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On the oscillation of the expected number of extreme points of a random set

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Abstract: Let EN_n be the expected number of extreme points among n i.i.d. points with a common radially symmetric distribution in the plane. We show that for any monotone sequence $\omega_n \uparrow \infty$ and for every $\varepsilon > 0$, there exists a radially symmetric distribution for which $EN_n \geqslant n/\omega_n$ infinitely often and $EN_n \leqslant 4+\varepsilon$ infinitely often. In addition, there exists a unimodal radially symmetric density such that $EN_n \geqslant n^{1/3}/\omega_n$ infinitely often and $EN_n \leqslant 4+\varepsilon$ infinitely often.

Keywords: Convex hull, conterexamples, symmetric distributions, stochastic geometry.

1. Introduction

Let $X_1, \ldots, X_n$ be i.i.d. random variables with a radially symmetric distribution in the plane, i.e. X_1 is distributed as $(R \cos \Theta, R \sin \Theta)$, where Θ is uniformly distributed on $[0, 2\pi]$, and R is independent of Θ and has a given distribution on the positive reals. We let N_n be the number of points on the convex hull formed by $X_1, \ldots, X_n$ (the convex hull is the subset of $X_1, \ldots, X_n$ consisting of all extreme points; and a point X_j is extreme if all X_i's belong to one halfspace defined by a hyperplane passing through X_j). We would like to show by means of a constructive counterexample that EN_n can oscillate wildly as a function of n, even for this restricted class of distributions.

Theorem 1. *Let $\omega_n \uparrow \infty$ be given, and let $\varepsilon > 0$ be arbitrary. Then there exists a radially symmetric distribution such that*

$$EN_n \geqslant n/\omega_n \quad \textit{infinitely often}$$

and

$$EN_n \leqslant 4+\varepsilon \quad \textit{infinitely often}.$$

In the example, EN_n oscillates infinitely often between approximately 4 and an upper bound which is $o(n)$ but as close to n as desired (because ω_n is allowed to increase at any slow rate). Note that the lower bound cannot be improved upon in view of asymptotic bounds for radially symmetric distributions obtained by Carnal (1970). Our examples are mixtures of absolutely continuous and singular distributions, but trivial modifications can be introduced that allow us to construct pure absolutely continuous distributions as well. For such distributions, Devroye (1981) has shown that $EN_n = o(n)$, so that the upper bound of Theorem 1 is also not improvable. For particular radially symmetric distributions, such as normal distributions, or the uniform distribution in the unit circle, EN_n is well studied. We refer for this to

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the pioneering work of Rényi and Sulanke (1963, 1964), and to the key contributions of Carnal (1970), who treated many distributions according to the rate of decay of the tail of the distribution of R. Other major contributions are due to Efron (1965), Raynaud (1970), Eddy (1980), Eddy and Gale (1981), Groeneboom (1988), Borgwardt (1987), Davis, Mulrow and Resnick (1987), Dwyer (1988, 1989), Brozius (1989) and Borgwardt, Gaffke, Jünger and Reinelt (1989). Excellent surveys are now available in Buchta (1985) and Schneider (1988).

The example of Theorem 1 uses a pathological distribution for R. We only selected it for convenience. The same manner of constructing examples can be employed for 'nicer' counterexamples. For example, assume that we would like to consider radially symmetric distributions with a unimodal density $g(\|x\|)$. It is known that for the uniform distribution on the unit circle, $EN_n \sim cn^{1/3}$ for some constant c (Rényi and Sulanke, 1963, 1964). For any unimodal radially symmetric distribution, we have in fact $EN_n = O(n^{1/3})$ (we could not find a reference for this, but we can show that for all n, $EN_n \leqslant 74 + \frac{3}{2}(8\pi^2 n)^{1/3}$). These results show that the following theorem cannot be improved upon:

Theorem 2. *Let $\omega_n \uparrow \infty$ be given, and let $\varepsilon > 0$ be arbitrary. Then there exists a radially symmetric distribution with a unimodal density such that*

$$EN_n \geqslant n^{1/3}/\omega_n \quad \textit{infinitely often}$$

and

$$EN_n \leqslant 4 + \varepsilon \quad \textit{infinitely often}.$$

Gruber (1983) (see also Bárány and Larman, 1988) has shown that for every ω_n as in Theorem 2, there exists a convex set of unit volume in $\mathbb{R}^d$ such that for the uniform distribution on this convex set, we have

$$EN_n < \omega_n \log^{d-1} n,$$

and

$$EN_n > (1/\omega_n) n^{(d-1)/(d+1)}.$$

This oscillation result is related to ours, and basically reflects that the extreme cases for EN_n are the polytopes (small EN_n) and the spheres (large EN_n).

2. Proof of Theorem 1

We can assume without loss of generality that $n/\omega_n \to \infty$ (otherwise, replace ω_n by $\min(\omega_n, \sqrt{n})$) and that ω_n is strictly increasing (otherwise, replace it by $\omega_n + 1 - 1/n$). Let us extend ω_n to a continuous strictly increasing function $\omega(x)$ on the positive reals, so that its inverse is well-defined. We also need a strictly decreasing function ρ on the positive reals such that $\rho(n) \geqslant (5/n)\log(20n/\varepsilon)$, where $\varepsilon > 0$ is arbitrary.

To construct our distribution, we need to describe three sequences of parameters, p_i, probabilities summing to one, r_i, certain radii, and n_i, certain sample sizes. The latter two sequences will be monotonically increasing, while the former is monotonically decreasing in i. We begin with $n_0 = 1$, $r_0 = 1$ and $p_1 = \frac{1}{2}$. Then we compute the other parameters by iterating through the following loop for i from one to ∞:

$$\begin{aligned}
&n_{2i-1} \leftarrow \lceil \omega^{\text{inv}}(3/(2p_{2i-1})) \rceil. \\
&p_{2i} \leftarrow \min(p_{2i-1}^2, 1/(15 n_{2i-1})). \\
&n_{2i} \leftarrow \lceil \max(\rho^{\text{inv}}(p_{2i}), i/p_{2i}) \rceil. \\
&p_{2i+1} \leftarrow \min(p_{2i}^2, 1/(i n_{2i}^2)). \\
&r_{2i-1} \leftarrow \max((3n_{2i-1})^{1/a} r_{2i-2}, r_{2i-2}(i n_{2i}^2)^{1/a}). \\
&r_{2i} \leftarrow 100 r_{2i-1}.
\end{aligned}$$

Since $p_i \leqslant 2^{-2^{i-1}}$ for all $i \geqslant 1$, we see that the probabilities sum to a constant $c < 0.8$. We set $p_0 = 1 - c$. To describe the distribution of (R, Θ), it helps to look at the probability space for the random vector (Z, R^*, Θ), which consists of three independent random variables:

(1) Z is a random integer with distribution determined by the vector of p_i's: $\boldsymbol{P}(Z = i) = p_i$, $i \geqslant 0$.

(2) R^* has density a/r^{1+a} on $[1, \infty)$, and $a > 0$ is a positive integer chosen as a function of ε only.

(3) Θ is uniformly distributed on $[0, 2\pi]$.

To put things together, define R by the following rule:

$$R = \begin{cases} r_1 & \text{if } Z = 0, \\ r_{2i-1} & \text{if } Z = 2i - 1,\ i \geqslant 1, \\ r_{2i} R^* & \text{if } Z = 2i,\ i \geqslant 1. \end{cases}$$

In other words, if Z is odd or zero, we define $R = r_Z$, and otherwise $R = r_Z R^*$. Note also that $\boldsymbol{P}(R \geqslant r \mid Z = 2i) = \boldsymbol{P}(R^* \geqslant r/r_{2i}) = (r_{2i}/r)^a$.

Theorem 1 is proved by showing that as $i \to \infty$, we have $\boldsymbol{E}N_{n_{2i-1}} \geqslant n_{2i-1}/\omega_{n_{2i-1}}$ for all i large enough, and $\boldsymbol{E}N_{n_{2i}} \leqslant 4 + \varepsilon$ for all i large enough. This is achieved basically by noting that at sample size n_{2i-1}, the convex hull is nearly always formed by the points on the rim of the circle of radius r_{2i-1}, while at sample size n_{2i}, the convex hull is nearly always determined by the points whose radius is distributed as $r_{2i}R^*$. The proof consists of showing that failures of these desirable events have on the average an asymptotically negligible influence.

Let N_{ni} and $N_{n,i}$ denote the number of data points which are sampled from the ith part in the mixture; thus, N_{ni} is binomial (n, p_i).

Lemma 1. *If $n = n_{2i-1}$, we have for all i large enough, $\boldsymbol{E}N_n \geqslant n/\omega_n$.*

Proof. We use an embedding argument. Let the data consist of i.i.d. triples (Z_k, R_k^*, Θ_k), each distributed as (Z, R^*, Θ), which were defined above. Recall that these three random variables are independent. Thus,

$$\begin{aligned}
\boldsymbol{E}N_n &\geqslant \boldsymbol{E}\left(N_{n,2i-1} I_{[\cap_{k=1}^n [R_k \leqslant r_{2i-1}]]}\right) \\
&= \boldsymbol{E}\left(\sum_{j=1}^n I_{[Z_j = 2i-1]} I_{[\cap_{k=1}^n [R_k \leqslant r_{2i-1}]]}\right) \\
&= n\boldsymbol{E}\left(I_{[Z_1 = 2i-1]} I_{[\cap_{k=1}^n [R_k \leqslant r_{2i-1}]]}\right) \\
&= np_{2i-1}\boldsymbol{E}\left(I_{[\cap_{k=2}^n [R_k \leqslant r_{2i-1}]]}\right) \quad (\text{since } R_1 = r_{2i-1} \text{ when } Z_1 = 2i-1) \\
&= np_{2i-1}\boldsymbol{P}\left(\bigcap_{k=2}^n [Z_k = 2i-1 \text{ or } Z_k \neq 2i-1,\ R_k \leqslant r_{2i-1}]\right) \\
&\geqslant np_{2i-1}\left(1 - \sum_{j=1}^{i-1} p_{2j}\left(r_{2j}/r_{2i-1}\right)^a - \sum_{j=2i}^{\infty} p_j\right)^n \\
&\geqslant np_{2i-1}\left(1 - \left(\frac{r_{2i-2}}{r_{2i-1}}\right)^a - \sum_{j=0}^{\infty} (p_{2i})^{2^j}\right)^n \\
&\geqslant np_{2i-1}\left(1 - n\left(\frac{r_{2i-2}}{r_{2i-1}}\right)^a - n\frac{p_{2i}}{1 - p_{2i}}\right) \\
&\geqslant np_{2i-1}\left(1 - \tfrac{1}{3} - 5np_{2i}\right) \geqslant np_{2i-1}\left(1 - \tfrac{1}{3} - \tfrac{1}{3}\right) \geqslant \tfrac{2}{3}n\frac{3}{2\omega(n)} = \frac{n}{\omega_n}. \qquad \square
\end{aligned}$$

Lemma 2. *There exists an $a > 0$ depending upon $\varepsilon > 0$ only such that if $n = n_{2i}$, then for all i large enough, $EN_n \leqslant 4 + \varepsilon$.*

Proof. Let us consider the collection C of indices $k \in \{1, 2, \dots, n\}$ for which $Z_k = 2i$. Clearly, C has cardinality $N_{n,2i}$. We need to partition the plane into five sectors of equal angles $2\pi/5$ around the origin: these will be called $S_1, \dots, S_5$. We would like to use the fact that with high probability, $N_n = N^*$, where N^* is the number of convex hull points among X_k, $k \in C$. In fact, if $N_n \neq N^*$, then one of three events must hold,

$$A_1 = \left[\sup_{j > 2i} N_{n,j} > 0\right] \quad \text{or} \quad A_2 = \bigcup_{1 \leqslant k \leqslant n} [Z_k < 2i][R_k > r_{2i-1}] \quad \text{or} \quad A_3 = \bigcup_{j=1}^{5} \bigcap_{k \in C} \left[X_k \notin S_j\right].$$

The last implication follows from the fact that if all five sectors capture at least one point with $R_k \geqslant r_{2i}$, then in view of $r_{2i} \geqslant 100\, r_{2i-1}$, no point with $R_k \leqslant r_{2i-1}$ can possibly be a convex hull point. Thus,

$$EN_n \leqslant EN^* + n\big(P(A_1) + P(A_2) + P(A_3)\big).$$

We look at each term in turn. First of all, by rescaling, N^* is distributed as the number of points on a convex hull defined by $N_{n,2i}$ i.i.d. points drawn from the distribution of $(R^* \cos \Theta,\ R^* \sin \Theta)$. From Carnal (1970) (see also Dwyer, 1988, 1989), we know that there exists a bounded function f with limit $L_a = 4\sqrt{\pi}\,\Gamma(a + \frac{1}{2})\Gamma^2(1 + \frac{1}{2}a)/(\Gamma(1 + a)\Gamma^2(\frac{1}{2}(a + 1)))$ such that

$$EN^* \leqslant Ef(N_{n,2i}).$$

But since $N_{n,2i}$ is binomial $(n,\ p_{2i})$ and $N_{n,2i}/(np_{2i}) \to 1$ in probability (this follows from $np_{2i} \geqslant i \to \infty$), we see immediately that $EN^* \to L_a$ as well. It is also known that as $a \downarrow 0$, $L_a \to 4$, so we can pick a so small that $L_a < 4 + \frac{1}{4}\varepsilon$. Thus, $EN^* < 4 + \frac{1}{4}\varepsilon + o(1)$. Also,

$$nP(A_1) \leqslant n^2 \sum_{j=2i+1}^{\infty} p_j \sim n^2 p_{2i+1} \leqslant \frac{1}{i}.$$

Next,

$$nP(A_2) \leqslant n^2 \sum_{j<i} p_{2j} \left(\frac{r_{2j}}{r_{2i-1}}\right)^a \leqslant n^2 \left(\frac{r_{2i-2}}{r_{2i-1}}\right)^a \leqslant \frac{1}{i}.$$

Finally,

$$nP(A_3) \leqslant 5n\big(1 - \tfrac{1}{5}p_{2i}\big)^n \leqslant 5n\, \mathrm{e}^{-np_{2i}/5} \leqslant 5n\, \mathrm{e}^{-np(n)/5} \leqslant \tfrac{1}{4}\varepsilon.$$

Combining all this, we see that $EN_n \leqslant 4 + \frac{1}{4}\varepsilon + o(1) + 2/i + \frac{1}{4}\varepsilon = 4 + \frac{1}{2}\varepsilon + o(1)$. □

Lemmas 1 and 2 together give us our result. □

3. Proof of Theorem 2

The family of distributions is not unlike that of Theorem 1. The data are determined by i.i.d. foursomes $(Z,\ R',\ R'',\ \Theta)$, where Z is integer-valued with probability vector p_i, $i \geqslant 0$, Θ is uniformly distributed on $[0, 2\pi]$, R' has density $2r$ on $[0, 1]$, and R'' has density $br/(1 + r^{2+a})$ on $[0, \infty)$, where $b > 0$ is a normalization constant and $a > 0$ is a constant depending upon ε only. If Z is odd or zero, we set $R = R' r_Z$ (inducing the uniform distribution on the unit circle), while otherwise $R = R'' r_Z$ (which leads to a unimodal distribution with polynomially decreasing tails). Since for fixed Z, $(R\cos\Theta, R\sin\Theta)$ is unimodal, the mixture distribution is also unimodal. As before, we define $N_{ni}=\sum_{k=1}^{n} I_{Z_k=i}$. We will make use of the inequality $\boldsymbol{P}(R''\geqslant r)\leqslant b/(ar^a)$, and of the fact that for $r\geqslant 1$, $\boldsymbol{P}(R''\leqslant r)\geqslant b/(2ar^a)$.

Now for the definition of p_i, r_i, n_i. We define $\omega(x)$ as in Theorem 1. Take a strictly decreasing function ρ on the positive reals with the property that $\rho(n)\geqslant \max(\frac{20}{3}, 10a/b)\log(5n^2)/n$. We have $p\geqslant\rho(x)$ when $x\geqslant\rho^{\text{inv}}(p)$. The iterative definition of our constants starts with $n_0=1$, $r_0=1$, $p_1=\frac{1}{2}$. We have, for i looping from 1 to ∞:

$n_{2i-1}\leftarrow\lceil\max((2i-1)/p_{2i-1}, \omega^{\text{inv}}(3/(Cp_{2i-1}^{1/3})), \rho^{\text{inv}}(p_{2i-1}))\rceil$, where $C>0$ is a universal constant to be defined later.
$p_{2i}\leftarrow\min(p_{2i-1}^2, 1/(2in_{2i-1}^2))$.
$n_{2i}\leftarrow\lceil\max(2i/p_{2i}, \rho^{\text{inv}}(p_{2i}))\rceil$.
$p_{2i+1}\leftarrow\min(p_{2i}^2, 1/(2in_{2i}^2))$.
$r_{2i-1}\leftarrow\max(((b/a)(2i-1)n_{2i-1}^2)^{1/a}100r_{2i-2}, 100r_{2i-2})$.
$r_{2i}\leftarrow 100r_{2i-1}$.

The sum of the p_i's sums to a constant $c<0.8$, and we set $p_0=1-c$ to obtain a proper distribution. Note in passing that $n_i\geqslant i$ for all i.

Let H_n be the hull defined by $X_1,\ldots,X_n$, and let H_{ni} be the hull defined by the subset of these points for which $Z_k=i$. The bounds we will use in our arguments are based on the obvious inequalities (valid for any i):

$$\boldsymbol{E}N_n\leqslant\boldsymbol{E}\,|H_{ni}|+n\boldsymbol{P}(H_n\neq H_{ni}),$$
$$\boldsymbol{E}N_n\geqslant\boldsymbol{E}\,|H_{ni}|-n\boldsymbol{P}(H_n\neq H_{ni}).$$

We will show that as i grows to ∞ along even integers, and $n=n_i$, then $\boldsymbol{E}\,|H_{ni}|\leqslant 4+\varepsilon$ for all i large enough, and $n\boldsymbol{P}(H_n\neq H_{ni})=o(1)$. Also, as i grows to ∞ along odd integers, and $n=n_i$, then $\boldsymbol{E}\,|H_{ni}|\geqslant n^{1/3}/\omega_n$ for all i large enough, and $n\boldsymbol{P}(H_n\neq H_{ni})=o(1)$. This would conclude the proof of Theorem 2.

For i even, we have $\boldsymbol{E}\,|H_{ni}|\leqslant\boldsymbol{E}f(N_{ni})$ where f is a bounded function with limit L_a (see proof of Lemma 2), where $L_a\downarrow 4$ as $a\downarrow 0$. Thus, $\boldsymbol{E}\,|H_{ni}|\leqslant 4+\frac{1}{4}\varepsilon+o(1)$ by our choice of a when $\boldsymbol{E}N_{ni}=np_i\to\infty$ (but this follows from $n_ip_i\geqslant i$).

For i odd, we have $\boldsymbol{E}\,|H_{ni}|\geqslant\boldsymbol{E}(CN_{ni}^{1/3})$ for some constant $C>0$ (Rényi and Sulanke, 1963, 1964). But

$$\begin{aligned}\boldsymbol{E}\left(CN_{ni}^{1/3}\right)&\geqslant C\lfloor np_i\rfloor^{1/3}\boldsymbol{P}(N_{ni}\geqslant\lfloor np_i\rfloor)\\&\geqslant\tfrac{1}{2}C\lfloor np_i\rfloor^{1/3}\\&\geqslant\tfrac{1}{3}C(np_i)^{1/3}\quad\text{(for all } i \text{ large enough)}\\&\geqslant n^{1/3}/\omega_n\end{aligned}$$

by an inequality due to Slud (1977), the fact that $np_i\geqslant i\to\infty$, and our definition of p_i for i odd.

For i odd, we have $H_n=H_{ni}$ when all five sectors S_j (see Lemma 2 for a definition) have at least one point with $Z_k=i$, $R_k\geqslant\frac{1}{2}r_i$, when no point has $Z_k>i$, when $r_{i-2}<\frac{1}{100}r_i$ (which is satisfied by definition), and when for all points with $Z_k=j<i$, j even, we have $r_jR_k''\leqslant\frac{1}{100}r_i$. Thus,

$$\begin{aligned}n\boldsymbol{P}(H_n\neq H_{ni})&\leqslant n^2\sum_{j>i}p_j+5n\left(1-\tfrac{1}{5}\tfrac{3}{4}p_i\right)^n+n^2\boldsymbol{P}\left(R''>\frac{r_i}{100r_{i-1}}\right)\\&\leqslant n^2(1+o(1))p_{i+1}+5n\,\mathrm{e}^{-3np_i/20}+n^2\frac{b}{a}\left(\frac{100r_{i-1}}{r_i}\right)^a\\&\leqslant\frac{1+o(1)}{i+1}+\frac{1}{i}+\frac{1}{i}=o(1).\end{aligned}$$

For i even, we have $H_n = H_{ni}$ when all five sectors S_j have at least one point with $Z_k = i$, $R_k \geqslant r_i$, when no point has $Z_k > i$, when $r_{i-1} < \frac{1}{100} r_i$ (which is satisfied by definition), and when for all points with $Z_k = j < i$, j even, we have $r_j R_k'' \leqslant \frac{1}{100} r_i$. Thus,

$$nP(H_n \neq H_{ni}) \leqslant n^2 \sum_{j>i} p_j + 5n\left(1 - \tfrac{1}{5}\frac{b}{2a} p_i\right)^n + n^2 \boldsymbol{P}\left(R'' > \frac{r_i}{100 r_{i-2}}\right)$$

$$\leqslant n^2(1 + o(1)) p_{i+1} + 5n\, e^{-bnp_i/(10a)} + n^2 \frac{b}{a}\left(\frac{100 r_{i-2}}{r_i}\right)^a$$

$$\leqslant \frac{1 + o(1)}{i+1} + \frac{1}{i} + \frac{1}{i} = o(1).$$

This concludes the proof of Theorem 2. □

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