ON EXACT SIMULATION ALGORITHMS FOR SOME DISTRIBUTIONS RELATED TO JACOBI THETA FUNCTIONS

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Abstract. We develop exact random variate generators for several distributions related to the Jacobi theta function. These include the distributions of the maximum of a Brownian bridge, a Brownian meander and a Brownian excursion, and distributions of certain first passage times of Bessel processes. The algorithms are based on the alternating series method. Furthermore, we survey various distributional identities and point out ways of dealing with generalizations of these basic distributions.

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Two probability laws related to Jacobi theta functions

The purpose of this note is to propose two simple random variate generation algorithms for some probability distributions that arise in the theory of Brownian and Bessel processes. The laws we are interested in are characterized by their Laplace transforms, and involve nonnegative random variables denoted here by J and J^* (J for Jacobi). The properties of these laws are carefully laid out by Biane, Pitman and Yor (2001). We define

$$\mathbf{E}\left\{e^{-\lambda J}\right\} = \frac{\sqrt{2\lambda}}{\sinh\left(\sqrt{2\lambda}\right)} \quad , \quad \mathbf{E}\left\{e^{-\lambda J^*}\right\} = \frac{1}{\cosh\left(\sqrt{2\lambda}\right)} .$$

Using Euler's formulae

$$\sinh z = z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{n^2\pi^2}\right) \quad , \quad \cosh z = \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{(n-1/2)^2\pi^2}\right),$$

it is easy to see that J and J^* are indeed positive random variables, and that they have the following representation in terms of i.i.d. standard exponential random variables $E_1, E_2, \ldots$:

$$J \stackrel{\mathcal{L}}{=} \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{E_n}{n^2} \quad , \quad J^* \stackrel{\mathcal{L}}{=} \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{E_n}{(n-1/2)^2} .$$

It is known that J^* is the first passage time of Brownian motion started at the origin for absolute value 1, and J is similarly defined for the Bessel process of dimension 3 (which is the square root of the sum of the squares of three independent Brownian motions). See, e.g., Yor (1992, 1997). The maximum of a Brownian meander on $[0,1]$ is distributed as twice the maximum absolute value of a Brownian bridge on $[0,1]$, and is equal in law to $\pi\sqrt{J}$ (Durrett, Iglehart and Miller, 1977; Kennedy, 1976; Biane and Yor, 1987; Borodin and Salminen, 2002).

We use the name Jacobi and the symbol J because the densities of J and J^* can be expressed in terms of the Jacobi theta function

$$\theta(x) = \sum_{n=-\infty}^{\infty} \exp\left(-n^2\pi x\right), x > 0.$$

It has the remarkable property that $\sqrt{x}\theta(x) = \theta(1/x)$, which follows from the Poisson summation formula, and more particularly from Jacobi's theta function identity

$$\frac{1}{\sqrt{\pi x}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{(n+y)^2}{x}\right) = \sum_{n=-\infty}^{\infty} \cos(2\pi n y) \exp\left(-n^2\pi^2 x\right), y \in \mathbf{R}, x > 0.$$

The density of J is

$$f(x) = \frac{d}{dx} \sum_{n=-\infty}^{\infty} (-1)^n \exp\left(-\frac{n^2\pi^2 x}{2}\right) = \sum_{n=1}^{\infty} (-1)^{n+1} n^2\pi^2 \exp\left(-\frac{n^2\pi^2 x}{2}\right).$$

The density of J^* is

$$f^*(x) = \pi \sum_{n=0}^{\infty} (-1)^n \left(n + \frac{1}{2}\right) \exp\left(-\frac{(n+1/2)^2\pi^2 x}{2}\right).$$

We note that all moments are finite, and are expressible in terms of the Riemann zeta function. We will not need them for the discussion that follows.

Random variate generation for J has already been treated in the literature. Indeed, it is easy to see that $(\pi/2)\sqrt{J}$ has distribution function

$$F(x) = \sum_{n=-\infty}^{\infty} (-1)^n e^{-2n^2x^2},$$

which is nothing but the law of the Kolmogorov-Smirnov statistic (Kolmogorov, 1933). We call it the Kolmogorov-Smirnov distribution and denote its random variable by K. The identity $K \stackrel{\mathcal{L}}{=} (\pi/2)\sqrt{J}$ was first observed by Watson (1961). Exact random variate generation for the Kolmogorov-Smirnov law was first proposed by Devroye (1981), who used the so-called alternating series method, which is an extension of von Neumann's rejection method. This method is useful whenever densities can be written as infinite sums,

$$f(x) = \sum_{n=0}^{\infty} (-1)^n a_n(x),$$

where $a_n(x) \geq 0$ and for fixed x, $a_n(x)$ is eventually decreasing in n. Jacobi functions are prime examples of such functions.

The second example in the literature is that of the theta distribution function, which is the limit law of the maximum of a positive Brownian excursion, and thus appears as the limit law of the height of random conditional Galton-Watson trees (see, e.g., Rényi and Szekeres, 1967, de Bruijn, Knuth and Rice, 1972, Chung, 1975, Kennedy, 1975, Meir and Moon, 1978, and Flajolet and Odlyzko, 1982). The distribution function is

$$G(x) = \sum_{n=-\infty}^{\infty} \left(1 - 2n^2x^2\right) e^{-n^2x^2}, x > 0,$$

and we warn that some authors use a different scaling: if we call a random variable with distribution function G a theta random variable, and denote it by T, then the maximum of Brownian excursion on the unit interval is distributed as $T/\sqrt{2}$. (For more on this, see, e.g., Pitman and Yor, 2001.) Let us write $K(1), K(2), \ldots$ for a sequence of i.i.d. copies of a Kolmogorov-Smirnov random variable K. As noted by Biane, Pitman and Yor (2001), the distribution function of the sum $J(1)+J(2)$ of two independent copies of J is given by

$$\sum_{n=-\infty}^{\infty} \left(1 - n^2\pi^2 x\right) e^{-n^2\pi^2 x/2}, x > 0.$$

Thus, we have the distributional identity

$$\frac{\pi^2}{2}(J(1) + J(2)) \stackrel{\mathcal{L}}{=} T^2.$$

Using $J \stackrel{\mathcal{L}}{=} (4/\pi^2)K^2$, we deduce

$$T \stackrel{\mathcal{L}}{=} \sqrt{2(K(1)^2 + K(2)^2)}.$$

This provides a route to the simulation of T via a generator for K. Devroye (1997) published a more direct exact algorithm that uses the principle of a converging series representation for the density.

It is also noteworthy that

$$J \stackrel{\mathcal{L}}{=} \frac{J(1)+J(2)}{(1+U)^2}$$

where U is uniform $[0,1]$ and independent of the $J(i)$'s (Biane, Pitman and Yor (2001, section 3.3)). Thus we have the further identities

$$J \stackrel{\mathcal{L}}{=} \frac{2T^2}{\pi^2(1+U)^2} \stackrel{\mathcal{L}}{=} \frac{4K^2}{\pi^2}.$$

Finally,

$$K \stackrel{\mathcal{L}}{=} \frac{T}{(1+U)\sqrt{2}}.$$

Further properties of K and of maxima of Bessel bridges are given by Pitman and Yor (1999).

This leaves us with the problem of simulating J^*, for which no convenient distributional relations linking it to K, T or J are available. An exact random variate generator for J^* was presented by Burq and Jones (2008), who used the rejection method (von Neumann, 1951) combined with the alternating series method. However, a crucial inequality in that paper is only verified numerically, which is especially problematic for inequalities between functions that touch. For the exact simulation of certain diffusions, Beskos and Roberts (2005) developed a method based on Brownian motion. For the exact simulation of stochastic differential equations, Chen (2008) built on their ideas and decomposed Brownian motion into pieces determined by first passage times of certain levels (which are distributed as J^*). These passage times are the cornerstone of Chen's algorithm, and it is thus important to have the best possible generators available for these random variables. In this paper, we provide a "natural" way of treating Jacobi functions, and thus provide a second example that can be used as a template to deal with other similar distributions in the future.

A THIRD LAW LINKING J AND J^*. Since

$$\frac{1}{\cosh z} = \frac{z}{\sinh z} \times \frac{\tanh z}{z},$$

we have, upon replacing z by $\sqrt{2\lambda}$, that

$$J^* \stackrel{\mathcal{L}}{=} J + J',$$

where $J' \geq 0$ is independent of J and has Laplace transform

$$\mathbb{E}\left\{e^{-\lambda J'}\right\} = \frac{\tanh\left(\sqrt{2\lambda}\right)}{\sqrt{2\lambda}}.$$

That J' is indeed a nonnegative random variable and that this is a valid Laplace transform follows from Euler's formulae and the representation

$$J' \stackrel{\mathcal{L}}{=} \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{E_n \xi_{1/n-1/(4n^2)}}{(n-1/2)^2},$$

where ξ_p is Bernoulli (p) and all random variables in the infinite sum are independent.

EXTENSIONS OF J AND J^*. Sums of independent copies of J, J^* and J' may also be considered. For example, for fixed parameter $\alpha > 0$, the Laplace transforms

$$\left(\frac{1}{\cosh\sqrt{2\lambda}}\right)^{\alpha}, \quad \left(\frac{\sqrt{2\lambda}}{\sinh\sqrt{2\lambda}}\right)^{\alpha}, \quad \left(\frac{\tanh\sqrt{2\lambda}}{\sqrt{2\lambda}}\right)^{\alpha}, \quad \lambda \ge 0 \,, \tag{1}$$

define three different families of distributions—Pitman and Yor (2003) study them in detail and call them infinitely divisible laws associated with hyperbolic functions. The density of the first one of the three, obtained by term by term inversion of the series expansion for the Laplace transform, is (see Biane, Pitman and Yor, 2001)

$$f_\alpha(x) = \frac{2^\alpha}{\Gamma(\alpha)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+\alpha)(2n+\alpha)}{\Gamma(n+1)\sqrt{2\pi x^3}} \exp\left(-\frac{(2n+\alpha)^2}{2x}\right), x > 0.$$

The question is whether one can derive an algorithm for generating such random variates X_α that is uniformly fast uniformly over all values of the parameter $\alpha > 0$. This will not be attempted in the present paper. However, it should be obvious that the infinite exponential sum formulas for J and J^* can now be written in terms of independent gamma (α) random variables $G_\alpha(1), G_\alpha(2), \ldots$: the first two random variables in (1) are, respectively, distributed as

$$\frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{G_\alpha(n)}{n^2} \quad \text{and} \quad \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{G_\alpha(n)}{(n-1/2)^2}.$$

However, this does not yield an exact finite time algorithm. In view of the tails of f_1, the limit law for X_α/α^2, as $\alpha \to \infty$, is the inverse of the square of a normal (or an extreme stable of parameter 1/2). Following the ideas of Devroye (1993) for a similar family (see below for more on this), it is probably best to attack this by considering the characteristic function of X_α/α^2 more directly instead of focusing on series representations for the density.

A generator for J^*

The starting point is a dual representation for the density f^* of J^*, noted by Ciesielski and Taylor (1962), based upon the properties of the Jacobi function:

$$f^*(x) = \begin{cases} \pi \sum_{n=0}^{\infty} (-1)^n \left(n+\frac{1}{2}\right) \exp\left(-(n+1/2)^2 \pi^2 x/2\right) & (1) \\ \left(\frac{2}{\pi x}\right)^{\frac{3}{2}} \pi \sum_{n=0}^{\infty} (-1)^n \left(n+\frac{1}{2}\right) \exp\left(-\frac{2\left(n+\frac{1}{2}\right)^2}{x}\right) & (2) \end{cases}$$

(see section 3 of Biane, Pitman and Yor, 2001). We generically write

$$f^*(x) = \sum_{n=0}^{\infty} (-1)^n a_n(x),$$

where the functions a_n are nonnegative.

LEMMA 1. *We have $a_0(x) \ge a_1(x) \ge a_2(x) \ge \cdots$ if either representation (1) is used and $x \ge (\log 3)/\pi^2$, or representation (2) is used and $x \le 4/\log 3$.*

PROOF. For (1), we have for $n \geq 0$,

$$\frac{a_{n+1}(x)}{a_n(x)} = \frac{2n+3}{2n+1} \exp\left(-(n+1)\pi^2 x\right) \leq 3 \exp\left(-\pi^2 x\right).$$

For (2), we have for $n \geq 0$,

$$\frac{a_{n+1}(x)}{a_n(x)} = \frac{2n+3}{2n+1} \exp\left(-(n+1)\, 4/x\right) \leq 3 \exp\left(-\frac{4}{x}\right). \square$$

Denote the partial sums as follows,

$$A_n(x) = \sum_{i=0}^{n} (-1)^i a_i(x).$$

Then Lemma 1 implies that for representation (1) and $x \geq (\log 3)/\pi^2$, or for representation (2) and $x \leq 4/\log 3$,

$$A_0(x) \geq A_2(x) \geq \cdots \geq f^*(x) \geq \cdots \geq A_3(x) \geq A_1(x).$$

This suggests that we can use the rejection method based upon the inequality (see Fig. 1)

$$f^*(x) \leq g(x) \stackrel{\text{def}}{=} \begin{cases} \frac{\pi}{2} \exp\left(-\frac{\pi^2 x}{8}\right), & \text{if } x \geq t, \\ \left(\frac{2}{\pi x}\right)^{\frac{3}{2}} \frac{\pi}{2} \exp\left(\frac{1}{2x}\right), & \text{if } x \leq t. \end{cases}$$

The threshold $t = 0.64$ is suggested such that the bounding curve is nearly indistinguishable from f^*, while being inbetween the range $[\log 3/\pi^2, 4/\log 3]$ suggested by Lemma 1 for monotonicity. We define $a_n(x)$ as in (1) for $x \geq t$, and as in (2) for $x \in (0, t)$. The area under the bounding curve has two components. We define

$$p = \int_t^\infty g(x)\, dx = \frac{4}{\pi} \exp\left(-\frac{\pi^2 t}{8}\right),$$

$$q = \int_0^t g(x)\, dx = 4\mathrm{P}\{N \geq 1/\sqrt{t}\},$$

where N is standard normal. With our choice of t, we have $p = 0.57810262346829443\ldots$ and $q = 0.422599094\ldots$. Note in particular that $p + q = 1.0007017178682944\ldots$, which means that less than one out of a thousand random variates result in any rejection! A random variate with density proportional to g is obtained as follows: with probability $p/(p+q)$, generate $t + 8E/\pi^2$, where E is standard exponential. With probability $q/(p+q)$, generate $1/N^2$, where N is a standard normal conditioned on being in $[1/\sqrt{t}, \infty)$. The tail-of-normal random variate N can be obtained by a method given in Devroye (1986, p. 382): keep generating independent pairs of independent standard exponentials (E, E') until $E^2 \leq 2E'/t$. Then return $N \leftarrow (1 + tE)/\sqrt{t}$.

With all these routine matters out of the way, we can now summarize the algorithm, using alternating series to successively reject and accept until, with probability one, a decision is made to either reject or accept.

```
Let t = 0.64, p = 0.57810262346829443, q = 0.422599094
Repeat Generate U,V uniform on [0,1]
        If U < p/(p+q) then set X ← t + 8E/π^2 with E exponential
                          else repeatedly generate independent exponentials (E,E')
                               until E^2 ≤ 2E'/t, and set X ← t/(1+tE)^2
        Set S ← a_0(X), Y ← VS, n ← 0
        Repeat until exit:
          n ← n+1
          if n is odd:  S ← S − a_n(X); if Y < S, then return X and halt
          if n is even:  S ← S + a_n(X); if Y > S, then exit loop
```

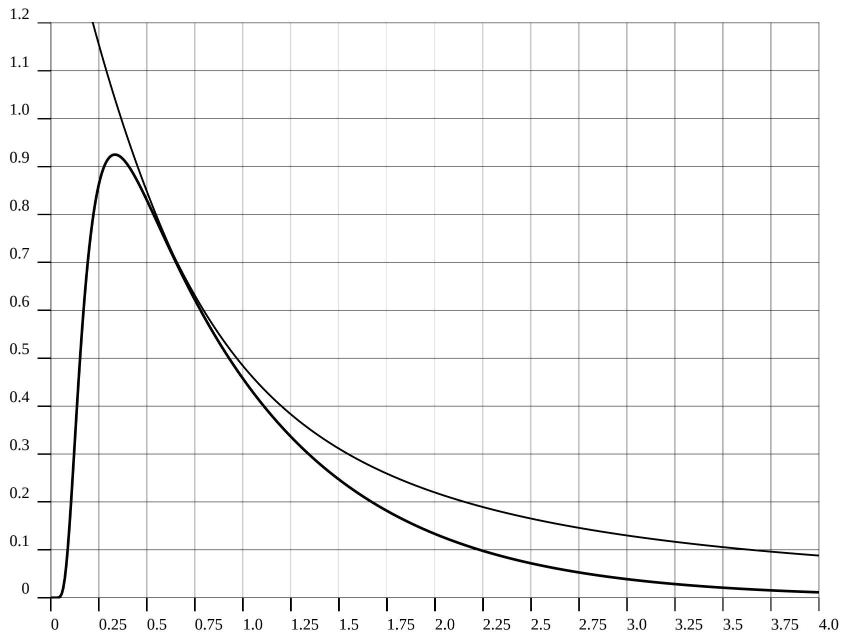

Figure 1. The density f^* of J^* is shown with a thick stroke. The function a_0 is shown for representations (1) and (2). For representation (1), it is exponential, and hugs f^* tightly to the right of the threshold point $t = 0.64$. For representation (2), it is inverse squared normal (like for an extreme stable law of parameter 1/2), and fits f^* precisely to the left of t, but has a bigger (polynomial, but unused) tail to the right of t.

The expected number of outer iterations before halting is $p+q < 1.000702$. For the inner loop, the expected number of iterations is analyzed as in Devroye (1981). Let X denote a random variate with density g. Then the number of calculations of an $a_k(X)$ function value is at least equal to $n+1$ with probability $a_n(X)/g(X)$. Given X, the expected number of such calculations (and thus the expected work done) is therefore not more than

$$1+\sum_{n=0}^{\infty}\frac{a_n(X)}{g(X)}.$$

Unconditioning, the expected work to process one candidate X is not more than

$$\mathbb{E}\left\{1+\sum_{n=0}^{\infty}\frac{a_n(X)}{g(X)}\right\}=1+\int g(x)\sum_{n=0}^{\infty}\frac{a_n(X)}{g(X)}\,dx=1+\sum_{n=0}^{\infty}\int a_n(x)\,dx\stackrel{\text{def}}{=}\rho.$$

By Wald's identity, the overall expected work (taking the outer loop into account as well) is not more than 1.000702 times ρ. Since a_n decreases exponentially, it is rather easy to verify that $\rho\in(2,3)$.

Generalized Jacobi laws.

The laws of J and J^* can be generalized as follows, using a slightly more convenient scaling. For parameter $\alpha\in(0,1/2]$, we define nonnegative random variables J_α and J^*_α via their Laplace transforms

$$\mathbb{E}\left\{e^{-\lambda J_\alpha}\right\}=\frac{\lambda^\alpha}{\sinh(\lambda^\alpha)}\quad,\quad\mathbb{E}\left\{e^{-\lambda J^*_\alpha}\right\}=\frac{1}{\cosh(\lambda^\alpha)}.$$

Note that $J=2J_{1/2}$, $J^*=2J^*_{1/2}$.

Observe that

$$J_\alpha\stackrel{\mathcal{L}}{=}S_{2\alpha}J_{1/2}^{\frac{1}{2\alpha}}\stackrel{\mathcal{L}}{=}S_{2\alpha}(J/2)^{\frac{1}{2\alpha}},$$

where S_α is an extreme (or unilateral) positive stable random variable with Laplace transform

$$\mathbb{E}\left\{e^{-\lambda S_\alpha}\right\}=e^{-\lambda^\alpha},\lambda\geq 0.$$

This can easily be verified from the Laplace transforms:

$$\mathbb{E}\left\{e^{-\lambda S_{2\alpha}J_{1/2}^{\frac{1}{2\alpha}}}\right\}=\mathbb{E}\left\{e^{-\lambda^{2\alpha}J_{1/2}}\right\}=\frac{\lambda^\alpha}{\sinh(\lambda^\alpha)}.$$

Similarly,

$$J^*_\alpha\stackrel{\mathcal{L}}{=}S_{2\alpha}{J^*_{1/2}}^{\frac{1}{2\alpha}}\stackrel{\mathcal{L}}{=}S_{2\alpha}(J^*/2)^{\frac{1}{2\alpha}}.$$

The entire family can be exactly simulated using this distributional identity and the algorithms described above.

Simple exact generators for S_α are easily available (see, e.g., Zolotarev (1986) for a broad study of stable distributions). A simple random variate generator for S_α has been suggested by Kanter (1975), who used an integral representation of Zolotarev (1966) (see Zolotarev (1959, 1981, or 1986, p. 74)), which states that the distribution function of $S_\alpha^{\alpha/(1-\alpha)}$ is given by

$$\frac{1}{\pi}\int_0^\pi e^{-\frac{A(u)}{x}}\,du,$$

where A is Zolotarev's function:

$$A(u)\stackrel{\text{def}}{=}\left\{\frac{(\sin(\alpha u))^\alpha(\sin((1-\alpha)u))^{1-\alpha}}{\sin u}\right\}^{\frac{1}{1-\alpha}}.$$

By taking limits, we note that $S_1 = 1$, so that the family is properly defined for all $\alpha \in (0,1]$. Zolotarev's integral representation implies that

$$S_\alpha \stackrel{\mathcal{L}}{=} \left(\frac{A(U)}{E}\right)^{\frac{1-\alpha}{\alpha}},$$

where U is uniform on $[0,\pi]$ and E is exponential. This is known as Kanter's method. A year after Kanter's paper, a similar generator was proposed by Chambers, Mallows and Stuck (1976), which was again based on Zolotarev's integral representation of stable distributions, but applicable to all stable laws. A different method based on the series expansion of the stable density was developed by Devroye (1986).

Hyperbolic secant, cosecant, and tangent distributions

The hyperbolic secant distribution has density

$$f(x) = \frac{1}{2}\operatorname{sech}(\pi x/2) = \frac{1}{2\cosh(\pi x/2)} = \frac{1}{e^{\pi x/2} + e^{-\pi x/2}}$$

(Baten, 1934, and Talacko, 1951). We write H^* for its random variable, and note that $\mathbb{E}H^* = 0$, $\mathbb{V}\{H^*\} = 1$, and its cumulative distribution function is

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(\operatorname{sech}(\pi x/2)) = \frac{2}{\pi}\arctan\left(\exp(\pi x/2)\right).$$

The inversion method implies that

$$H^* \stackrel{\mathcal{L}}{=} \frac{2}{\pi}\log\left(\tan\left(\frac{\pi U}{2}\right)\right),$$

where U is uniform $[0,1]$. The characteristic function is

$$\frac{1}{\cosh(t)} = \frac{2}{e^t + e^{-t}} = \frac{2}{e^{|t|} + e^{-|t|}}.$$

In the first section, we introduced J and J^*; their counterparts here are H and H^*. The random variable H with the hyperbolic cosecant distribution has characteristic function

$$\frac{t}{\sinh(t)} = \frac{2t}{e^t - e^{-t}} = \frac{2|t|}{e^{|t|} - e^{-|t|}}.$$

Its density is $\pi/(2\cosh(x\pi/2))^2$. Integrating this density and applying inversion shows that

$$H \stackrel{\mathcal{L}}{=} \frac{S}{\pi}\log\left(\frac{1+U}{1-U}\right),$$

where S is a random equiprobable sign and U is uniform $[0,1]$. Of particular interest here is that if N is standard normal and independent of K, then $|N|K$ has distribution function $\tanh x$ (Pitman and Yor, 1999), and is thus distributed as $(1/2)\log((1+U)/(1-U))$. Therefore, $2NK/\pi \stackrel{\mathcal{L}}{=} H$. Many more relationship exist between the random variables introduced in this paper.

Jurek and Yor (2004) provide an in-depth study of the self-decomposability and other properties of H and H^*. Further information can be found in Pitman and Yor (2003). Jurek and Yor look at a third random variable H' with characteristic function $\tanh(t)/t$, called the hyperbolic tangent distribution.

Trivially, $H^* \stackrel{\mathcal{L}}{=} H + H'$ (multiply the characteristic functions). Furthermore, by Euler's formulae, if $L_1, L_2, \ldots$ denote i.i.d. Laplace random variables,

$$H \stackrel{\mathcal{L}}{=} \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{L_n}{n} \quad , \quad H^* \stackrel{\mathcal{L}}{=} \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{L_n}{n - 1/2}$$

(see, e.g., Laha and Lukacs, 1960). Since a Laplace random variable is the difference of two independent exponential random variables, the relationship with J and J^* is intriguing. Nevertheless, it is not clear how this relationship will aid in developing other or better methods for simulating J or J^*. We remark that

$$H' \stackrel{\mathcal{L}}{=} \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{L_n \xi_{1/n - 1/(4n^2)}}{n - 1/2},$$

where ξ_p is Bernoulli (p), and all random variables in the infinite sum are independent.

We call random variables H_α and H_α^* $(0 \le \alpha \le 1)$ stable hyperbolic cosecant and stable hyperbolic secant if they have characteristic functions

$$\mathbb{E}\left\{e^{itH_\alpha}\right\} = \frac{|t|^\alpha}{\sinh(|t|^\alpha)} = \frac{2|t|^\alpha}{e^{|t|^\alpha} - e^{-|t|^\alpha}} \quad , \quad \mathbb{E}\left\{e^{itH_\alpha^*}\right\} = \frac{1}{\cosh(|t|^\alpha)} = \frac{2}{e^{|t|^\alpha} + e^{-|t|^\alpha}}.$$

Note that $H_1^* \stackrel{\mathcal{L}}{=} H^*$, $H_1 \stackrel{\mathcal{L}}{=} H$. For simulation, the following (trivial) Lemma is useful. Denote by $S_{\alpha,0}$ the symmetric stable random variable with characteristic function $\exp(-|t|^\alpha)$, and let C be a standard Cauchy random variable,

Lemma 2. *Let $0 < \alpha \le 1$. Then*

$$H_\alpha \stackrel{\mathcal{L}}{=} S_{\alpha,0} H^{\frac{1}{\alpha}} \stackrel{\mathcal{L}}{=} C S_\alpha H^{\frac{1}{\alpha}} \quad , \quad H_\alpha^* \stackrel{\mathcal{L}}{=} S_{\alpha,0} {H^*}^{\frac{1}{\alpha}} \stackrel{\mathcal{L}}{=} C S_\alpha {H^*}^{\frac{1}{\alpha}}.$$

Also,

$$H_\alpha \stackrel{\mathcal{L}}{=} S_{2\alpha,0} {J_{1/2}}^{\frac{1}{2\alpha}} \quad , \quad H_\alpha^* \stackrel{\mathcal{L}}{=} S_{2\alpha,0} {J_{1/2}^*}^{\frac{1}{2\alpha}}.$$

Finally, for $\alpha \in (0, 1/2]$,

$$H_\alpha \stackrel{\mathcal{L}}{=} C J_\alpha \stackrel{\mathcal{L}}{=} C S_{2\alpha} {J_{1/2}}^{\frac{1}{2\alpha}} \quad , \quad H_\alpha^* \stackrel{\mathcal{L}}{=} C J_\alpha^* \stackrel{\mathcal{L}}{=} C S_{2\alpha} {J_{1/2}^*}^{\frac{1}{2\alpha}}.$$

Proof. The first part follows by computing the characteristic functions. For example,

$$\mathbb{E}\left\{e^{itS_{\alpha,0}{H^*}^{\frac{1}{\alpha}}}\right\} = \mathbb{E}\left\{e^{-|t|^\alpha H^*}\right\} = \frac{1}{\cosh(|t|^\alpha)}.$$

The second part follows similarly:

$$\mathbb{E}\left\{e^{itS_{2\alpha,0}{J_{1/2}^*}^{\frac{1}{2\alpha}}}\right\} = \mathbb{E}\left\{e^{-|t|^{2\alpha} J_{1/2}^*}\right\} = \frac{1}{\cosh(|t|^\alpha)}.$$

All statements involving C follow from the fact that if $X \ge 0$ has Laplace transform $L(\lambda)$, then CX has characteristic function $L(|t|)$. In particular, for $\alpha \le 1/2$, $S_{2\alpha,0} \stackrel{\mathcal{L}}{=} C S_{2\alpha}$. $\square$

Subordination

Let X_α be the generic notation for the three random variables defined by (1). If N denotes a standard normal random variable, then it is easy to check that $N\sqrt{X_\alpha}$ has characteristic function $\mathbb{E}\{\exp(itN\sqrt{X_\alpha})\}$ given, respectively by

$$\left(\frac{1}{\cosh t}\right)^\alpha \ , \ \left(\frac{t}{\sinh t}\right)^\alpha \ , \ \left(\frac{\tanh t}{t}\right)^\alpha .$$

These families, also studied by Pitman and Yor (2003), are related to the GHS (generalized hyperbolic secant) distribution discussed in the next section—in fact, the first one is GHS. So, while simulating X_α in general is a challenge, the normal scale mixtures $N\sqrt{X_\alpha}$ are more easily dealt with, as we will now explain.

Further related distributions

The GHS (generalized hyperbolic secant) distribution has characteristic function

$$\varphi(t) = \left(\frac{1}{\cosh t}\right)^\alpha = (\operatorname{sech} t)^\alpha = \left(\frac{2}{e^t + e^{-t}}\right)^\alpha \ ,$$

where $\alpha > 0$ is the parameter. It is symmetric about 0, has mean 0 and variance α, is unimodal with mode at 0, and possesses exponentially decaying tails. For $\alpha = 1$, we obtain the hyperbolic secant distribution. For integer α, a GHS random variate G is distributed as

$$H^*(1) + \cdots + H^*(\alpha) \ ,$$

where the $H^*(i)$'s are i.i.d. copies of H^* (Harkness and Harkness, 1968). The time taken by the naive method that exploits this property grows linearly with α. Devroye (1993) derives a rejection method which takes expected time uniformly bounded over all values of α.

REMARK: SPECIAL CASES. For the sake of completeness, we mention three special cases in which explicit forms of the density f_α of a GHS random variate are known (see Harkness and Harkness, 1968):

$$f_2(x) = \frac{x}{2\sinh\frac{\pi x}{2}} = \frac{x}{e^{\pi x/2} - e^{-\pi x/2}} \ ;$$

$$f_{2n+1}(x) = \frac{2^{2n-1}}{(2n)!\cosh\frac{\pi x}{2}} \times \prod_{j=1}^{n} \left(\frac{x^2}{4} + \left(\frac{2j-1}{2}\right)^2\right) \ ;$$

$$f_{2n}(x) = \frac{4^{n-1}x}{2\,(2n-1)!\sinh\frac{\pi x}{2}} \times \prod_{j=1}^{n-1} \left(\frac{x^2}{4} + j^2\right) \ .$$

REMARK: AN ASYMMETRIC GENERALIZATION. There are several generalizations of the GHS distribution that introduce asymmetry. The Laha–Lukacs distribution is defined via its characteristic function

$$\varphi(t) = (\cosh t - i\mu \sinh t)^{-\alpha} \ ,$$

where $\alpha > 0$, $\mu \in \mathbb{R}$ (Laha and Lukacs, 1960). No exact universally fast random variate generators are known for this family.

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