Statistics & Probability Letters 2 (1984) 257–261
North-Holland

October 1984

METHODS FOR GENERATING RANDOM VARIATES WITH POLYA CHARACTERISTIC FUNCTIONS

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Received February 1984

Abstract: Polya has shown that real even continuous functions that are convex on $(0,\infty)$, 1 for $t=0$, and decreasing to 0 as $t\to\infty$ are characteristic functions. Dugué and Girault (1955) have shown that the corresponding random variables are distributed as Y/Z where Y is a random variable with density $(2\pi)^{-1}(\sin(x/2)/(x/2))^2$, and Z is independent of Y and has distribution function $1-\phi+t\phi'$, $t>0$. This property allows us to develop fast algorithms for this class of distributions. This is illustrated for the symmetric stable distribution, Linnik's distribution and a few other distributions. We pay special attention to the generation of Y.

Keywords: random variate generation, Polya characteristic function, symmetric stable distribution, convexity, algorithms.

1. Introduction

Some distributions are best described by their characteristic functions. In some cases, other descriptions (densities, distribution functions, etc.) are not known in a simple analytical form. We are thinking for example about the symmetric stable distribution with parameter $\alpha\in(0,2]$: it has characteristic function $\exp(-|t|^\alpha)$. Yet, except for $\alpha=\frac{1}{2}$, 1 and 2, the density can only be expressed by a convergent series, or an asymptotic expansion, or an integral (Bergstrom, 1952; Feller, 1966; Zolotarev, 1964; see Lukacs (1970) for a comprehensive survey). For this distribution, algorithms can be derived that are based upon the integral representation of Zolotarev (see Chambers, Mallows and Stuck, 1976): for $\alpha\neq1$, random variates can be generated as

$$\frac{\sin(\alpha U)}{(\cos U)^{1/\alpha}}\left(\frac{\cos((1-\alpha)U)}{E}\right)^{(1-\alpha)/\alpha}.$$

where U is uniform $[\pi/2, \pi/2]$ and E is exponential and independent of U. For $\alpha=1$, replace this by $\tan U$. It seems possible to apply the Bergstrom–Feller series to obtain algorithms via the series method (Devroye, 1981a). Since this method requires good truncation bounds, the paper by Bartels (1981) would probably be very helpful. Both solutions are 'ad hoc': there is no general principle behind their development. One would like to have methods that are applicable to large classes of characteristic functions.

One attempt in this direction (Devroye, 1981b) required the knowledge and finiteness of $\int|\phi|$ and $\int|\phi''|$ where ϕ is the characteristic function in question. In the algorithm, at least one inversion integral

$$f(x)=(2\pi)^{-1}\int e^{-ixt}\phi(t)\,dt$$

is needed. A necessary condition for the finiteness of these integrals is that f, the density, be bounded and $O(x^{-2})$ as $|x|\to\infty$. Of course, the fact that an integral has to be evaluated is a serious drawback.

In this note, we would like to point out that for distributions with Polya-characteristic functions, i.e. real even continuous functions ϕ with $\phi(0)=1$,

Research supported by NSERC Grant A3456 and FCAC Grant EQ-1679.

0167-7152/84/$3.00 © 1984, Elsevier Science Publishers B.V. (North-Holland)

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Amsterdam

$\lim_{t\to\infty}\phi(t)=0$, convex on $(0,\infty)$, there exists a simple general method for random variate generation. The procedure requires the explicit knowledge of ϕ and at least one derivative of ϕ. It leads to fast competitive algorithms for very specific classes of distributions such as the symmetric stable distribution. The examples that we will consider throughout this note are

(A) $\phi(t)=\mathrm{e}^{-|t|^\alpha}$, $0<\alpha\leqslant 1$ (for $\alpha=1$, this is the Cauchy distribution);

(B) $\phi(t)=1/(1+|t|^\alpha)$, $0<\alpha\leqslant 1$ (for $\alpha\in(0,2]$, this is a characteristic function of a unimodal density; see Linnik (1953) and Lukacs (1970, pp. 96–97));

(C) $\phi(t)=(1-|t|)^\alpha$, $|t|\leqslant 1$, $\alpha\geqslant 1$;

(D) $\phi(t)=1-|t|^\alpha$, $|t|\leqslant 1$, $0<\alpha\leqslant 1$.

The point of this note is that for these distributions, we can generate random variates without ever trying to compute the density or distribution function.

2. A property of Polya-type characteristic functions

The attentive reader is urged to read Section 4.3 of Lukacs (1970). Dugué and Girault (1955) and Girault (1954) have shown that Polya-type characteristic functions can be decomposed as follows:

$$\phi(t)=\int_0^\infty\left(1-\left|\frac{t}{s}\right|\right)_+\mathrm{d}F(s),\quad t>0,$$

$$\phi(t)=-\phi(t),\quad t<0,$$

where $(\cdot)_+$ is the positive part of $(\cdot)$ and F is a distribution function with $F(0)=0$:

$$F(s)=1-\phi(s)+s\phi'(s),\quad s>0.$$

Here ϕ' is the right-hand derivative of ϕ (which exists everywhere).

But $(1-|t|)_+$ is the characteristic function of the Fejer–de la Vallee Poussin (FVP) density

$$(2\pi)^{-1}\left(\frac{\sin(x/2)}{x/2}\right)^2.$$

Thus, because we have a very simple mixture, we can conclude that if Y is an FVP random variable, and Z is an independent random variable with distribution function F, then $X=Y/Z$ has characteristic function ϕ.

The generation of Y is a trivial problem, and will be discussed in Section 4. The intriguing property here is that the distribution function F is a simple function of ϕ and ϕ'. In a sense, we have switched from t-space to the real line. All of this can be summarized as follows:

Property. *Let ϕ be a Polya-type characteristic function with right-hand derivative ϕ'. Then $X=Y/Z$ has this characteristic function, where Y, Z are independent random variables: Y has the FVP density, and Z has distribution function*

$$F(s)=1-\phi(s)+s\phi'(s),\quad s>0,\qquad F(0)=0.$$

In addition, if $\phi(s)-s\phi'(s)$ is absolutely continuous, then Z has density given by

$$g(s)=s\phi''(s),\quad s>0.$$

This property leads to yet another integral representation of the density of X, but this matter won't be pursued here.

3. Examples

In this section, we consider the examples (A)–(E) of Section 1. For these distributions, we will derive F and mention how random variates with distribution function F can be obtained.

A. Symmetric stable distribution

It can easily be verified that Z has density g given by

$$g(s)=\left(\alpha^2 s^{2\alpha-1}+\alpha(1-\alpha)s^{\alpha-1}\right)\mathrm{e}^{-s},\quad s>0.$$

But we note that Z^α has density

$$\alpha(s\mathrm{e}^{-s})+(1-\alpha)(\mathrm{e}^{-s}),\quad s>0,$$

i.e. a mixture of a gamma (2) and an exponential density. Thus, Z is distributed as

$$\left(E_1+E_2 I_{[U<\alpha]}\right)^{1/\alpha}$$

where E_1, E_2 and U are independent random variables: E_1 and E_2 have an exponential density, and U is uniformly distributed on [0,1]. To save a uniform [0,1] random variate, we can replace

$E_2 I_{[U<\alpha]}$ by $\max(E_2 + \log \alpha, 0)$. By using the property that an exponential random variable is distributed as minus the logarithm of a uniform [0,1] random variable, we obtain the result that Z is distributed as

$$\log^{1/\alpha}\left(\max\left(\frac{\alpha}{U_1 U_2}, \frac{1}{U_1}\right)\right).$$

where U_1 and U_2 are independent uniform [0,1] random variables. In some cases, the U_1, U_2 implementation is faster than the E_1, E_2 implementation. In most cases, the generation of X as Y/Z is competitive with the method of Chambers, Mallows and Stuck (1976). It is also worthwhile pointing out that a Cauchy random variable is distributed as $Y \log^{-1}(1/(U_1 U_2))$ and as $Y(E_1 + E_2)^{-1}$.

Mitra (1981) has shown that if $\alpha = 2^{2-n}$, $n \geqslant 3$, the random variable $N_1/(2^{2^{n-2}-1} N_2 (N_3^2)^1 \cdots (N_n^2)^{2^{n-3}})$ has characteristic function $\exp(-|t|^\alpha)$ when $N_1, \ldots, N_n$ are independent normal (0,1) random variables. Our method for the symmetric stable distribution can be used for all α, and is fast because the distribution of Y is fixed (α-independent) and the generation of Z is fast because it does not involve any sin or cos evaluations. Unfortunately, there is no straightforward extension to all stable distributions, including those with skewness parameter $\beta \neq 0$. For $\beta = 1$, $\alpha = 2^{-m}$, see Brown and Tukey (1946); for $\beta = 1$ and all α, see Kanter (1975) who has a method based upon an integral representation of Ibragimov and Chernin (1959). For general (α, β), the method of Chambers, Mallows and Stuck (1976) remains unchallenged.

B. Linnik's distribution

We verify that Z has density g given by

$$g(s) = \left((\alpha^2 + \alpha) s^{2\alpha-1} + (\alpha - \alpha^2) s^{\alpha-1}\right)(1 + s^\alpha)^{-3},$$

$$s > 0.$$

It is perhaps easier to work with the density of Z^α:

$$\frac{s(\alpha+1) + (1-\alpha)}{(1+s)^3}, \quad s > 0.$$

This latter density has distribution function

$$1 - \frac{1+\alpha}{1+s} + \frac{\alpha}{(1+s)^2},$$

and this is easy to invert. Thus, a random variate Z can be generated as

$$\left(\frac{\alpha + 1 - \sqrt{(\alpha+1)^2 - 4\alpha U}}{2U} - 1\right)^{1/\alpha},$$

where U is a uniform [0,1] random variate. If speed is important, the square root can be avoided if we use the rejection method for the density Z^α, with dominating density $1/(1+u)^2$ (the density of $(1/U) - 1$). A little work shows that Z can thus be generated as follows:

Repeat Generate two independent uniform [0,1] random variates U, V. Set $X \leftarrow (1/U) - 1$.
Until $2\alpha U < V$. (Now, X is distributed as Z^α.)
Exit with $X \leftarrow X^{1/\alpha}$. ($X$ is distributed as Z.)

The average number of loops is $1 + \alpha$.

C. $\phi(t) = (1 - |t|)_+^\alpha$

For $\alpha > 1$, $\phi - s\phi'$ is absolutely continuous. Thus, Z has density $g(s) = \alpha(\alpha - 1)s(1-s)^{\alpha-2}$, $0 \leqslant s \leqslant 1$. This is the beta (2, $\alpha - 1$) density. Z can be obtained directly by means of a fast beta generator, or as $G/(G + G^*)$ where G, G^* are independent gamma (2) and gamma $(1+\alpha)$ random variates. For beta and gamma generators, we refer to the surveys of Schmeiser (1980), Schmeiser and Babu (1980) and Tadikamalla and Johnson (1981), where further references are found to the algorithms of Ahrens and Dieter, Best, Cheng, Marsaglia and others.

D. $\phi(t) = (1 - |t|^\alpha)_+$

This example was chosen to illustrate the fact that F does not have to be absolutely continuous. In fact, $F(s) = (1-\alpha)s^\alpha$ on (0,1) and $F(1) = 1$. Thus, F has an atom of weight α at 1, and has an absolutely continuous part of weight $1 - \alpha$ on (0,1). The absolutely continuous part has density $\alpha s^{\alpha-1}$, $0 \leqslant s \leqslant 1$, which is the density of $U^{1/\alpha}$, where U is uniform on [0,1]. Thus, we have:

$$Z = \begin{cases} 1 & \text{with probability } \alpha, \\ U^{1/\alpha} & \text{with probability } 1-\alpha. \end{cases}$$

Here too, we can use the standard trick of recuperating part of the uniform [0,1] random variate used to make the 'with probability α' choice.

4. The Fejer-de la Vallee Poussin density

In this section, we will briefly describe a relatively fast algorithm for generating a random variate Y with the FVP density

$$(2\pi)^{-1}\left(\frac{\sin(x/2)}{x/2}\right)^2.$$

It is clear that this random variate is distributed as $2/W$ where W has density

$$f(x) = \frac{1}{\pi}\sin^2\left(\frac{1}{x}\right).$$

The density f is very tractable in view of

$$f(x) \leqslant \min\left(\frac{1}{\pi}, \frac{1}{\pi x^2}\right) = \frac{4}{\pi}\min\left(\tfrac{1}{4}, \tfrac{1}{4}x^{-2}\right) = \frac{4}{\pi}h(x),$$

where h is the density of V^B where V is uniform $[-1,1]$ and B is $+1$ and -1 with equal probability $\frac{1}{2}$, and B and V are independent. Thus, simple rejection with dominating density h gives:

Repeat Generate (U, V) uniformly in $[-1,1]^2$.
If $U<0$, set $V \leftarrow 1/V$. (V now has density h.)
Until $|U|\min(1, 1/V^2) < \sin^2(1/V)$.
Exit with $W \leftarrow V$.

A slight improvement can be obtained by implementing this as follows:

Repeat Generate (U, V) uniformly in $[-1,1]^2$.
If $U<0$, set $(U, V) \leftarrow (-UV^2, 1/V)$.
Until $U < \sin^2(1/V)$.
Exit with $W \leftarrow V$.

In both cases, the average number of iterations is $4/\pi$. If the average time must be reduced by all means, it pays to avoid the sin computation. For an argument x in $[0, \pi/2]$, $\sin^2 x$ is bounded from above by

$$x^2 \quad \left(\text{useful in range } \left[0, \frac{\pi}{4}\right]\right)$$

and

$$\left(1 - \tfrac{1}{2}y^2 + \tfrac{1}{24}y^4\right)^2, \quad y = \frac{\pi}{2} - x \quad \left(\text{useful in range } \left[\frac{\pi}{4}, \frac{\pi}{2}\right]\right),$$

and from below by

$$\left(x - \tfrac{1}{6}x^3\right)^2 \quad \text{and} \quad \left(1 - \tfrac{1}{2}y^2\right)^2.$$

These bounds can be used to accept or reject most of the time without having to evaluate the sinus. Arguments outside $[0, \pi/2]$ can of course always be reduced to arguments in this range. The point here is that generating W (and thus Y) can be made very fast, so that in most cases, the cost of generating $X = Y/Z$ is nearly equal to the cost of generating Z.

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