On the Convergence of Statistical Search

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Abstract—The convergence of statistical (random) search for the
minimization of an arbitrary multimodal functional $Q(w)$ is dealt with
by using the theorems of convergence of random processes of Braverman
and Rozonoer. It is shown that random search can be regarded as a
gradient algorithm in the $Q$-domain. Using this gradient to define the
minimum of the functional, the convergence to this minimum is discussed
at length. The theorems proved in this paper apply as well to discrete
as to continuous optimization problems and as such, the developed tech-
nique is competitive with stochastic automata with a variable structure.
The optimality of the scheme follows from the convergence in probability
of the average performance to the minimum. The freedom in the or-
ganization of the search within the boundaries outlined by the conditions of
convergence is emphasized. Finally, it is pointed out how various mixed
random search and hierarchical search systems fall into the domain of
application of the theorems.

INTRODUCTION

The problem of optimizing a multimodal and unknown functional $Q(w)$ with respect to a set of $m$
parameters $w$ is solved through direct search techniques. Usually, random search [6], [7], [11], [12], [16], [18],
[19], [27] or stochastic automata with a variable structure [1], [4], [8], [17] are used when the functional (performance
index) is extremely complicated; e.g., $Q(w)$ has many discontinuities, has many local minima, is very nonlinearly
shaped, is not differentiable, etc. If more information about the functional is available (usually in terms of differenti-
tiability, smoothness, etc.), combinations of the above-mentioned methods with local hill-climbing methods can
be considered [4], [5], [15], [16], [23], [24]. Most of these local hill-climbing methods (for a recent survey, see [25])
are derived from the Kiefer–Wolfowitz stochastic approximation method [14], [25] and stochastic gradient
algorithms [2], [3], [13], [21].

While the convergence of most of the hill-climbing algorithms has been investigated at length in the literature,
there are only a few papers that deal with the convergence of random search, mainly because of the mathematical
intractability of the algorithm and the difficulty of dealing with multimodal functionals [6], [7], [18], [19]. Most of the
research in the field of random search has been concerned with the organizational aspect of the search [13],
[15], [16], [19], [23], [24], and hence heuristic techniques are legion. In comparison, this paper is a unified presenta-
tion dealing with different modes of convergence (convergence with probability one, in probability, of the mean) for
different quantities (the sequence of estimates of the minimum, the average performance, etc.). The strongest
result obtained until now was the convergence in probability to its minimal value for an accumulated average performance [18].

The mathematical difficulties are overcome here by projecting the whole problem in the $Q(w)$ domain. It is shown
that there exists a unimodal convex function in this domain with respect to which the random search algorithm can be
regarded as a gradient algorithm. The convergence is proved by using the theorems of convergence of random
processes in machine learning [9]. Old results [6], [7], [18], [19] are strengthened and generalized at the same
time. The class of functionals to be allowed includes "anomalous functionals" (an anomalous functional is
such that the value of $Q(w)$ does not convey any information as to the value of any other point $w^* \neq w$). In order to
allow such general performance indices, a new definition of the minimum has to be introduced. The stationary points
of the convex function derived in this paper uniquely define the minimal value of the performance index.

The random search algorithm of Matyas [19] and Gurin [6] is generalized in order to obtain a gradual reduction
of the overall average performance. It is emphasized that within the bounds dictated by the conditions of conver-
gence, the organization of the search depends only upon the imagination of the designer. This freedom can be used
to develop convergent nonheuristic hierarchical search systems, mixed random and nonrandom search systems,
search systems with subsequent linear search in special directions (as in deterministic optimization), etc.

PROBLEM FORMULATION

Let $w \in W \subseteq E^m$, $m$-dimensional Euclidean space. $Q(w)$ is the expected value of a random variable $\zeta/w$ conditioned
on $w$. The pdf of $\zeta/w$ is $\pi(\zeta/w)$:

$$Q(w) \triangleq E\{\zeta/w\} = \int \pi(\zeta/w) \, d\zeta. \tag{1}$$

$\zeta$ will be referred to as a measurement of the performance index $Q(w)$. $Q(w)$ is sometimes called a noisy surface or
stochastic performance index. If $\pi(\zeta/w) = \delta(\zeta - Q(w))$, $Q(w)$ is said to be deterministic. In analogy with the termin-
ology used in automata theory, $\pi(\zeta/w)$ defines a random environment. The environment is a $P$-model environment
if $\zeta \in [0,1]$ and an $S$-model environment if $\zeta \in [0,1]$. In general, the environment is called a $Q$-model environment.

The presented procedure is iterative with iteration counter $j$. The "state" of the system (minimization procedure) is
denoted by $X_j \in X$. $X$ is the state space. Essential in the class of probabilistic procedures discussed in this paper is
the concept of basepoint $w_j$ (best estimate of the minimum up to the $j$th iteration) which is included in $X_j$. The tech-
nique consists of a suitable rule for updating \( w_j \) and \( X_j \) in such a way that convergence is guaranteed. \( X = \{X_j\}_{j \geq 0} \) is called the state sequence. \( X_j \), which might contain parameters to be learned or adapted during the search, is a random element defined on some (usually growing) probability space. Functions of the state that map \( X \) into \( \mathbb{R}^1 \) are thus random variables on this probability space.

The procedure is defined as follows.

1. \( X_0 \) (and thus \( w_0 \)) is given or selected in \( X \). The initial distribution of \( w_0 \) in \( W \) is denoted by \( g_B(w) \).

2. \( w_j \) is applied to the environment; \( \lambda_{Bj} \geq 1 \) iid (independent identically distributed) measurements \( \zeta_j/w_j \) are observed (measured), totalled, and averaged. The average is denoted by \( \bar{\zeta}_j \).

3. A trial point \( w_{j+1}^* \) is generated. With probability at least equal to \( \alpha_j \in [0,1] \), \( w_{j+1}^* \) is purely randomly generated according to a given fixed pdf \( g_B(w) \) concentrated on \( W \).

4. The trial point \( w_{j+1}^* \) is applied to the environment; \( \lambda_{Tj} \geq 1 \) iid measurements \( \zeta_j/w_{j+1}^* \) are observed, totalled, and averaged. The average is denoted by \( \bar{\zeta}_j \).

5. The new state \( X_{j+1} \) is computed. This includes the adaptation of the basepoint by the decision rule:

\[
g(w|X_j) = \alpha_j g_B(w) + (1 - \alpha_j) g^*(w|X_j).
\]

The number of auxiliary measurements needed in this stage is denoted by \( \lambda_{Gj} \).

6. Operations 2)—5) constitute one basic cycle or “iteration” of the search process.

A flow chart of a generalized random search algorithm is shown in Fig. 1.

Remark 1: \( \{\lambda_B\}, \{\lambda_G\}, \{\lambda_T\}, \{\alpha\}, \{\lambda_i\} \) are treated in the following as number sequences. Notice however that adaptive parameter sequences can be allowed if there are lower and upper adaptation boundaries that are number sequences satisfying the conditions of convergence to be established in this paper.

Remark 2: In the \( j \)th cycle, the number of measurements equals \( \lambda_{Bj} + \lambda_{Tj} + \lambda_{Gj} \). The properties of convergence, however, are with respect to \( j \), while the cost of optimization is often proportional to the number of performance index evaluations.

Remark 3: \( g_B(w) \) is the basic random law of generation of test points and defines \( W \). If \( g_B(w) \) is concentrated on a finite set of points in \( \mathbb{R}^m \), then \( W \) is a finite set, and the minimization problem is reduced to a strategy selection problem. If a minimum is to be found in an \( n \)-dimensional hypercube, \( g_B(w) \) could be the uniform pdf in this hypercube. If \( W \) is unbounded but at the same time there are strong indications that the minimum is close to \( w^0 \), then \( g_B(w) \) might be centered at \( w^0 \) and have a Gaussian form with a fixed positive definite covariance matrix. Thus \( g_B(w) \) reflects the \textit{a priori} knowledge about the problem and is considered to be given.

Remark 4: It is assumed that \( w_{j+1}^* \in W_j \) for all \( X_j \in X \).

Remark 5: The nature of \( g^*(w|X_j) \) and \( X_j \) is of no major importance for establishing the convergence of the algorithm. Hence, to increase the efficiency of the procedure, techniques from direct deterministic optimization theory (linear search in a special direction, creeping random search, partan techniques, etc. [29]), hierarchical search systems, all kinds of “almost”-gradient stochastic minimization algorithms [2], [3], [12], [21], [25], [26] and combined global search and stochastic approximation methods [4], [5], [15], [24], [30] can be incorporated in the search scheme. Notice that due to the option that \( \alpha_j \) might tend to zero, \( g^*(w|X_j) \) might tend to \( g^*(w|X_j) \) on which no restrictions are imposed. The purely random way of generating trial points will only be needed in the beginning to scan the complete search domain.

Remark 6: Gurin’s scheme [6] is obtained for \( \lambda_{Tj} = \lambda_{Bj} = \lambda_j, \lambda_{Gj} = 0, \alpha_j = \epsilon > 0 \).

Remark 7: Constraints are easily dealt with in random search [29] and cause such minor trouble that they are not explicitly discussed here. The most popular ways of dealing...
with constraints are the rejection method (reject all the trial points that fall outside the permitted region \( W \)) and the variable-weight Lagrange multiplier method \([31], [32]\).

**NOTATIONS AND DEFINITIONS**

The problem will now be translated into the \( q \)-domain. Notice that every pdf in \( W \) induces a corresponding pdf in the \( q \)-domain; \( p_b(q) \) defined by (4) will be called the basic search pdf and \( p(q/X_j) \) induced by \( g(w/X_j) \) will be referred to as the search pdf:

\[
p_b(q) = \int_W \delta(q - Q(w)) g_b(w) \, dw
\]

\[
p(q/X_j) = \int_W \delta(q - Q(w)) g(w/X_j) \, dw.
\]

The corresponding cdf's (cumulative distribution functions) are \( P_b(q) \) and \( P(q/X_j) \) for which:

\[
P_b(q) = \int_0^q p_b(u) \, du = \int_W \omega(Q(w),q) g_b(w) \, dw
\]

\[
= \text{Prob} \{Q(w_{j+1}) \leq q | x_j = 1\}
\]

\[
P(q/X_j) = \int_0^q p(u/X_j) \, du = \int_W \omega(Q(w),q) g(w/X_j) \, dw
\]

\[
= \text{Prob} \{Q(w_{j+1}) \leq q/X_j\}
\]

where \( \omega(a,b) \triangleq 1 \) for \( a \leq b \), \( \omega(a,b) \triangleq 0 \) for \( a > b \). An important inequality independent of \( X_j \) is (8), which stems from combining (2) with (6)–(7):

\[
P(q/X_j) \geq \omega_j \cdot P_b(q) \quad p(q/X_j) \geq \omega_j \cdot p_b(q).
\]

Thus \( P(q/X_j) \) is the distribution function of \( Q(w_{j+1}) \) conditioned on \( X_j \).

**State functions** of some importance in classical optimization theory include the value of the performance index at the basepoint (9) and the indicator of the event \( \{Q(w_j) \leq q_{\text{min}} + \eta\} \), where \( \eta > 0 \) is some constant and \( q_{\text{min}} \) is the minimum of \( Q(w) \) which will be defined later in a rigorous way:

\[
q_j \triangleq Q(w_j)
\]

\[
\omega(q_j, q_{\text{min}} + \eta) \triangleq \text{ind} \{Q(w_j) \leq q_{\text{min}} + \eta\}.
\]

In automata theory and in several applications it is not only important that \( q_j \) tends to \( q_{\text{min}} \) as \( j \to \infty \) but also that the average measured performance tends to this value. Therefore, define the \textit{mean of the search pdf} as

\[
\rho(X_j) \triangleq \int q \cdot p(q/X_j) \, dq = E\{Q(w_{j+1})/X_j\}
\]

and the mean over all the expected values of the auxiliary measurements made in the \( w_{j+1} \) generating process if \( \alpha_j = 0 \) as \( \rho_0(X_j) \). The \textit{average measured performance} \( \psi(X_j) \) is then defined by

\[
\psi(X_j) \triangleq \frac{\lambda_b \cdot q_j + \lambda_T \cdot \rho(X_j) + \lambda_g \cdot \rho_0(X_j)}{\lambda_b + \lambda_T + \lambda_g}.
\]

The \textit{degree of concentration} \( D(X_j, \eta) \) on \( (-\infty, q_{\text{min}} + \eta) \) is defined by

\[
D(X_j, \eta) \triangleq \frac{P_{\text{g}}(q_{\text{min}} + \eta/X_j)}{P_{\text{g}}(q_{\text{min}} + \eta)}.
\]

where \( P_{\text{g}}(q_{\text{min}} + \eta/X_j) \) denotes the probability that if one of the \( \lambda_g \) auxiliary measurements is picked at random, the value of the performance index at that point is not greater than \( q_{\text{min}} + \eta \). \( D(X_j, \eta) \) denotes the probability that if one of the \( \lambda_b \) and \( \lambda_T \) measurements made at the \( j \)th cycle is picked at random, the corresponding expected value of the performance index at the point of measurement is less than or equal to \( q_{\text{min}} + \eta \).

Of theoretical interest will be the integral

\[
\int_{-\infty}^{q_{\text{min}}} P_b(u) \, du
\]

to be studied later. Finally, introduce the bar operator \( \bar{a} \) with the following meaning: \( \bar{a} = \max \{a, q_{\text{min}}\} \). Then, before investigating the properties of convergence of the cited state functions, \( q_{\text{min}} \) will be defined.

**A NEW DEFINITION OF THE MINIMUM**

Consider the positive goal function \( I(q) \) and assume that \( \lim_{q \to -\infty} q \cdot P_b(q) = 0 \):

\[
I(q) = \int_{-\infty}^{q_{\text{min}}} \int_{-\infty}^{q} (q - u) \cdot p_b(u) \, du \, dx
\]

\[
= \int_{-\infty}^{q_{\text{min}}} \int_{-\infty}^{q} \left[ \frac{q - \int_{-\infty}^{q} u \cdot p_b(u) \, du}{P_b(q)} \right] \, dx
\]

This goal function is convex (the second derivative is \( P_b(q) \geq 0 \), nonnegative and continuous. Its gradient is given by

\[
I'(q) = \int_{-\infty}^{q_{\text{min}}} P_b(u) \, du = \int_{-\infty}^{q} (q - u) \cdot p_b(u) \, du
\]

\[
= p_b(q) \left[ q - \int_{-\infty}^{q} u \cdot \frac{p_b(u)}{p_b(q)} \, du \right].
\]

The gradient is always positive except if either \( P_b(q) = 0 \) or the mean over all \( Q(w) \leq q \) equals \( q \) when \( w \) has pdf \( g_b(w) \).

The \textit{minimal value} (minimum) of \( Q(w) \) with respect to \( g_b(w) \) is \( q_{\text{min}} \): \( q_{\text{min}} = \inf \{q \in R \mid I'(q) > 0\} \). Trivially, \( q_{\text{min}} \) satisfies:

\[
\int_{-\infty}^{q_{\text{min}}} P_b(u) \, du = 0.
\]
To each \( \eta > 0 \), there exists a number \( \delta(\eta) > 0 \) such that

\[
\int_{-\infty}^{q_{\min} + \eta} P_B(u) \, du \geq \delta(\eta) > 0.
\]  

(18)

The minimum is such that the set \( \{ w \in W : Q(w) < q_{\min} \} \) has a zero measure with respect to \( g_B(w) \). This follows directly from definition (17) and property (4). It can also be shown that the given definition is in accordance with most of the definitions encountered in the literature for continuous or piecewise continuous functions.

Notice that it is not necessarily true that \( P_B(q_{\min}) = 0 \).

One can easily imagine a staircase function \( Q(w) \) with a set of points having a nonzero measure with respect to \( g_B(w) \) and for which \( Q(w) = q_{\min} \). Clearly, \( P_B(q_{\min}) > 0 \).

It is not hard to see that

\[
q_{\min} = \inf \{ q \in \mathbb{R} \mid P_B(q) > 0 \}
\]  

(19)

and therefore, to each \( \eta > 0 \), there exists a number \( \theta(\eta) > 0 \) such that

\[
P_B(q_{\min} + \eta) \geq \theta(\eta) > 0.
\]  

(20)

**CONVERGENCE OF THE STATE FUNCTIONS**

Assign the following letters to denote the convergence of the corresponding state function:

A) convergence of \( \bar{q}_j \) to \( q_{\min} \)

B) convergence of \( \omega(q_j, q_{\min} + \eta) \) to 1, for every \( \eta > 0 \)

C) convergence of \( D(X_j, \eta) \) to 1, for every \( \eta > 0 \)

D) convergence of \( \int_{-\infty}^{q_{\min}} P_B(u) \, du \) to 0

E) convergence of \( \tilde{W}(X_j) \) to \( q_{\min} \).

A second index added to it concerns the mode of convergence:

1) convergence with probability one
2) convergence in probability
3) convergence in the mean.

Finally, introduce the conditions:

\[
|Q(w)| \leq q_{\max} < \infty, \quad \text{for all } w \in W
\]  

(21)

\[
\lim_{\eta \to \infty} \frac{\lambda_{Gj} + \lambda_{Tj}}{\lambda_{Bj} + \lambda_{Gj} + \lambda_{Tj}} = 0.
\]  

(22)

Then the following relations exist between the properties of convergence of the given random variables.

1) \( Y_1 \to Y_2, (Y = A,B,C,D,E) \) since convergence with probability one implies convergence in probability.

2) \( Y_3 \to Y_2, (Y = A,B,C,D,E) \) since convergence in the mean implies convergence in probability.

3) \( Y_2 \to Y_3, (Y = B,C) : 1 \geq D(X_j, \eta) \geq 0 \). Then \( E\{D(X_j, \eta)\} \geq (1 - \eta') \cdot \text{Prob} \{D(X_j, \eta) \geq 1 - \eta'\} \) with \( \eta' > 0 \). Since \( D(X_j, \eta) \to 1 \) in probability, \( E\{D(X_j, \eta)\} \to 1 \) also. Convergence in the mean follows since one is the upper limit of \( D(X_j, \eta) \). The proof for \( \omega(q_j, q_{\min} + \eta) \) is similar.

4) \( (21) \oplus Y_2 \to Y_3, Y = A,D,E \). The proof is similar if it is noted that all the random variables converge to their lower limit and are deterministically bounded from above.

Indeed, if \( \bar{W}(X_j) \leq q_{\max} ; \bar{q}_j \leq q_{\max} \) and integral (14) is never greater than \( (q_{\max} - q_{\min}) \).

5) \( (21) \oplus (22) \oplus Ay \to Ey, y = 1,2,3 \):

\[
q_{\min} \leq \bar{W}(X_j) \leq \lambda_{Bj} \cdot \bar{q}_j + (\lambda_{Tj} + \lambda_{Gj}) \cdot q_{\max} \to \bar{q}_j,
\]  

for \( j \to \infty \)

by using definition (12), (21), and (22). The rest of the proof is trivial.

6) \( (22) \oplus By \to Cy, y = 1,2,3 \): The proof is as under 5) if it is noted that

\[
1 \geq D(X_j, \eta) \geq \lambda_{Gj} \cdot \omega(q_j, q_{\min} + \eta) + 0 \cdot (\lambda_{Tj} + \lambda_{Gj})
\]  

\[
= \omega(q_j, q_{\min} + \eta), \quad \text{for } j \to \infty.
\]

7) \( Ay \leftrightarrow By, y = 1,2,3 \). Indeed, take \( y = 1 \) for instance.

Then from \( Ay \):

\[
\lim_{j \to \infty} \text{Prob} \{\max_{1 \leq j} \bar{q}_i \leq q_{\min} + \eta\}
\]

= \lim_{j \to \infty} \text{Prob} \{\min_{1 \leq j} \omega(q_i, q_{\min} + \eta) = 1\} = 1
\]

which is exactly the definition of \( B1 \), since this holds for every \( \eta > 0 \) by supposition.

8) \( Ay \leftrightarrow Dy, y = 1,2,3 \). Indeed, note that

\[
\int_{-\infty}^{q_{\min} + \eta} P_B(u) \, du = \int_{-\infty}^{q_{\min} + \eta} P_B(u) \, du
\]

is a continuous function of \( \bar{q}_j - q_{\min} \) only vanishing for \( \bar{q}_j - q_{\min} = 0 \). The same applies for \( \bar{q}_j - q_{\min} \) and thus, applying lemma 2 of [22], the convergence either with probability one or in probability of one of both random variables to zero implies the convergence in the same mode of the other random variable to zero.

The relations between the properties of convergence of the different state functions for \( y = 1,2 \), are shown in Fig. 2.

Using these properties it is seen that the convergence of

\[
\bar{q}_j, \int_{-\infty}^{q_{\min} + \eta} P_B(u) \, du
\]

or \( \omega(q_j, q_{\min} + \eta) \), for all \( \eta > 0 \), is crucial. Therefore, we will focus our attention on the convergence of the latter state functions.
THEOREMS OF CONVERGENCE OF STATISTICAL SEARCH

The decision process (3) can mathematically be formulated in terms of the random variable \( \xi(w^*_j, w^*_{j+1}) \) given \( X_j \):
\[
\xi(w_j, w^*_j) \triangleq \begin{cases} 
1, & \text{if } w^*_{j+1} = w^*_j + 1 \\
0, & \text{if } w^*_{j+1} = w_j 
\end{cases} 
\]
(23)
and thus \( q_{j+1} = Q(w^*_{j+1}) = u \).

Notice that
\[
E\{\xi(w_j, w^*_j)\} = \text{Prob} \{ w_{j+1} = w^*_j + 1 | w_j, w^*_j, X_j \}
\]
does depend upon \( \lambda_{BJ}, \lambda_{TJ} \) and \( \varepsilon_j \). Thus, given \( X_j \):
\[
w_{j+1} = \begin{cases} 
\text{w}_j \text{ with probability } 1 - \int_w E\{\xi(w_j, w^*)\} \\
\text{w}^* \text{ with probability } E\{\xi(w_j, w^*)\} 
\end{cases} 
\]
(24)
The pdf \( f(w | X_j) \) of \( w_{j+1} \) conditioned on \( X_j \) is given by
\[
E\{\xi(w_j, w^*)\} \cdot g(w | X_j) + \left( 1 - \int_w E\{\xi(w_j, w^*)\} \right) \cdot g(w^* | X_j) \cdot \delta(w - w_j).
\]
Define now the random variable \( \xi^*(q_j, u) \in \{0, 1\} \) such that
\[
E\{\xi^*(q_j, u)\} = \left\{ \begin{array}{ll}
\inf_{\{w \in \mathcal{W}: Q(w) = Q(w^*_j)\}} & \text{if } q_j \geq u \\
\sup_{\{w \in \mathcal{W}: Q(w) = Q(w^*_j)\}} & \text{if } q_j < u.
\end{array} \right.
\]
(25)
Clearly,
\[
E\{\xi^*(q_j, u)\} \geq \text{Prob} \{ w_{j+1} = w^*_j + 1 | w_j, w^*_j, q_j \geq u, X_j \}
\]
\[
E\{\xi^*(q_j, u)\} \leq \text{Prob} \{ w_{j+1} = w^*_j + 1 | w_j, w^*_j, q_j < u, X_j \}.
\]
Thus
\[
\{ q_{j+1} - q_{\text{min}} / X_j \} \leq (\bar{q}_j - \mu) \cdot p(u | X_j) \, du.
\]
(27)
After introducing the step function:
\[
\phi(q_j, u) \triangleq \begin{cases} 
1, & \text{if } u \leq q_j \\
0, & \text{if } u > q_j
\end{cases}
\]
(28)
(27) can be rewritten as
\[
E\{q_{j+1} - q_{\text{min}} / X_j \} 
\leq (\bar{q}_j - q_{\text{min}}) - \int_{-\infty}^{+\infty} \phi(q_j, u) \cdot p(u | X_j) \cdot (\bar{q}_j - \mu) \, du
\]
\[
+ \int_{\bar{q}_j}^{+\infty} (q_j - u) \cdot E\{\xi^*(q_j, u) - \xi^*(q_j, u)\} \cdot p(u | X_j) \, du.
\]
(29)
The third term on the right side in (29) can be considered as positive noise and disappears only if \( \xi^*(q_j, u) \equiv \phi(q_j, u) \), for all \( q_j \) and \( u \), which is only the case for deterministic performance indices \( Q(w) \) if \( \varepsilon_j = 0 \). Furthermore,
\[
\int_{-\infty}^{+\infty} \phi(q_j, u) \cdot p(u | X_j) \cdot (\bar{q}_j - \mu) \, du
\]
\[
= \int_{q_{\text{min}}}^{\bar{q}_j} (q_j - u) \cdot p(u | X_j) \, du
\]
\[
+ \int_{q_{\text{min}}}^{+\infty} (q_j - u) \cdot p(u | X_j) \, du
\]
\[
= (\bar{q}_j - q_{\text{min}}) \cdot P(q_{\text{min}} / X_j)
\]
\[
+ \int_{q_{\text{min}}}^{+\infty} (q_j - u) \cdot p(u | X_j) \, du
\]
\[
= (\bar{q}_j - q_{\text{min}}) \cdot P(q_{\text{min}} / X_j) + \int_{q_{\text{min}}}^{q_j} P(u | X_j) \, du
\]
(30)
which, using inequality (8) is not less than
\[
\alpha \cdot \left[ (\bar{q}_j - q_{\text{min}}) \cdot p_\beta(q_{\text{min}}) + \int_{q_{\text{min}}}^{\infty} P_\beta(u) \, du \right]
\]
(31)
since
\[
\int_{q_{\text{min}}}^{q_{\text{min}}+1} P_\beta(u) \, du = 0.
\]
Furthermore,
for \( q_j \geq u : \bar{q}_j - u \leq q_j - u \) and \( E\{\phi(q_j, u) - \xi^*(q_j, u)\} \geq 0 \)
for \( q_j < u : q_j - \bar{q}_j \geq q_j - u \) and \( E\{\phi(q_j, u) - \xi^*(q_j, u)\} \leq 0 \).
Together, if the second term in (29) is replaced by (31) and
if in the third term \( (\bar{q}_j - \mu) \) is replaced by \( (q_j - u) \), the following stronger inequality is obtained:
\[
E\{U(X_{j+1}) / X_j \} \leq U(X_j) - \alpha_j \cdot V(X_j) + Z(X_j)
\]
\[
U(X_j) = \bar{q}_j - q_{\text{min}}
\]
\[
V(X_j) = (\bar{q}_j - q_{\text{min}}) \cdot P_\beta(q_{\text{min}}) + \int_{q_{\text{min}}}^{q_j} P_\beta(u) \, du
\]
\[
Z(X_j) = \int_{q_{\text{min}}}^{+\infty} (q_j - u) \cdot E\{\phi(q_j, u) - \xi^*(q_j, u)\} \cdot p(u | X_j) \, du.
\]
The functions $U$, $V$, and $Z$ are all nonnegative, and hence, the inequality (32) is of the type studied by Braverman and Rozonoer [9]. Note that both $U(X_j)$ and $V(X_j)$ are continuous functions of $q_j - q_{\text{min}}$ and only vanish for $\overline{q}_j = q_{\text{min}}$. Thus, if $U \to 0$, $V \to 0$ and vice versa. Then, if $E\{U(X_0)\}$ and $E\{V(X_0)\}$ exist,

$$\sum_{x_j \in X} x_j = \infty, \sum_{x_j \in X} Z(x_j) < \infty$$

there holds:

$$U(X_j) \to 0 \quad V(X_j) \to 0, \quad \text{as } j \to \infty.$$  

Theorem 1, proved in the Appendix, reduces to a mere check of the condition on $Z(X_j)$.

Define $\mu_k$:

$$\mu_k \triangleq \sup_{w \in W} \{E[1 - Q(w)]/w\}$$

$$= \sup_{w \in W} \left\{ \int \zeta\cdot Q(w) \cdot P(\zeta) d\zeta \right\}, \quad k > 0. \quad (35)$$

**Theorem 1:** Let there be satisfied (H1)-(H5)

$$E[q_{\text{min}}] < \infty \quad \text{(H1)}$$

$$\alpha_j \in [0,1], \quad \sum_{j=1}^{\infty} \alpha_j = \infty \quad \text{(H2)}$$

$$\varepsilon_j \geq 0, \quad \sum_{j=1}^{\infty} \varepsilon_j < \infty \quad \text{(H3)}$$

$$\lambda_{B_j} \geq 1, \quad \lambda_{T_j} \geq 1, \quad \lambda_{A_j} \geq 0 \quad \text{(H4)}$$

and the environment be either deterministic:

$$\mu_2 = 0 \quad \text{(H6)}$$

or $Q$-model random with (H7) and (H8) or (H9) holding:

$$\sum_{j=1}^{\infty} \frac{1}{\varepsilon_j - 1} < \infty \quad \text{(H7)}$$

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_{B_j}^{1/2}} < \infty \quad \text{(H8)}$$

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_{T_j}^{1/2}} < \infty \quad \text{(H9)}$$

or $S$-model random or $P$-model random:

$$\zeta \in [0,1] \quad \text{($S$-model)} \quad \zeta \in [0,1] \quad \text{($P$-model)} \quad \text{(H10)}$$

$$\sum_{j=1}^{\infty} (\varepsilon_j \cdot \lambda_{B_j})^{-1} \cdot e^{-\lambda_{B_j} \cdot \varepsilon_j / \alpha_j} < \infty \quad \text{(H11)}$$

then the state sequence $\mathcal{X}$ generated by the above given procedure is such that

$$q_{\text{min}} \quad \text{as } j \to \infty. \quad (36)$$

The proof of convergence based upon the theorems of Braverman and Rozonoer is given in the Appendix.

**Remark 1.1:** The conditions are satisfied by taking $\lambda_{B_j} = \lambda_{T_j} = 1$, $\varepsilon_j = 0$ for deterministic environments. Since $Q(w_j) = q_j$ is exactly known from previous measurements, it should not be measured any more at time $j$, and eventually $\lambda_{B_j} = 0$ is allowable, in which case the algorithm of Matyas is obtained [19].

**Remark 1.2:** For all the random environments $\lambda_{B_j}$ and $\lambda_{T_j}$ should diverge for $j \to \infty$ (see also [6], [18]), and the rate of increase depends upon the rate of decrease of $\varepsilon_j$ and upon the conditions imposed upon the nature of the environment (H7), (H10). What makes Theorem 1 interesting is the option of making $\varepsilon_j$ equal 0, which is completely new for this type of random search algorithm.

**Remark 1.3:** The divergence of $\alpha_j$ (H3) insures that the whole search domain will be “searched.” However, (H3) does not prevent $\alpha_j$ from decreasing to 0 as $j \to \infty$, which makes this scheme more flexible than the schemes of Gurin [6], Cockrell [16], etc.

An alternative way of dealing with the convergence of $\{\overline{q}_j\}$ is through the indicator function (10) $\omega(q_j q_{\text{min}} + \eta)$, for all $\eta > 0$. In a way similar to the derivation of (27) it can be shown:

$$E\{1 - \omega(q_j + 1, q_{\text{min}} + \eta)/X_j\} \leq 1 - \omega(q_j q_{\text{min}} + \eta) - \int E[\xi^*(q_j, u)] \cdot \omega(u, q_{\text{min}} + \eta) \cdot p(u/X_j) du$$

$$= (1 - \omega(q_j q_{\text{min}} + \eta)) - \int_{-\infty}^{q_{\text{min}} + \eta} E[\xi^*(q_j, u)]/q_j \leq q_{\text{min}} + \eta)$$

$\cdot p(u/X_j) du + \int_{q_{\text{min}} + \eta}^{\infty} E[\xi^*(q_j, u)]/q_j \leq q_{\text{min}} + \eta)$$

$\cdot p(u/X_j) du.$

(37)

and from (8), reducing the integration interval in the second term gives

$$E\{1 - \omega(q_j + 1, q_{\text{min}} + \eta)/X_j\} \leq (1 - \omega(q_j q_{\text{min}} + \eta))$$

$$\cdot (1 - \zeta_j \cdot \int_{q_{\text{min}} + \eta}^{q_{\text{min}} + \eta}) E[\xi^*(q_j, u)]/q_j > q_{\text{min}} + \eta)$$

$$\cdot p(u/X_j) du + \int_{q_{\text{min}} + \eta}^{\infty} E[\xi^*(q_j, u)]/q_j \leq q_{\text{min}} + \eta)$$

$\cdot p(u/X_j) du.$

(38)

Introducing

$$Z_{1,j} \triangleq \inf_{\theta \in (q_{\text{min}} + \eta, \infty)} E[\xi^*(q_j, \theta)/q_j > q_{\text{min}} + \eta)$$

$$Z_{2,j} \triangleq \sup_{\theta \in (q_{\text{min}} + \eta, \infty)} E[\xi^*(q_j, \theta)/q_j \leq q_{\text{min}} + \eta)$$

we have

$$E\{(1 - \omega(q_j + 1, q_{\text{min}} + \eta))/X_j\} \leq (1 - \omega(q_j q_{\text{min}} + \eta))$$

$$\cdot (1 - \zeta_j \cdot Z_{1,j} \cdot p(q_{\text{min}} + \eta/2)) + Z_{2,j}.$$

(40)
In view of property (20), for all $\eta > 0$: $P_\eta(q_{\text{min}} + \eta/2) > 0$. The theorems of Braverman and Rozonoer applied to (40) insure that

$$\omega(q_j, q_{\text{min}} + \eta) \xrightarrow{\text{with probability one}} 1, \quad \text{as } j \to \infty$$

if

$$\alpha_j \cdot Z_{1j} \geq 0 \quad \sum_{j=1}^{\infty} \alpha_j \cdot Z_{1j} = \infty \quad \sum_{j=1}^{\infty} Z_{2j} < \infty \quad (41)$$

Further,

$$\omega(q_j, q_{\text{min}} + \eta) \xrightarrow{\text{in probability}} 1, \quad \text{as } j \to \infty$$

if

$$\alpha_j \cdot Z_{1j} \geq 0 \quad Z_{2j} \geq 0 \quad \lim_{j \to \infty} \frac{Z_{2j}}{\alpha_j \cdot Z_{1j}} = 0 \quad \sum_{j=1}^{\infty} \alpha_j \cdot Z_{1j} = \infty. \quad (42)$$

However, as was proved in the foregoing section, the convergence, for all $\eta > 0$, of $\omega(q_j, q_{\text{min}} + \eta)$ to 1 (with probability one, in probability) is equivalent with the convergence (with probability one, in probability) of $\bar{q}_j$ to $q_{\text{min}}$. Theorem 2, also proved in the Appendix, reduces to a mere check of conditions (41) and (42) by finding explicit expressions for $Z_{1j}$ and $Z_{2j}$.

**Theorem 2:** Let there be satisfied (H2), (H3), (H5), (H12):

$$\epsilon_j \geq 0 \quad \lim_{j \to \infty} \epsilon_j = 0 \quad (H12)$$

and let the environment be either deterministic (H6) or Q-model random with (H7) and (H13) holding:

$$\sum_{j=1}^{\infty} \frac{1}{e_j \cdot \beta_{ij} \cdot k^2} < \infty \quad \sum_{j=1}^{\infty} \frac{1}{e_j \cdot \beta_{ij} \cdot T_j \cdot k^2} < \infty \quad (H13)$$

or S-model random or P-model random (H10) with:

$$\sum_{j=1}^{\infty} e^{-\lambda_{ij} \cdot s_j^2 / 2} < \infty \quad \sum_{j=1}^{\infty} e^{-\lambda_{ij} \cdot s_j^2 / 2} < \infty \quad (H14)$$

then the state sequence $\mathcal{X}$ generated by the above given procedure is such that

$$\bar{q}_j \xrightarrow{\text{with probability one}} q_{\text{min}}, \quad \text{as } j \to \infty.$$  

If (H13) and (H14) are replaced by (H15) and (H16), respectively,

$$\lim_{j \to \infty} \frac{1}{\alpha_j \cdot e_j \cdot \beta_{ij} \cdot k^2} = 0 \quad \lim_{j \to \infty} \frac{1}{\alpha_j \cdot e_j \cdot \beta_{ij} \cdot T_j \cdot k^2} = 0 \quad (H15)$$

$$\lim_{j \to \infty} \alpha_j^{-1} \cdot e^{-\lambda_{ij} \cdot s_j^2 / 2} = 0 \quad \lim_{j \to \infty} \alpha_j^{-1} \cdot e^{-\lambda_{ij} \cdot s_j^2 / 2} = 0 \quad (H16)$$

then

$$\bar{q}_j \xrightarrow{\text{in probability}} q_{\text{min}}, \quad \text{as } j \to \infty. \quad (43)$$

**Remark 2.1:** The conditions of Theorem 1 and Theorem 2 mainly differ in that the strong condition $\sum \epsilon_j < \infty$ (H4) is replaced by the weaker condition (H12). Since the rate of decrease of $\epsilon_j$ directly influences the conditions imposed on $\lambda_{ij}$ and $\lambda_{Tj}$, Theorem 2 can, for some environments, be of more value than Theorem 1.

### The Accumulated Average Performance

The theorems do not impose upper bounds on the sampling frequencies $\lambda_{ij}$, $\lambda_{Tj}$, and $\lambda_{Gj}$. In order to extend the theorems to include convergence with respect to the total number of measurements, it is necessary to introduce such upper bounds. Still, the theorems are concerned with the convergence of some state function, let us say $f(X_j)$, such as the average measured performance, etc.

Another more elegant way of dealing with the convergence with respect to the total number of measurements is through accumulated state functions. The idea of proving the convergence for an accumulated average performance was first mentioned by Gurin [6] and later investigated by Saridis et al. [18]. Let $\lambda_j$ denote $\lambda_{ij} + \lambda_{Tj} + \lambda_{Gj}$. Then the accumulated state function $f_{\lambda_j} = f_\lambda(X_0, X_1, \ldots, X_j)$ is defined as

$$f_{\lambda_j} = f_\lambda(X_0, X_1, \ldots, X_j) \leq \frac{\sum_{i=0}^{j} \lambda_i \cdot f(X_i)}{\sum_{i=0}^{j} \lambda_i}. \quad (44)$$

For all the state functions treated in this paper, it is true that

$$f_0 \leq f(X_i) \leq f_{\text{max}} < \infty, \quad X_i \in \mathcal{X}. \quad (45)$$

Assume further that

$$\lim_{j \to \infty} E\{f(X_j)\} = f_0 \quad (46)$$

$$\sum_{i=1}^{\infty} \lambda_i = \infty \quad (47)$$

then

$$\lim_{j \to \infty} E\{f_\lambda(X_0, \ldots, X_j)\} = f_0. \quad (48)$$

The proof can be found in the Appendix. Under the same conditions it can be proved that convergence with probability one (in probability) of $f(X_j)$ to $f_0$ induces convergence with probability one (in probability) of $f_{\lambda_j}$ to $f_0$.

### Conclusion

The main advantage of random search (statistical search) over other direct search techniques is its general applicability, i.e., there are almost no conditions concerning $Q(w)$ (continuity, unimodality, convexity, etc.).

If the problem is completely transferred to the Q-domain, even $W$ is unspecified. Therefore, random search is the basic building block for any multimodal search routine, and hence, mixed search techniques are legion. Theorems 1 and 2 deal with global convergence of such procedures both in deterministic and stochastic environments. The main purpose was to discuss a procedure that would include most of the well-known random search algorithms as a special case. In fact, the scheme is so flexible that mixed search pro-
presented scheme due to the presence of the condition $Frob \{ I'J - \mu_l \sim e \} \sim 2e^{-2}$. Matyas [19] is not deducable as a special case of the state, more sophisticated adaptation algorithms can be derived. It is desired to derive a lower bound for the theorems of convergence. The scheme is too general and should be efficiently adapted for use in specific environments (deterministic or stochastic; anomalous or smooth; finite set $W$ or Euclidean spaces; unimodal or multimodal; $P$, $S$ or $Q$-model; etc.). Particularly, the scheme is competitive with SAVS for use in finite set environments. This comparison deserves more attention, both theoretically and experimentally.

3) The search parameters are number sequences which is as big a disadvantage as it is for stochastic approximation, for instance, because the scheme deals with “global” search. Intuitively, it is felt that “optimal” number sequences, if they exist, will strongly depend upon the form of $p_B(q)$ for anomalous $Q(w)$. Because almost all the basic pdf’s appearing in real problems are continuous in $q$ (even if $Q(w)$ is not continuous in $W$!), it is possible to make some hypotheses about $p_B(q)$ and to derive these theoretically important “optimal” number sequences.

4) A completely new way of proving the convergence of random search is presented here. By extending the state space and finding other Lyapunov-type functions of the state, more sophisticated adaptation algorithms can be treated. Algorithms with many “basepoints” seem interesting in view of their application in multimodal function optimization with several parallel local hill-climbing processes.

5) The deterministic random optimization scheme of Matyas [19] is not deducable as a special case of the presented scheme due to the presence of the condition $\lambda_{Bj} \geq 1$. This suggests that there must exist a statistical search scheme in which, instead of $\zeta_j$, there is another estimate of $Q(w_j)$ which is based upon the mean of some or all of the previous measurements made at the basepoint while $\lambda_{Bj}$ might be zero. Such a scheme with an “aging” basepoint is under investigation now.

6) The core of the scheme is the averaging process to obtain estimates of $Q(w)$. For some environments however such mean estimates are out of the question (for instance, because the variance of the noise is infinite) or inefficient when compared with other statistics. If the noise satisfies some regularity conditions, quantile estimators (median, etc.) can be used instead. In heavy noise situations, the use of thresholds to keep $\zeta$ within certain bounded limits is to be studied. For general purpose computer programs, even “robust” estimation should be given attention.

7) The experimental comparison of this group of techniques is an enormous task and until now, the author only performed serious experiments on anomalous environments ($W$ unspecified), finite set $Q$-model environments, and deterministic environments.

8) The stochastic analog of creeping random search [7], [11], [19], [29] for the local hill-climbing of continuous $Q(w)$ has been studied by Poznyak [12], but the conditions imposed on the noise are not realistic. It seems possible to derive inequalities in the $W$ domain using our approach from which the convergence can be proved—under the most general conditions—using the theorems of Braverman and Rozonoer [9], [22].

**APPENDIX**

In the proofs, some inequalities for sums of independent random variables will be needed. Consider that $\zeta_j$ is the average over $\lambda_j$ iid random variables with pdf $\pi(\zeta)$ and mean $\mu_1$. It is desired to derive a lower bound for $\text{Prob} \{ |\zeta_j - \mu_1| \leq \varepsilon \}, \varepsilon > 0$. Three environments are considered.

1) **Q-model environment**: If all the moments $\mu_k$ up to $k \geq 2$ (even $k$) exist, then the Markov inequality reads:

$$\text{Prob} \{ |\zeta_j - \mu_1| \leq \varepsilon \} \geq 1 - \frac{L_k \cdot \mu_k}{\lambda_j \cdot \varepsilon^k} \tag{A1}$$

where $L_k$ is a constant depending only upon $k$. Also, if $\mu_2$ exists:

$$\text{Prob} \{ |\zeta_j - \mu_1| \leq \varepsilon \} \geq 1 - \frac{L_1 \cdot \mu_1^*}{\lambda_j \cdot \varepsilon} \geq 1 - \frac{\sqrt{\mu_2}}{\lambda_j \cdot \varepsilon} \tag{A2}$$

where

$$\mu_1^* \triangleq E\{ |\zeta - \mu_1| \} = \int |\zeta - \mu_1| \cdot \pi(\zeta) d\zeta$$

in view of $\mu_1^* \leq \sqrt{\mu_2}$ and $L_1 = 1$.

2) **P-model environment** ($\zeta \in \{0,1\}$) and 3) **S-model environment** ($\zeta \in [0,1]$): Hoeffding’s inequality for sums of iid bounded random variables reads:

$$\text{Prob} \{ |\zeta_j - \mu_1| > \varepsilon \} \leq 2e^{-2\lambda_j \varepsilon^2} \tag{A3}$$
If $\zeta \in [A,B]$, $|A| < \infty$, $|B| < \infty$, (A4) holds:

$$\text{Prob} \{ |\zeta_j - \mu_j | > \varepsilon \} \leq 2e^{-2\lambda_j(\theta - \mu_j)^2}$$  \hfill (A4)

**Proof of Theorem 1**

$Z(X_j)$ defined in (32) is maximal for $\rho(u/X_j) = \delta(u - \theta)$, where $\theta$ is specially selected to maximize $Z(X_j)$. Thus

$$Z_j = \sup_{X_j \in X} \{ \zeta_j - \theta \} \cdot E [ \phi(q_j, \theta) - \xi_j^*(q_j, \theta) \}].$$  \hfill (A5)

Consider three cases:

1) $\theta > q_j$
2) $q_j \geq \theta > q_j - \varepsilon_j - \rho$
3) $q_j - \varepsilon_j - \rho \geq \theta$

where $\rho$ is a freely chosen positive quantity (which may depend upon $j$).

**Case 1:** $\theta > q_j$; $\phi(q_j, \theta) = 0$.

**1a) Deterministic environment:** $\mu_2 = 0$, and thus

$$\xi_j^*(q_j, \theta) = \phi(q_j, \theta) = Z(X_j) = 0.$$  \hfill (A6)

**1b) Random environment:** There holds

$$E [ \xi_j^*(q_j, \theta) ] \leq \text{Prob} \{ \zeta_j < q_j - \frac{e_j - \theta - q_j}{2} \}$$

$$+ \text{Prob} \{ \zeta_j^* > \theta + \frac{e_j + \theta - q_j}{2} \}$$

$$\leq \text{Prob} \{ \zeta_j - q_j \geq \frac{e_j + \theta - q_j}{2} \}$$

$$+ \text{Prob} \{ \xi_j^* - \theta \geq \frac{e_j + \theta - q_j}{2} \}$$  \hfill (A6)

where $\zeta_j$ and $\zeta_j^*$ are averages over $\lambda_{Bj}$ and $\lambda_{Tj}$ iid random variables with pdf $\pi(\zeta_j)$ and $\pi(\zeta_j)$ and respective means $Q(w_j) = q_j$ and $Q(w_j) = \theta$. By definition of $E [ \xi_j^*(q_j, \theta) ]$, the inequalities (A1)-(A4), (A6) can be sharpened: first, for $Q$-model environments with $\mu_k$ existing ($k \geq 2$), (A1) and (A2) hold. Thus

$$Z_j = \sup_{\rho} \phi(q_j, \theta) \cdot \left( \frac{L_k \cdot \mu_k}{(e_j + \theta - q_j)^2} \right)^k \cdot (\lambda_{Bj}^{1/2} + \lambda_{Tj}^{1/2}).$$  \hfill (A7)

Since $\mu_2$ exists, (A7) also holds for $k = 1$, in which case $\mu_1$ has to be replaced by $\sqrt{\mu_2}$. Expression (A7) is maximal for

$$\theta = \infty, \quad \text{for } k = 1.$$  \hfill (A8)

Clearly, substituting $\theta$ in (A7) by its value computed in (A8), one obtains

$$Z_j = \left( \frac{L_k \cdot \mu_k}{(e_j)^2} \right) \cdot (\lambda_{Bj}^{-1/2} + \lambda_{Tj}^{-1/2}).$$  \hfill (A9)

Secondly, for $P$ and $S$-model environments, using (A3)-(A4):

$$Z_j = \sup_{\theta} \left( \frac{L_k \cdot \mu_k}{(q_j - \theta)^2} \right) \cdot (\lambda_{Bj}^{-1/2} + \lambda_{Tj}^{-1/2})$$

$$+ e^{-\lambda_{Tj}^{-1/2} \cdot (q_j - \theta + \varepsilon_j)^2}.$$  \hfill (A10)

Consider the first term in (A10) with $a = \sqrt{\lambda_{Bj}^{-1} + 1}, b = \alpha \varepsilon_j$ and $x = \theta - q_j > 0$. Since $xe^{-(a + b)x^2}$ has a maximal value for $ax + b = -1/2ax$ of $e^{-1/4a^2x^2}$ and since the positive root $x_0$ of $(ax)^2 + b(ax) = 1 = 0$ is $x_0 = (b/a) \cdot (1 + 2/b^2 - 1) = (Hab)$, we have by the monotonicity of $e^{-1/4a^2x^2}$ that the first term is always smaller than $(ab)^{-1} e^{-b/4}$. Maximizing the second term separately gives

$$Z_j = 4(\lambda_{Bj} \varepsilon_j)^{-1} e^{-\lambda_{Bj} \varepsilon_j^2/8} + 4(\lambda_{Tj} \varepsilon_j)^{-1} e^{-\lambda_{Tj} \varepsilon_j^2/8}.$$  \hfill (A11)

Case 2: $q_j \geq \theta > q_j - \varepsilon_j - \rho \geq \rho$; $\phi(q_j, \theta) = 1$. No sharper bound can be found for $E [ \xi_j^*(q_j, \theta) ]$ than $E [ \xi_j^*(q_j, \theta) ] \geq 0$. For all the environments, we have

$$Z_j = \sup_{\rho} \left( \frac{L_k \cdot \mu_k}{(q_j - \theta)^2} \right) \cdot (\lambda_{Bj}^{-1/2} + \lambda_{Tj}^{-1/2}).$$  \hfill (A12)

because $q_j - \theta < \varepsilon_j + \rho$ in the given interval and $\rho = \varepsilon_j$.

Case 3: $\theta \leq q_j - \varepsilon_j - \rho$. $\phi(q_j, \theta) = 1$. Furthermore, by definition, $\phi(q_j, \theta) = 1$:

$$E [ \xi_j^*(q_j, \theta) ] \geq \text{Prob} \left\{ \left| \zeta_j - \frac{q_j - \theta + \varepsilon_j}{2} \right| \right\}$$

$$\cdot \text{Prob} \left\{ \left| \zeta_j^* - \theta \right| \leq \frac{q_j - \theta + \varepsilon_j}{2} \right\}.$$  \hfill (A13)

**3a) Deterministic environment:** Clearly, $Z(X_j) = 0$.

**3b) Q-model random environment:** Replacing the probabilities in (A13) by the bounds given in (A1) and (A2), there holds:

$$E [ \xi_j^*(q_j, \theta) ] \geq \left( 1 - \frac{L_k \mu_k}{(q_j - \theta + \varepsilon_j)^2} \right) \cdot \left( \lambda_{Bj}^{-1/2} + \lambda_{Tj}^{-1/2} \right).$$  \hfill (A14)

Disregarding the product term and substituting the last expression in (A5) yields

$$Z_j = \sup_{\theta} \left( \frac{L_k \mu_k}{(q_j - \theta + \varepsilon_j)^2} \right) \cdot (\lambda_{Bj}^{-1/2} + \lambda_{Tj}^{-1/2})$$

which is maximal on the boundary of the interval, i.e., for $q_j - \theta = 2\varepsilon_j$ since $\rho = \varepsilon_j$. Thus

$$Z_j = \frac{L_k \mu_k}{(q_j - \theta + \varepsilon_j)^2} \cdot (\lambda_{Bj}^{-1/2} + \lambda_{Tj}^{-1/2})$$  \hfill (A15)

**3c) S,P-model environments:** Using (A3), by a similar procedure as under 3b):

$$Z_j = \sup_{\theta} \left( \frac{L_k \mu_k}{(q_j - \theta + \varepsilon_j)^2} \right) \cdot (\lambda_{Bj}^{-1/2} + \lambda_{Tj}^{-1/2})$$

$$+ e^{-\lambda_{Tj}^{-1/2} \cdot (q_j - \theta + \varepsilon_j)^2}.$$  \hfill (A16)
and (A16) is never greater than (A17):
\[ Z_j = 4e_j^{-1} \cdot (\lambda_{B_j}^{-1} \cdot e^{-e_j/4}) + \lambda_{T_j}^{-1} \cdot e^{-e_j/4}. \] (A17)
Together, safe upper limits \( Z_j \) are obtained by summing the expressions in each interval. For \( S,P \)-model environments, (A17) is identical to (A11) and can be dropped. Thus, for deterministic environments:
\[ Z_j = 2e_j \]
for \( Q \)-model random environments:
\[ Z_j = B_k \left( \frac{1}{d_j^k} \cdot \lambda_{B_j}^{-k} + \frac{1}{d_j^{-1} \cdot \lambda_{T_j}^{-k}} \right) + 2e_j \]
\[ B_k = L_k \cdot \mu_k \cdot \max \left( (k - 1)^{k-1} \cdot \left( \frac{k}{2} \right)^{-k} ; 2^{k+1} \cdot 3^{-k} \right) \]
and for \( S,P \)-model random environments:
\[ Z_j = 8e_j^{-1} \cdot (\lambda_{B_j}^{-1} \cdot e^{-e_j/4}) + \lambda_{T_j}^{-1} \cdot e^{-e_j/4} \]
Equation (A18) together with (33) leads to the conditions formulated in Theorem 1.

**Proof of Theorem 2**

Notice that
\[ E \left\{ \xi^*(q_j, \theta) / q_j > q_{\text{min}} + \eta, \theta \geq q_{\text{min}} + \eta \right\} \]
\[ \geq \Pr \left\{ |\xi_j - q_j| \leq \frac{(q/2) - \varepsilon_j}{2} \right\} \]
\[ \cdot \Pr \left\{ \xi_j^* - \theta | \leq \frac{\eta}{2} \right\} \cdot \omega \left( e_j, \frac{\eta}{4} \right) \]
\[ \geq \Pr \left\{ |\xi_j - q_j| > \frac{\varepsilon_j}{2} \right\} + \Pr \left\{ \xi_j^* - \theta | > \frac{\varepsilon_j}{2} \right\} \]
\[ \geq \Pr \left\{ |\xi_j - q_j| > \frac{\varepsilon_j}{2} \right\} \]
\[ \geq \omega \left( e_j, \frac{\eta}{4} \right) \]
\[ \geq \frac{3}{4} \omega \left( e_j, \frac{\eta}{4} \right) \cdot \omega \left( e_j, \frac{\eta}{4} \right) \cdot \omega \left( (\lambda_{T_j}^{-1} \cdot 1/2)^{1/3} \right) \]
\[ \cdot \omega \left( (\lambda_{B_j}^{-1} \cdot (\eta/8)^2 \right) \]
(\text{A20})

Since \( \theta \to q_{\text{min}} + \eta \) is the worst case for all the environments (see (A1)-(A3)). Expressions (A19) and (A20) are useful in combination with (39), where \( Z_{1j} \) and \( Z_{2j} \) are defined. Each type of environment is now considered separately.

1) **Deterministic environments:** Clearly, (A20) and thus \( Z_{2j} \) are 0. The right side in (A19) is equal to \( \omega(e_j, \eta/4) \). Condition (41) reads:
\[ \sum_{j=1}^{\infty} \xi_j \cdot \omega \left( e_j, \frac{\eta}{4} \right) = \infty \] (A21)
which holds in view of (H3) and (H12), for every \( \eta > 0 \).

2) **Q-model random environments:** Combining (A1) and (A2) with (A20) gives a bound for \( Z_{2j} \):
\[ Z_{2j} = \frac{L_k \cdot \mu_k}{(e_j/2)^k} \cdot (\lambda_{B_j}^{-k/2} + \lambda_{T_j}^{-k/2}). \] (A22)
Combining (A1) and (A2) with (A19) gives a bound for \( Z_{1j} \):
\[ Z_{1j} = \left( 1 - \frac{L_k \cdot \mu_k}{(\eta/8)^k} \right) \cdot \left( \lambda_{B_j}^{-k/2} + \lambda_{T_j}^{-k/2} \right) \cdot \omega \left( e_j, \frac{\eta}{4} \right) \]
\[ \geq \frac{3}{4} \omega \left( e_j, \frac{\eta}{4} \right) \cdot \omega \left( \lambda_{B_j}^{-1}, \frac{(\eta/8)^2}{(3L_k \cdot \mu_k)^2/8} \right) \]
(A23)

and this lower bound tends in the limit to 1/3, since \( e_j \to 0 \), \( \lambda_{Bj} \to \infty \), and \( \lambda_{Tj} \to \infty \). Using (A22) and (A23), there is no problem in deducing the conditions of convergence stated in Theorem 2 from (41) or (42).

3) **S,P-model random environments:** Combining (A3) and (A4) with (A20) gives a bound for \( Z_{2j} \):
\[ Z_{2j} = 2(e^{-e_j/2} \cdot B_j/2 + e^{-e_j/2} \cdot T_j/2). \] (A24)
Combining (A3) and (A4) with (A19) gives a bound for \( Z_{1j} \):
\[ Z_{1j} \geq \left( 1 - 2e^{-q_j \cdot (\eta/16} - 2e^{-q_j \cdot (\eta/16} \right) \cdot \omega \left( e_j, \frac{\eta}{4} \right) \]
\[ \geq \frac{3}{4} \omega \left( e_j, \frac{\eta}{4} \right) \cdot \omega \left( \lambda_{B_j}^{-1}, \frac{(\eta/8)^2}{(3L_k \cdot \mu_k)^2/8} \right) \]
(A25)

and this lower bound tends in the limit to 1/3, since \( e_j \to 0 \), \( \lambda_{Bj} \to \infty \), and \( \lambda_{Tj} \to \infty \). Using (A24) and (A25), there is again no problem in deducing the conditions of convergence stated in Theorem 2 from (41) or (42).

As is shown in the text, the convergence, for all \( \eta > 0 \), of \( \omega(q_j, q_{\text{min}} + \eta) \) to 1 is linked with the convergence of \( \bar{q}_j \) to \( q_{\text{min}} \). This completes the proof.

**Proof of (48)**

Equation (47) and \( E\{f(X_j)\} \to f_0 \) imply that
\[ E\{f(X_j)\} = L_j^{-1} \cdot \sum_{i=1}^{\infty} \lambda_i \cdot E\{f(X_i)\} \to f_0 \]
where
\[ L_j = \sum_{i=1}^{\infty} \lambda_i \]
by Toeplitz's lemma (see [10, p. 238]).

**References**


