.i.d. gamma(k) random variables. By an inequality found in Devroye (1981, Lemma 3.1), if G^*_{n+1} is a gamma($n+1$) random variable,

$$\begin{aligned} P(G^*_{n+1} \geqq n + n^{3/4}) &= P\left(\frac{G^*_{n+1} - (n+1)}{n+1} \geqq \frac{n^{3/4} - 1}{n+1}\right) \\ &\leqq \exp\left(-\tfrac{1}{2}(n^{3/4}-1)^2 (n+1)^{-1}\left(1 - \frac{n^{3/4}-1}{n+1}\right)\right) \\ &\leqq \exp(-\tfrac{1}{4}\sqrt{n}), \qquad n \geqq 16. \end{aligned}$$

Furthermore, the second term in our upper bound is exactly equal to

$$(1 - \psi_k(u_n(n+n^{3/4})))^{\left[\frac{n}{k}\right]} \leqq \exp\left(-\left[\frac{n}{k}\right] \psi_k(n+n^{3/4}))\right).$$

The second inequality in Lemma 8 can be obtained as follows: let S_i be the i-th uniform k-spacing determined by $X_1, \ldots, X_n$, $i = 0, 1, \ldots, n+1-k$. Then

$$\begin{aligned} &P(M_{n+1} < u_{n+1}, M_n \geqq u_n) \\ &\quad \leqq P(X_{n+1} \in [A_n, B_n],\ B_n - A_n \leqq 2u_{n+1},\ \text{second largest of} \\ &\qquad S_0, S_k, S_{2k}, \ldots \text{ is } < u_{n+1}) \\ &\quad \leqq 2u_{n+1} P(\text{second largest of } S_0, S_k, \ldots \text{ is } < u_n) \\ &\quad \leqq 2u_n \left(P\left(\sum_{i=0}^{n} E_i \geqq n + n^{3/4}\right) + P(\text{Second largest of } G_1, \ldots, G_{\left[\frac{n}{k}\right]} \text{ is} \right. \\ &\qquad\qquad \left. < u_n(n+n^{3/4}))\right) \\ &\quad \leqq 2u_n \left(\exp(-\tfrac{1}{4}\sqrt{n}) + (1-\psi_k)^{\left[\frac{n}{k}\right]} + \left[\frac{n}{k}\right] \psi_k (1-\psi_k)^{\left[\frac{n}{k}\right]-1}\right), \qquad n \geqq 16, \end{aligned}$$

where $\psi_k = \psi_k(u_n(n+n^{3/4}))$. We obtain the desired result now by elementary observations.

Lemma 9. *Let u_n be a nonincreasing sequence of positive numbers, such that $u_n \sim \frac{\log n}{n}$. Then, if for some $\varepsilon > 0$, and all n large enough,*

$$\left[\frac{n}{k}\right] \psi_k(u_n(n+n^{3/4})) \geqq (2+\varepsilon) \log\log n,$$

we have $P(M_n < u_n \text{ i.o.}) = 0$.

Proof of Lemma 9. By Lemma 8, for n large

$$P(M_n < u_n) \leqq \exp(-\tfrac{1}{4}\sqrt{n}) + \exp(-(2+\varepsilon)\log\log n) \to 0.$$

Also, by the monotonicity of the function $(1+u)e^{-u}$, $u \geqq 0$, we have

$$\begin{aligned} &P(M_{n+1} < u_{n+1}, M_n \geqq u_n) \\ &\quad \leqq 3\frac{\log n}{n} \exp\left(-\frac{1}{4}\sqrt{n}\right) + 3\frac{\log n}{n}(1+(2+\varepsilon)\log\log n)\, e^{1-(2+\varepsilon)\log\log n} \end{aligned}$$

for n large enough. But since this upper bound is summable in n, we have $\sum_n P(M_{n+1} < u_{n+1}, M_n \geqq u_n) < \infty$. Thus, once again we may employ Barndorff-Nielsen's Borel-Cantelli type Theorem and conclude that

$$P(M_n < u_n \text{ i.o.}) = 0.$$

Proof of Theorem 8. We verify the condition of Lemma 9. First, as in the proof of Theorem 7, it suffices to look at $\frac{n}{k}\psi_k(nu_n)$. Assume first that $k \uparrow \infty$. Then, in Lemma 6, the probability $\psi_k(nu_n)$ can be estimated by setting

$$z_n = (1-\delta)\frac{k-1}{\log n}(k-1)\log\left(\frac{e\log n}{k}\right) - \tfrac{1}{2}\log(2\pi k) - \log k - \log\log\log n - \varepsilon,$$

so that

$$\begin{aligned} \log\left(\frac{n}{k}\psi_k(nu_n)\right) &= 1 + o(1) + \varepsilon + \log\log\log n \\ &\quad + (k-1)\left(\log\left(1 + \frac{z_n}{\log n} + \frac{b_n}{\log n}\right) - (1-\delta)\frac{b_n}{\log n}\right) \end{aligned}$$

where

$$b_n = (k-1)\log\left(\frac{e\log n}{k}\right).$$

Now, we are done if we can show that for all n large enough,

$$\log\left(1 + \frac{z_n + b_n}{\log n}\right) - (1-\delta)\frac{b_n}{\log n} \geqq 0.$$

But clearly, by Lemma 1, the first term in this sum $\sim \frac{z_n+b}{\log n}$, so that the difference $\sim (z_n+\delta b_n)/\log n$. (Since $z_n=o(b_n)$ and $b_n=o(\log n)$, we need not consider higher order terms to obtain a correct asymptotic expansion.) This concludes the proof of Theorem 8 for the case $k \uparrow \infty$. When $k=K<\infty$, we note that the $1+\varepsilon$ in the asymptotic expression for $\log\left(\frac{n}{k}\psi_k(n u_n)\right)$ must be replaced by $R(K)+\frac{1}{2}\log(2\pi K)+\varepsilon$. For this to be greater than $\log 2$, we need a universal lower bound for $R(K)+\frac{1}{2}\log(2\pi K)$: for $K=1$, we have $\frac{1}{2}\log(2\pi)>\frac{2}{3}$. For $K>1$, a lower bound is given by $(K-\frac{1}{2})\log\left(\frac{K}{K-1}\right)-\frac{1}{12(K-1)}$. But since $\log\left(1+\frac{1}{K-1}\right)\geqq\frac{1}{K}$, we obtain the weaker lower bound $1-(2K)^{-1}-(12(K-1))^{-1}\geqq 1-4^{-1}-12^{-1}=\frac{2}{3}$. Thus, in all cases, the choice $\varepsilon>\log 2-\frac{2}{3}$ will do.

4. Bottom Bound. Inner Class Sequence

We have already proved Theorems 2 and 5. Thus, there is no harm in assuming from now on that $k=0(\sqrt{\log n})$. Using the fact that $P(M_n<u_n)\to 1$ implies that $P(M_n<u_n\ i.o.)=1$, and reconsidering the proof of Theorem 7, we notice that $P(M_n<u_n\ i.o.)=1$ for the following sequence u_n:

$$u_n=\frac{1}{n}\left(\log n+(k-1)\log\left(\frac{e\log n}{k}\right)-\tfrac{1}{2}\log(2\pi k)\right.$$
$$\left.+(1+\delta)\frac{k-1}{\log n}(k-1)\log\left(\frac{e\log n}{k}\right)+\eta_n\right)$$

where $\eta_n\to\infty$, and $\delta>0$ is arbitrary. This, together with Theorem 8 is strong enough to obtain the second half of Theorem 4 at least for the case $\log\log\log n=o(\log k)$, $k=o(\sqrt{\log n})$. But because we want to see the "$\log\log\log n$" term in u_n, just as in Theorem 8, in the hope to cover both Theorems 3 and 4 simultaneously, we are forced to use a more sophisticated argument. We will follow the "small k" technique developed in Devroye (1981).

We begin with a useful inequality.

Lemma 10. *Let $X_1,\ldots,X_n$ be independent random variables, and let $J_1,\ldots,J_m$ be subsets of $\{1,\ldots,n\}$, where m is another integer. Then, for all integers $k_1,\ldots,k_m$,*

$$P\Big(\sum_{i\in J_1}X_i\geqq k_1,\ldots,\sum_{i\in J_m}X_i\geqq k_m\Big)\geqq\prod_{j=1}^{m}P\Big(\sum_{i\in J_j}X_i\geqq k_j\Big).$$

Proof of Lemma 10. We argue by induction on n. For $n=1$, it is clear that for any m

$$P\left(\bigcap_{j=1}^{m}[X_1\geqq k_j]\right)\geqq\prod_{j=1}^{m}P(X_1\geqq k_j).$$

Assuming that the inequality is valid for $n-1$, we need to show that for all $1 \leqq l \leqq m$ and all m,

$$P(X_n + \sum_{i \in J_1} X_i \geqq k_1, \ldots, X_n + \sum_{i \in J_l} X_i \geqq k_l, \sum_{i \in J_{l+1}} X_i \geqq k_{l+1}, \ldots, \sum_{i \in J_m} X_i \geqq k_m)$$
$$\geqq \prod_{j=l+1}^{m} P(\sum_{i \in J_j} X_i \geqq k_j) \prod_{j=1}^{l} P(X_n + \sum_{i \in J_j} X_i \geqq k_j)$$

(which we shall call inequality *) and this, for all possible subsets J_j, $1 \leqq j \leqq m$, of $\{1, \ldots, n-1\}$. But, by conditioning on n and using our induction hypothesis, we see that the left hand side of (*) is at least equal to

$$\prod_{j=l+1}^{m} P(\sum_{i \in J_j} X_i \geqq k_j) E\left(\prod_{j=1}^{l} P(\sum_{i \in J_j} X_i \geqq k_j - X_n | X_n)\right).$$

The random variables in $\prod_{j=1}^{l}$ are all of the form $f_j(X_n)$ where the f_j's are nonnegative nondecreasing functions. But, by Gurland's inequality,

$$E\left(\prod_{j=1}^{l} f_j(X_n)\right) \geqq \prod_{j=1}^{l} E(f_j(X_n)).$$

Now, taking expectations again gives the right hand side of (*). This concludes the proof of Lemma 10.

This lemma allows us to obtain the following rather crude lower bound for $P(M_n < u_n)$:

Lemma 11. *For $u > 0$ and $\lambda \in (0, 1)$, we have*

$$P(M_n < u) \geqq (1 - \psi_k(n u(1-\lambda)))^{n+1} - \exp(-n \lambda^2/2).$$

If $\lambda = \lambda_n$, $k = k_n$ and $u = u_n$ are such that $\lambda \in (0, 1)$, all n, and

$$\text{(i)} \quad n \psi_k^2(n u(1-\lambda)) \to 0;$$
$$\text{(ii)} \quad \psi_k(n u(1-\lambda))/\lambda^2 \to 0,$$

then

$$P(M_n < u_n) \geqq (1 + 0(1)) \exp(-n \psi_k(n u(1-\lambda))).$$

Proof of Lemma 11. We will use Lemma 2 again, together with its notation. Clearly, if $J_j \leqq \{j, j+1, \ldots, j+k-1\}$, then

$$P(M_n < u) \geqq P(\operatorname*{Max}_{\substack{0 \leqq j \\ j \leqq n+1-k}} \sum_{i \in J_j} E_i < u n(1-\lambda)) - P\left(\sum_{i=0}^{n} E_i \leqq n(1-\lambda)\right)$$
$$\geqq \prod_{j=0}^{n+1-k} P(\sum_{i \in J_j} E_i < u n(1-\lambda)) - \exp(-\tfrac{1}{2} n \lambda^2)$$
$$= (1 - \psi_k(u n(1-\lambda)))^{n+2-k} - \exp(-\tfrac{1}{2} n \lambda^2)$$

which gives us the top inequality. The second part of Lemma 11 follows from the first part after noting that $\log(1+z)=z+0(z^2)$ as $z \to 0$.

The main result of this section can now be announced as follows:

Theorem 9. *If* $k=o(\sqrt{\log n})$, *and*

$$u_n=\frac{1}{n}\left(\log n+(k-1)\log\left(\frac{e\log n}{n}\right)+1-\log\log\log n-(\tfrac{1}{2}-\varepsilon)\log(2\pi k)\right), \qquad \varepsilon>0,$$

then

$$P(M_n<u_n \text{ i.o.})=1.$$

Theorem 9 can of course be improved for fixed k, and the constant 2 is chosen for practical convencience only. Nevertheless, even in this crude form, it is powerful enough to give us the "lim inf" parts of Theorems 3 and 4, when considered together with the outer class result of Theorem 8.

Proof of Theorem 9. We employ the technique for k fixed of Devroye (1981), that is, we show that

$$P(M^*_{n_i}<u_{n_i} \text{ i.o. (in } i))=1$$

where $M^*_{n_i}$ is the maximal k-spacing formed by $X_{n_{i-1}+1}, \ldots, X_{n_i}$, and $1 \leqq n_1<n_2<\ldots$ is an increasing sequence of integers. We will choose $n_i=[\exp(2i\log i)]$. (Technical note: the reason for choosing such a sequence, roughly speaking, is that n_i-n_{i-1} and n_i are of the same order of magnitude, so "forgetting" a subsample of size n_{i-1} does not matter too much.)

Since we need only verify that

$$\sum_{i=1}^{\infty} P(M^*_{n_i}<u_{n_i})=\infty,$$

it is important to have a reasonable lower bound for the i-th probability. The lower bound of Lemma 11 will be used here with

$$k=k_{n_i}, \qquad u=u_{n_i}, \qquad \lambda=(n_i-n_{i-1})^{-1/4}, \qquad n=n_i-n_{i-1}.$$

By elementary calculus, we verify first that

$$n_i/n_{i-1} \sim (e\,i)^2, \qquad (n_i-n_{i-1})=n_i\left(1-\frac{1+o(1)}{e^2 i^2}\right)$$

(see also Devroye (1981, Lemma 5.2)). If we can show now that for all i large enough,

$$(n_i-n_{i-1})\,\psi_{k_i}((n_i-n_{i-1})\,u_{n_i}(1-(n_i-n_{i-1})^{-1/4})) \leqq \log i,$$

then obviously, (i) and (ii) of Lemma 11 are satisfied, so that the last inequality of Lemma 11 can be used. But this gives a lower bound of $(1+0(1))\exp(-\log(i))$ for the i-th probability, and this is not summable in i.

The argument of ψ_{k_i} is

$$n_i u_{n_i}\left(1-\frac{1+0(1)}{e^2 i^2}\right)\left(1-\frac{1+0(1)}{e^{i\log i/2}}\right)=n_i u_{n_i}(1-(1+0(1))(e\,i)^{-2}).$$

This can be rewritten as

$$\log n_i+(k_i-1)\log\left(\frac{e\log n_i}{k_i}\right)-\log\log\log n_i+1$$
$$-(\tfrac{1}{2}-\varepsilon)\log(2\pi k_i)-(1+0(1))\frac{2i\log i}{e^2 i^2}.$$

Assume first that $k+\infty$, so that we may apply the first estimate of Lemma 6. We have, omitting the argument of ψ_{k_i},

$$(n_i-n_{i-1})\psi_{k_i}(.)\sim n_i\psi_{k_i}(.)\sim\exp(1-\varepsilon\log(2\pi k_i)+\log\log\log n_i-1+o(1)$$
$$+(k_i-1)\log(1+v_i/\log n_i))$$

where $v_i=(k_i-1)\log\left(\frac{e\log n_i}{k_i}\right)-\log\log\log n_i+1+o(1)-(\frac{1}{2}-\varepsilon)\log(2\pi k_i)$. By adding $\log\log\log n_i$ to v_i, we obtain an upper bound for our expression. We check quickly that $v_i=o(\log n_i)$ and that $k_i v_i/\log n_i=o(\log k_i)$ (here we need the fact that $k=o(\sqrt{\log n})$). Thus, an upper bound for our expression is

$$\exp(o(1)-(\varepsilon+o(1))\log(2\pi k_i)+\log\log\log n_i)\sim(\log i)\cdot e^{-(\varepsilon+o(1))\log(2\pi k_i)},$$

and we are done. When $k=K<\infty$, exactly the same upper bound is valid. To see this, note that the quantity $R(k)+\frac{1}{2}\log(2\pi k)\leqq 1$ for all k (see Lemma 6 for the definition of $R(k)$, and use the fact that $\log(k-1)!\geqq(k-1)\log\left(\frac{k-1}{e}\right)$ $+\frac{1}{2}\log(2\pi(k-1))$, $k>1$, and that $\log(1+z)\leqq z$, to obtain the bound $(k-\frac{1}{2})/(k-1)\leqq 1$, $k>1$).

5. Top Bound. Inner Class Sequence

Theorem 10. *Assume that* $k=o(\sqrt{\log n})$ *and that*

$$u_n=\frac{1}{n}\left(\log n+(k-1)\log\left(\frac{e\log n}{k}\right)+(2-\varepsilon)\log\log n-\tfrac{1}{2}\log(2\pi k)-\log(k)\right)$$

for some $\varepsilon>0$. *Then*

$$P(M_n>u_n\ i.o.)=1.$$

Theorem 10 together with Theorem 7 imply the "lim sup" results found in Theorems 3 and 4. In the proof we make heavy use of the fact that k changes very infrequently.

Proof of Theorem 10. Consider a strictly increasing sequence of integers n_i. We will show that $P(M_{n_i}>u_{n_i}\ i.o.)=1$ by a technique that could be called "bridging between the lim inf and lim sup bounds" (see Devroye, 1981, for another example of this technique). Let us define

$$v_n = \frac{1}{n}\left(\log n + (k-1)\log\left(\frac{e\log n}{k}\right) - \tfrac{1}{2}\log(2\pi k) - \log k - \log\log\log n - 1\right),$$

and recall from Theorem 8 that $M_n > v_n$ *f.o.* almost surely.

Now, if $v_{n_i} \geqq u_{n_{i+1}}$ for all i large enough, then we are done if $P(C_i$ *i.o.* (in i)) $=1$ where

$$C_i = \bigcap_{j=n_i+1}^{n_{i+1}} [X_j \notin [A_{j-1}, A_{j-1} + u_j]].$$

But, by the conditional independence version of the Borel-Cantelli lemma, this is true if

$$\sum_{i=1}^{\infty} P(C_i) = \infty.$$

In view of

$$\begin{aligned} P(C_i) = (1-u_{n_{i+1}})^{n_{i+1}-n_i} &\geqq \exp(-u_{n_{i+1}}(n_{i+1}-n_i) - u_{n_{i+1}}^2(n_{i+1}-n_i)) \\ &\sim \exp(-u_{n_{i+1}})(n_{i+1}-n_i)) \end{aligned}$$

for i large enough, if

$$\log^2 n_{i+1} \cdot \frac{n_{i+1}-n_i}{n_{i+1}^2} \to 0,$$

we see that it is only necessary to find a subsequence n_i verifying the following conditions:

(i) $v_{n_i} \geqq u_{n_{i+1}}$, all i large enough;

(ii) $\log^2 n_{i+1} = o(n_{i+1}^2/(n_{i+1}-n_i))$;

(iii) $\sum_{i=1}^{\infty} \exp(-u_{n_{i+1}}(n_{i+1}-n_i)) = \infty$.

In our hunt for such a subsequence, the sequence $n_i = [\exp(\sqrt{2i\log i})]$ seems to fulfill the role, were it not for the fact that (i) is not satisfied when k increases in the i-th block between n_i and n_{i+1}. To circumvent this problem, we collect all integers i for which $k_{n_{i+1}} = k_{n_i}$ in a set G, and all other integers in its complement, G^c. Since (ii) is obviously satisfied, we see that we are done if (iv)–(vi) hold:

(iv) $v_{n_i} \geqq u_{n_{i+1}}$, all i large enough, $i \in G$;

(v) $u_{n_{i+1}}(n_{i+1}-n_i) \leqq \log i + 0(1)$;

(vi) $\sum_{i \in G} \frac{1}{i} = \infty$.

We will proceed in reverse order, starting with (vi). Let N_m be the number of integers $i \leqq m$ in G^c. Then,

$$\sum_{i \in G} \frac{1}{i} \geqq \liminf_{m\to\infty} \sum_{N_m}^{m} \frac{1}{i} = \infty$$

when $\log(m/N_m) \to \infty$, i.e. when $N_m = o(m)$. Now, by the monotonicity of k,

$$N_m \leqq k_{n_{m+1}} = 0(\log(n_{m+1})) = 0(\sqrt{m \log m}),$$

and (vi) follows.

By a simple exercise in analysis, we see that $n_{i+1} - n_i \sim n_i \sqrt{\frac{\log i}{2i}}$, and thus that $n_{i+1} \sim n_i$. We note that (v) is nearly proved since

$$u_{n_{i+1}}(n_{i+1} - n_i) \sim \frac{\log n_{i+1}}{n_{i+1}} n_i \cdot \sqrt{\frac{\log i}{2i}} \sim \left(\frac{2i \log i \cdot \log i}{2i}\right)^{1/2} = \log i.$$

Unfortunately, this is not good enough, and we are forced to bound things with more care. First, by a standard Taylor series bounding technique, we have

$$n_{i+1} - n_i \leqq 1 + \frac{n_{i+1}}{\sqrt{2i \log i}} (1 + \log i).$$

Also, for i large enough,

$$\begin{aligned} u_{n_{i+1}} \cdot n_{i+1} &\leqq \log n_{i+1} + 2\sqrt{\log n_{i+1}} \log\log n_{i+1} \\ &\leqq \log n_{i+1} + 2(3i \log i)^{1/4} \cdot \log i. \end{aligned}$$

Combining this, (v) follows if

$$\frac{1 + \log i}{\sqrt{2i \log i}} (\log n_{i+1} + i^{1/3}) \leqq \log i + 0(1).$$

But $\log n_{i+1} \leqq \sqrt{2(i+1)\log(i+1)} = \sqrt{2i \log i} \cdot \left(1 + \frac{1}{i}\right)^{1/2} \left(\frac{\log(i+1)}{\log i}\right)^{1/2}$. The $i^{1/3}$ contribution to our product is $o(1)$, and the contribution of the factor 1 in "1 $+ \log i$" is 0(1). Thus, we need only show that

$$\log i \cdot \left(1 + \frac{1}{i}\right)^{1/2} \left(\frac{\log(i+1)}{\log i}\right)^{1/2} \leqq \log i + 0(1).$$

This follows from the inequalities $\left(1 + \frac{1}{i}\right)^{1/2} \leqq 1 + \frac{1}{2i}$ and $\frac{\log(i+1)}{\log i} \leqq 1 + \frac{1}{i}$. This concludes the proof of (v).

For the proof of (iv), we note that $n_{i+1}/n_i = 1 + (1 + o(1)) \left(\frac{\log i}{2i}\right)^{1/2}$. Now, for $i \in G$, i.e. $k_{i+1} = k_i$, and writing $\log_j$ for the j times iterated logarithm,

$$\begin{aligned} \frac{v_{n_i}}{u_{n_{i+1}}} &= \frac{n_{i+1}}{n_i} \\ &\cdot \frac{\log n_i + (k_i - 1) \log\left(\frac{e \log n_i}{k_i}\right) - \frac{1}{2}\log(2\pi k_i) - \log(k_i) - 1 - \log_3 n_i}{\log n_{i+1} + (k_i - 1) \log\left(\frac{e \log n_{i+1}}{k_i}\right) - \frac{1}{2}\log(2\pi k_i) - \log(k_i) + (2 - \varepsilon) \log\log n_{i+1}} \\ &= \frac{n_{i+1}}{n_i} \cdot \frac{a_i}{b_i}. \end{aligned}$$

Now, for i large enough, $a_i < b_i$. Also, the term $(k_i - 1)\log\left(\frac{e \log n_i}{k_i}\right) - \frac{1}{2}\log(2\pi k_i)$ $-\log(k_i)$, which is positive for all i large enough, can be subtracted from numerator and denominator. This gives a strictly smaller ratio (a lower bound thus):

$$\frac{n_{i+1}}{n_i} \cdot \frac{\log n_i - 1 - \log_3 n_i}{\log n_{i+1} + (2-\varepsilon)\log\log n_i + (k_i - 1)\log\left(\frac{\log n_{i+1}}{\log_i}\right))}.$$

But

$$\log(n_{i+1}/n_i) = (1+o(1))\left(\frac{\log i}{2i}\right)^{1/2},$$

$$\log_2 n_{i+1} - \log_2 n_i = \log\left(1 + \frac{1+o(1)}{\log n_i}\left(\frac{\log i}{2i}\right)^{1/2}\right) = (1+o(1))/(2i).$$

Thus,

$$\frac{v_{n_i}}{u_{n_i+1}} \geq \left(1 + (1+o(1))\left(\frac{\log i}{2i}\right)^{1/2}\right) \cdot \frac{1 - \frac{(1+o(1))\log\log i}{\sqrt{2i\log i}}}{1 + \frac{(2-\varepsilon)\frac{1}{2}\log(2i\log i)(1+o(1))}{\sqrt{2i\log i}}}$$

$$= 1 + \left(\frac{\log i}{2i}\right)^{1/2}\left(\frac{\varepsilon}{2} - o(1)\right),$$

which is all we need for (iv).

6. An Erdös-Rényi Type Theorem

Deheuvels and Devroye (1983) have shown the following

Theorem 11. *If $k = [c \log n]$ for a positive constant c, and if a is the unique positive solution of the equation*

$$e^{-\frac{1}{c}} = (1+a)\,e^{-a},$$

then

$$V_n = \frac{n M_n - (1+a)(c \log n)}{\log\log n} \to -\frac{1}{2} \cdot \frac{a+1}{a} \quad \textit{in probability.}$$

This result was obtained as a special case of a much more general result on the oscillations of partial sums of i.i.d. random variables. From it, we can only deduce that almost surely, the lim sup of V_n is at least equal to $-\frac{1}{2} \cdot \frac{a+1}{a}$, and the lim inf of V_n is at most equal to the same number. Theorem 6 stated in the Introduction is valid for less strict conditions on k, and gives also inner class sequences. Since it can be proved without a lot of extra effort, its inclusion in this paper seems natural. To obtain sharp inner class results however that improve on Theorem 11, quite a bit of new heavy machinery seems necessary.

First of all, we will need a large deviation result for the gamma density in the spirit of Lemma 5:

Lemma 12. (Large deviation result for the gamma density.) *If $a>0$ is a constant, and a_k, b_k are sequences of real numbers satisfying $a_k=(1+a)k+b_k$, $b_k=0(\sqrt{k})$, then*

$$\psi_k(a_k)\sim\frac{1}{a\sqrt{2\pi k}}\exp\left(k(\log(1+a)-a)-\frac{a}{1+a}b_k\right)\quad \textit{as } k\to\infty.$$

Proof of Lemma 12. Consider first the series

$$(k-1)!\sum_{j=0}^{k-1}a_k^{-j}(k-1-j)!^{-1}=\sum_{j=0}^{M}+\sum_{j=M+1}^{k-1}\qquad\text{(for integer } M\leqq k-1).$$

For fixed M, the first sum is at least equal to

$$\sum_{j=0}^{M}\left(\frac{k-M}{a_k}\right)^j\sim\sum_{j=0}^{M}\left(\frac{1-M/k}{1+a}\right)^j=\frac{1-((1-M/k)/(1+a))^{M+1}}{1-((1-M/k)/(1+a))}\sim\frac{1-(1+a)^{-(M+1)}}{1-(1+a)^{-1}},$$

and is at most equal to

$$\sum_{j=0}^{M}\left(\frac{k}{a_k}\right)^j\sim\sum_{j=0}^{M}(1+a)^{-j}=\frac{1-(1+a)^{-(M+1)}}{1-(1+a)^{-1}}.$$

The second term is at most equal to

$$(1+o(1))\sum_{j=M+1}^{\infty}(1+a)^{-j}=(1+a)^{-(M+1)}\cdot\frac{1+o(1)}{1-(1+a)^{-1}}\to 0\qquad\text{as } M\to\infty.$$

Since M was arbitrary, we conclude that the series converges to $\dfrac{1+a}{a}$ as $k\to\infty$. Next, by elementary calculus,

$$\begin{aligned}\int_{a_k}^{\infty}\frac{y^{k-1}e^{-y}}{(k-1)!}dy\Big/\frac{a_k^{k-1}e^{-a_k}}{(k-1)!}&=a_k\int_1^{\infty}\left(\frac{y}{a_k}\right)^{a-1}e^{-(y-a_k)}\,d\left(\frac{y}{a_k}\right)\\&=a_k\int_0^{\infty}(1+u)^{k-1}e^{-a_k u}\,du\\&=a_k\sum_{j=0}^{k-1}\int_0^{\infty}\binom{k-1}{j}u^j e^{-a_k u}\,du\\&=a_k\sum_{j=0}^{k-1}\int_0^{\infty}\binom{k-1}{j}z^j e^{-z}\,dz\cdot a_k^{-(j+1)}\\&=\sum_{j=0}^{k-1}\frac{(k-1)!}{(k-1-j)!}a_k^{-j}\\&\to\frac{1+a}{a}\qquad\text{as } k\to\infty.\end{aligned}$$

Next, by Stirling's formula,

$$\begin{aligned}\psi_k(a_k) &\sim \frac{1+a}{a} \frac{a_k^{k-1} e^{-a_k}}{(k-1)!} \\ &\sim \frac{1+a}{a} \left(\frac{(1+a)\,k\,e}{k-1}\right)^{k-1} \left(1+\frac{b_k}{(1+a)\,k}\right)^{k-1} e^{-(1+a)k-b_k} (2\pi k)^{-1/2} \\ &\sim \frac{(1+a)^k}{a} \exp\left(k-(1+a)\,k+\frac{1}{1+a} b_k - b_k\right) (2\pi k)^{-1/2},\end{aligned}$$

which concludes the proof of Lemma 12.

Proof of Theorem 6. We will first show that $P(M_n > u_n \text{ i.o.})=0$ for

$$u_n = \frac{1}{n}\left((1+a)(c\log n) + (\tfrac{3}{2}+\varepsilon)\frac{a+1}{a}\log\log n\right), \qquad \varepsilon>0.$$

We can apply Lemma 4 again since u_n is eventually nonincreasing. Applying Lemma 7 to the conditions of Lemma 4 leaves us with the simple task of showing that

$$\text{(i)} \quad n\,\psi_k(u_n(n-n^{3/4})) \to 0,$$

$$\text{(ii)} \quad \sum_n \log n \cdot \psi_k(u_n(n-n^{3/4})) < \infty.$$

Because $u_n = u_n(\varepsilon)$ satisfies: $u_n(\varepsilon)(n-n^{3/4}) > u_n\left(\frac{\varepsilon}{2}\right) n$ for n large enough, we see that in (i) and (ii) the argument of ψ_k can be replaced by

$$u_n\left(\frac{\varepsilon}{2}\right) n.$$

We write $u_n\left(\frac{\varepsilon}{2}\right) n$ as $(1+a)\,k+b_n$ (in the notation of Lemma 12) where

$$\begin{aligned}b_k &= (1+a)(c\log n) + \left(\frac{3}{2}+\frac{\varepsilon}{2}\right)\frac{a+1}{a}\log\log n - (1+a)\,k \\ &= \left(\frac{3}{2}+\frac{\varepsilon}{2}+o(1)\right)\frac{a+1}{a}\log\log n.\end{aligned}$$

Because obviously $n = \exp\left(\frac{k}{c} + o(\log\log n)\right)$, and $b_k = o(\sqrt{k})$, we have by Lemma 12,

$$\begin{aligned}n\,\psi_k\left(n\,u_n\left(\frac{\varepsilon}{2}\right)\right) &= \exp\left(-\left(\frac{3}{2}+\frac{\varepsilon}{2}+o(1)\right)\log\log n - \tfrac{1}{2}\log\log n\right) \\ &= (\log n)^{-2-\frac{\varepsilon}{2}-o(1)}.\end{aligned}$$

It is clear that (i) and (ii) hold for each $\varepsilon>0$.

Next, we will prove that $P(M_n < u_n \text{ i.o.}) = 0$ when

$$u_n = \frac{1}{n}\left((1+a)(c\log n) - (\tfrac{3}{2}+\varepsilon)\frac{a+1}{a}\log\log n\right), \qquad \varepsilon > 0.$$

Since u_n is eventually monotone, and $u_n \sim (1+a)c\frac{\log n}{n}$, we are in a position to apply Lemma 9. For n large enough, we have

$$\begin{aligned}
&\frac{n}{k}\psi_k(u_n(\varepsilon)(n+n^{3/4})) \geqq \frac{n}{k}\psi_k\left(u_n\left(\frac{\varepsilon}{2}\right)n\right)\\
&\quad\sim \frac{1}{a\sqrt{2\pi k}}\exp\left(\frac{k}{c}+o(\log\log n) - \log\log n + \left(\frac{3}{2}+\frac{\varepsilon}{2}+o(1)\right)\log\log n - \frac{k}{c}\right)\\
&\quad= \exp\left(\left(\frac{\varepsilon}{2}+o(1)\right)\log\log n\right),
\end{aligned}$$

which satisfies the inequality of Lemma 9 for all $\varepsilon > 0$. This concludes the proof of Theorem 6.

Remark. We have used the fact that Theorem 11 holds under the weaker conditions on k stated in Theorem 6. Note however that if the $o(\log\log n)$ terms is replaced by a bigger term in the condition for k, the estimate obtained via Lemma 12 will be influenced to such an extent that the conclusion of Theorem 6 may no longer be valid.

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