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RANDOM VARIATE GENERATION IN ONE LINE OF CODE

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ABSTRACT

A random variate with a given non-uniform distribution can often by generated in one assignment statement if a uniform source and some simple functions are available. We review such one-line methods for most of the key distributions.

1 A MODEL OF COMPUTATION

Random variate generators that are conceptually simple and quick to program become invariably popular, even if they are not as efficient as some more complicated methods. We explore and survey the simplest end of the spectrum—the generators that can be implemented in one line of code. We assume throughout that an unlimited source of i.i.d. uniform $[0,1]$ random variates $U_1, U_2, \ldots$ is available. When discussing one-liners, we must distinguish between two situations: in the ordinary case, each request of a uniform variate is fulfilled by another number from this sequence. In the extended case, we may index our requests by U_1 , U_2, and so forth, so that repetitions of the same uniform variate within the code are possible. This will be called an *extended one-liner.*

The standard operators $+, -, *, /$ are available, as are mod, round, $\lfloor . \rfloor$, sign, $\lceil . \rceil$, $|.|$, sin, cos, exp, log, tan, atan. Many functions may be derived from these using only a constant number of combinations. For example, the indicator function $I_{x>0}$ is simply $I_{x>a} = 2\,\mathrm{sign}(x-a) - 1$, and $I_{a>x>b} = (\mathrm{sign}(x-a) - \mathrm{sign}(x-b))/2$. Furthermore, max is included as

$$\max(a,b) = a + (b-a) I_{b>a} \ .$$

Alternatively,

$$\max(a,b) = \frac{a+b}{2} + \left| \frac{a-b}{2} \right| \ .$$

Some may include more complicated functions such as Γ or ζ, but these will not be required for the discussion below.

One may think of a one-liner as an expression tree in which the leaves are uniform $[0,1]$ random variables or constants, and the internal nodes are the operators or functions in the accepted class of operators, which we shall call $\mathcal{F}$. In a *simple one-liner*, each leaf has a different uniform random variate associated with it. In an extended one-liner, repetitions may occur. We may put this differently. Each expression may be represented as a directed acyclic graph (or "dag"), in which the leaf nodes contain constants or U_i's, but each U_i occurs only once. If multiple U_i nodes are disallowed, Extended one-liners are implementable by dags, while simple one-liners are implementable by trees. We should point out here that some smart compilers may transform expressions with repetitions into dags before machine translation.

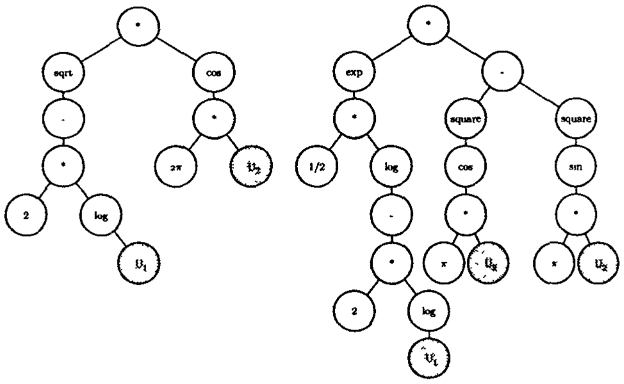
The well-known Box-Muller formula for normal random variates,

$$X = \sqrt{-2 \log U_1} \cos(2\pi U_2) \ ,$$

is thus a simple one-liner. However, the equivalent form

$$X = \exp((1/2)\log(-2\log U_1)) \left(\cos^2(\pi U_2) - \sin^2(\pi U_2)\right)$$

is an extended one-liner. The figure below depicts the expression trees for both forms of the Box-Muller formula.



When we have a family of distributions with parameter(s) θ, then it is of interest to have expression trees that have a fixed structure, independent of θ. The value θ appears at best in one or more leaves. For example, if a gamma (k) random variable is generated in one line by summing k independent exponential random variates, then the structure itself of the expression tree changes with k, and indeed, its size grows proportionally with k. For a family of distributions, we define a *fixed one-liner* as one whose expression tree has a given structure, whose internal nodes have fixed operators, and whose leaves have constants, the value θ, or uniform random variates. The operator "take the i-th component of a vector (such as θ)" is in the set of accepted operators.

With the previous set-up, if the family of operators and functions has k members (which is fixed of course), and if each member is unary or binary say, then the number of possible trees on n internal nodes does not exceed

$$\frac{1}{n+1}\binom{2n}{n} \times n^k \approx n^k = \Theta\left(n^{k-3/2}4^n\right) .$$

This limits the numbers of families we may construct using the basic operations. Still, by making n even moderately large, the possibilities are virtually unlimited. We will take a little tour of the popular distributions and exhibit a number of (mostly known) one-liners. Notable exceptions are the gamma and Poisson distributions.

2 NOTATION

We give different symbols for different random variables. For example, U is uniform $[0,1]$, N is standard normal (with density $e^{-x^2/2}/\sqrt{2\pi}$), A is arcsine (with density $1/(\pi\sqrt{1-x^2})$ on $[-1,1]$), $B_{a,b}$ is beta (a,b) (with density $x^{a-1}(1-x)^{b-1}/B(a,b)$ on $[0,1]$ where $a,b>0$), G_a is gamma (a) (with density $x^{a-1}e^{-x}/\Gamma(a)$ on $[0,\infty)$, where $a>0$), E is exponential (with density e^{-x} , $x>0$), L is Laplace (with density $e^{-|x|}/2$), C is Cauchy (with density $1/(\pi(1+x^2))$), T_a is Student t(a) (with density

$$1/(B(a/2,1/2)\sqrt{a}(1+x^2/a)^{\frac{a+1}{2}}) ,$$

where $a>0$).

Some densities are best defined in terms of their characteristic functions φ. A partial list follows below (note that $K(\alpha)=\alpha-2I_{\alpha>1}$): $S_{\alpha,0}$ is symmetric stable with $\varphi(t)=e^{-|t|^\alpha}$, $0<\alpha\le 2$, $S_{\alpha,\beta}$ is stable (α,β) with $\varphi(t)=e^{-|t|^\alpha e^{-i(\pi}}$, $0<\alpha\le 2$, $\alpha\ne 1$, $\beta\in[-1,1]$, $S_{1,\beta}$ is stable $(1,\beta)$ with $\varphi(t)=e^{-|t|(\pi/2+i\beta\,\mathrm{sign}(t)\log|t|)}$, $-1\le\beta\le 1$, $S_{\alpha,1}$ is positive extreme stable with $\varphi(t)=e^{-|t|^\alpha e^{-i(\pi}}$, $0<\alpha<1$, G_a is gamma (a) with $\varphi(t)=1/(1-it)^a$, $a>0$, L is Laplace with $\varphi(t)=1/(1+t^2)$, M_a is Mittag-Leffler with $\varphi(t)=1/(1+(-it)^a)$, $a\in(0,1]$. $P_{a,b}$ is Pillai with $\varphi(t)=1/(1+(-it)^a)^b$, $a\in(0,1], b>0$. L_a is Linnik with $\varphi(t)=1/(1+|t|^a)$, $a\in(0,2]$.

3 THE INVERSION METHOD

The inversion method is based upon the property that $F^{\mathrm{inv}}(U)$ has distribution function F if U is uniformly distributed on $[0,1]$. It leads to one-liners only if F is explicitly invertible in terms of functions that are in $\mathcal{F}$. In the table below, a, b and c are positive constants that serve as parameters.

In this manner, we note that $E\stackrel{\mathcal{L}}{=}-\log U$ as $F(x)=1-e^{-x}$, $B_{1,a}\stackrel{\mathcal{L}}{=}U^{1/a}$, as $F9x)=x^a$ $(0<x<1)$, $B_{a,1}\stackrel{\mathcal{L}}{=}1-U^{1/a}$, $A\stackrel{\mathcal{L}}{=}\cos(\pi U)$, as $F(x)=1-\arccos(x)/\pi$, and $C\stackrel{\mathcal{L}}{=}\tan(\pi U)$ as $F(x)=1/2+\arctan(x)/\pi$.

Other notable examples include the logistic $(F(x)=1/(1+e^{-x}))$ which can be obtained as $-\log(U/(1-U))$. $(U^{-1/a}-1)^{1/c}$ yields a Burr XII random variate $(F(x)=-1/(x^c+1)^a$ $(x>0))$, and $(U^{-1/a}-1)^{-1/c}$ yields a Burr III random variate $(F(x)=1/(x^{-c}+1)^a$ $(x>0))$. A Fréchet or Weibull random variate with $F(x)=1-e^{-x^a}$ $(x>0)$ may be obtained as $\log^{1/a}(1/U)$. For the Gumbel distribution $(F(x)=e^{-ae^{-x}})$, we suggest $-\log((\log(1/U)/a))$. A Pareto or Pearson XI $(F(x)=a/x^{a+1}$ $(x\ge 1))$ may be obtained by $U^{-1/a}$. A tail of the Rayleigh distribution has distribution function $F(x)=1-e^{\frac{a^2-x^2}{2}}$ $(x\ge a>0)$, so that random variates may be obtained as $\sqrt{a^2-2\log U}$. The hyperbolic secant distribution function is $F(x)=1-(2/\pi)\arctan(e^{-\pi x/2})$. Random variates may be obtained as $(2/\pi)\ \log\tan(\pi U/2)$.

4 COMBINATIONS OF TWO OR MORE RANDOM VARIABLES

Mixtures of the form $X = Y$ with probability p and $X = Z$ with probability $1-p$ are easily taken care of in one-liners by setting

$$X = YI_{U<p} + ZI_{U\geq p} = Y + (Z-Y)I_{U\geq p}$$

where U is uniform $[0,1]$ and independent of (Y,Z). However, countably infinite mixtures are not easy to transform into one-liners.

Special distributional properties often lead to elegant one-liners. The triangular density provides a textbook example:

THE TRIANGULAR DENSITY. Assume that we wish to obtain a one-liner for the triangular density with support on $[a,b]$ and mode at m. This could be achieved in a number of ways, but two possibilities are

$$X = m + (a + U_1(b-a) - m)\sqrt{U_2}\ ,$$

and

$$X = m + (a + U_1(b-a) - m)\max(U_2, U_3)\ .$$

Here all the U_i's are i.i.d. uniform $[0,1]$ random variables.

5 THE POLAR METHOD

In the standard polar method, one generates a random pair (X,Y) as $(R\cos\Theta, R\sin\Theta)$, where R and Θ are (random) polar coordinates. Typically, Θ is uniformly distributed on $[0,2\pi]$, and R has a given distribution that is easy to sample from. Often, R is independent of Θ. In the context of one-liners, we thus have

$$X = R\cos(2\pi U)$$

where U is uniform $[0,1]$. Another way of writing this is $X = RA$, where A is a random variable on $[-1,1]$ with the arcsine density (note: our arcsine density is in fact a linear transformation of the standard arcsine density $2/(\pi\sqrt{z(1-z)})$, $0 < z < 1$). A simple exercise in analysis shows that if R and A are independent, and R has density f on $[0,\infty)$, then $X = RA$ has density

$$\int_0^\infty \frac{f(\sqrt{x^2+y^2})}{\pi\sqrt{x^2+y^2}}\,dy\ .$$

EXAMPLE 1: THE NORMAL DENSITY. If

$$f(r) = re^{-r^2/2}\ ,\ r > 0\ ,$$

which is the Rayleigh density (so that $R = \sqrt{-2\log(U)}$ has density f when U is uniform $[0,1]$), then the density of RA is

$$\int_0^\infty \frac{e^{-x^2/2}e^{-y^2/2}}{\pi}\,dy = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\ .$$

We thus rediscover the Box-Muller method given in the introduction:

$$N \stackrel{\mathcal{L}}{=} \sqrt{-2\log U_1}\cos(2\pi U_2)\ .$$

EXAMPLE 2: THE SYMMETRIC BETA DISTRIBUTION. In the above context, define

$$f(r) = 2cr(1-r^2)^{c-1}\ ,\ 0 < r < 1\ ,$$

where $c > 0$ is a parameter. The distribution function is $F(r) = 1-(1-r^2)^c$, so that, by the inversion method, R is distributed as $\sqrt{1-U^{1/c}}$ when U is uniform $[0,1]$. The density of RA is supported on $[-1,1]$ and is given by

$$\begin{aligned} h(x) &= \int_0^{\sqrt{1-x^2}} \frac{2c(1-x^2-y^2)^{c-1}}{\pi}\,dy \\ &= \frac{2c(1-x^2)^{c-1/2}}{\pi} \times \int_0^1 (1-u^2)^{c-1}\,du \\ &= \frac{2c(1-x^2)^{c-1/2}}{\pi} \times \frac{2^{2c-1}\Gamma^2(c)}{\Gamma(2c)} \\ &= \frac{\Gamma(2c+1)(1-x^2)^{c-1/2}}{2^{2c}\Gamma^2(c+1/2)}, \end{aligned}$$

in which we recognize a shifted version of the symmetric beta density. In the last step, we used a property due to Binet (property 23 on page 261 of Whittaker and Watson, 1927). The last density is that of $2B_{c+1/2,c+1/2} - 1$. Thus, our one-liner for symmetric betas with parameter $a > 1/2$ is based on

$$B_{a,a} \stackrel{\mathcal{L}}{=} \frac{1 + \sqrt{1 - U_1^{\frac{2}{2a-1}}}\cos(2\pi U_2)}{2}\ .$$

This is Ulrich's formula (Ulrich, 1984). For $a = 1/2$, note $B_{1/2,1/2} \stackrel{\mathcal{L}}{=} (1+\cos(2\pi U_2))/2 = \cos^2(\pi U_2)$. Just as for the normal distribution, equivalent respresentations are easy to construct. We note here that the density h plays a central role in nonparametric estimation theory. The case $c = 3/2$ leads to the Epanechnikov or Bartlett density. The case $c = 5/2$ is usually referred to as the quartic kernel. Note that Ulrich's formula is not valid for $a < 1/2$. A fixed one-liner for the entire symmetric beta family is given in the next section.

EXAMPLE 3: THE HALF-BETA DISTRIBUTION. The half-beta density is the density of $B_{1/2,a}$. If X has density h from the previous section, then on $[0,1]$, $Y = X^2$ has density

$$\frac{\Gamma(2c+1)(1-y)^{c-1/2}}{2^{2c}\Gamma^2(c+1/2)\sqrt{y}} = \frac{\Gamma(c+1)(1-y)^{c-1/2}}{\Gamma(1/2)\Gamma(c+1/2)\sqrt{y}} ,$$

which is the density of $B_{1/2,c+1/2}$. Thus, for $a \geq 1/2$,

$$B_{1/2,a} \stackrel{\mathcal{L}}{=} \left(1 - U_1^{\frac{2}{2a-1}}\right) \cos^2(2\pi U_2) .$$

EXAMPLE 4: THE T DISTRIBUTION. If B is beta (a,a), then

$$T_{2a} \stackrel{\mathcal{L}}{=} \frac{\sqrt{2a}(B-1/2)}{2\sqrt{B(1-B)}} .$$

This method yields an extended fixed one-liner for the t distribution. Another fixed one-liner is developed in the next section.

6 THE POLAR METHOD AND NORMAL SCALE MIXTURES

Assume that X is a normal scale mixture random variable, i.e., X can be written as YN, where N is standard normal, and Y is an arbitrary positive random variable, independent of N. If N_1, N_2 are two independent normals independent of Y, then

$$(X_1, X_2) \stackrel{\text{def}}{=} (YN_1, YN_2) \stackrel{\mathcal{L}}{=} (Z_1, Z_2) \times (Y\sqrt{2E}) ,$$

where E is exponential, and (Z_1, Z_2) is uniform on the unit circle and independent of E and Y. This says that (X_1, X_2) has a radially symmetric distribution with random radius distributed as $Y\sqrt{2E}$. Generating X_1 based upon this formula might be called a polar method for X. In some cases, the distribution of $Y\sqrt{2E}$ is very simple. For example, if $Y \stackrel{\mathcal{L}}{=} \sqrt{a/2G_{a/2}}$, then simultaneously

$$Y\sqrt{2E} = \sqrt{\frac{aE}{G_{a/2}}} \; ; \; YN \stackrel{\mathcal{L}}{=} T_a .$$

The last relationship in fact describes the genesis of the t-distribution. The quantity under the square root in the first equation has a beta II (or F) distribution. Routine calculations show that

$$Y\sqrt{2E} \stackrel{\mathcal{L}}{=} \sqrt{a(U^{-2/a}-1)} .$$

One could thus generate T_a in one line using two independent uniform $[0,1]$ random variates U_1 and U_2 since

$$T_a \stackrel{\mathcal{L}}{=} \sqrt{a(U_1^{-2/a}-1)} \cos(2\pi U_2) .$$

This method for t-variates was pointed out by Bailey (1994). This could also have been obtained by the method of the previous section, but the method above requires less integration work. We do not make claims that this is the fastest method for generating t variates. There is another interesting observation here: the t density is a mixture of bimodal densities with infinite peaks (arcsine densities)!

ANOTHER ONE-LINER FOR SYMMETRIC BETAS. Best (1978) has proved that

$$B_{a,a} \stackrel{\mathcal{L}}{=} \frac{1}{2}\left(1 + \frac{T_{2a}}{\sqrt{2a+T_{2a}^2}}\right) .$$

Thus, if S denotes a random sign ($S = I_{V \leq 1/2}$ for V uniform $[0,1]$), the following distributional identity yields a one-liner for all symmetric beta distributions:

$$B_{a,a} \stackrel{\mathcal{L}}{=} \frac{1}{2} + \frac{S}{2\sqrt{1+\frac{1}{\left(U_1^{-1/a}-1\right)\cos^2(2\pi U_2)}}} .$$

This (new) method is applicable for all values of a—Ulrich's formula required $a > 1/2$.

ANOTHER ONE-LINER FOR HALFBETAS. As $(2B_{a,a}-1)^2 \stackrel{\mathcal{L}}{=} B_{1/2,a}$, the previous paragraph suggests yet another one-liner for halfbetas:

$$B_{1/2,a} \stackrel{\mathcal{L}}{=} \frac{T_{2a}^2}{2a+T_{2a}^2} = \frac{1}{1+2a/T_{2a}^2} .$$

Thus, if S denotes a random sign ($S = I_{V \leq 1/2}$ for V uniform $[0,1]$), the following distributional identity yields a one-liner for all halfbeta distributions:

$$B_{1/2,a} \stackrel{\mathcal{L}}{=} \frac{1}{1+\frac{1}{\left(U_1^{-1/a}-1\right)\cos^2(2\pi U_2)}} .$$

This (new) method is applicable for all values of a.

7 DISTRIBUTIONS DEFINED AS ONE-LINERS

Systems of distributions such as Pearson's usually have simple analytic formats. Yet, random variate generation may cause problems. The Pearson IV family, for example, requires quite a bit of work (see Devroye, 1986, p. 480). In modeling, it may be useful to define a distribution by specifying first a one-liner (so that generation is easy), and then worrying about the choice of the parameters and the fine-tuning of the model. There are literally hundreds of such attempts. Tukey (1960) defined a symmetric family by

$$X = \frac{U^\lambda - (1-U)^\lambda}{\lambda} ,$$

where $\lambda \in \mathbb{R}$ is a parameter and U is uniform $[0,1]$. This was later generalized by Ramberg and Schmeiser (1974) by using different λ's for the exponents.

Omitting location and scale parameters, the **Schmeiser-Deutch family** (Schmeiser and Deutch, 1977) is the family of distributions of the random variables

$$X = \begin{cases} -(\lambda - U)^\mu , & \text{if } U \le \lambda \\ (U-\lambda)^\mu , & \text{if } U > \lambda , \end{cases}$$

where λ and μ are shape parameters.

In hydrology, one uses the Wakeby distribution (Johnson and Kotz, 1988, vol.9, p.513) because it is a versatile five-parameter distribution, gives occasionally outliers, and is easy to simulate. A random variate is obtained as

$$X = a + \frac{b}{c}\left(1-(1-U)^c\right) - \frac{b'}{c'}\left(1-(1-U)^{c'}\right) ,$$

where U is uniform $[0,1]$, $b' \ge 0$, $b+b' \ge 0$, and either $c+c' > 0$ or $c+b' = c' = 0$.

Burr (1942) (see Tadikamalla, 1980), Johnson (1949) and many others since then have invented their own families of distributions based on this convenient principle. Tadikamalla (1980) reviews many systems. For example, in the **Tadikamalla-Johnson system** Tadikamalla and Johnson (1990), we begin with a logistic random variate

$$Z = \log \frac{U}{1-U} ,$$

and define three random variables:

$$Y_L = \xi + \lambda e^{(Z-\gamma)/\delta} ,$$
$$Y_B = \xi + \lambda / \left(1 + e^{-(Z-\gamma)/\delta}\right) ,$$
$$Y_U = \xi + \lambda \sinh((Z-\gamma)/\delta) .$$

Just as with the Johnson family, the family covers the entire skewness-kurtosis plane. And the one-liners are fixed as well for the family.

8 REPRESENTATION THEOREMS

Sometimes densities, distribution functions or characteristic functions can be written as integrals, which, upon closer inspection, reveal some method for generating random variates. Two examples follow that lead to useful one-liners.

A NEW ONE-LINER FOR LINNIK'S DISTRIBUTION. Kawata (1972, pp. 396–397) derives the following representation for the density f of the Linnik distribution with parameter $a \in [1,2]$:

$$f(x) = f(-x) = \frac{1}{\pi}\sin\left(\frac{\pi a}{2}\right)\int_0^\infty u e^{-xu} g(u)du, x > 0,$$

where

$$g(u) = \frac{1}{\pi}\sin\left(\frac{\pi a}{2}\right)\frac{u^{a-1}}{1+2\cos(\frac{\pi a}{2})u^a + u^{2a}}, u > 0.$$

Thus, a Linnik random variate can be generated as SE/W where S is a random sign, E is exponentially distributed, and W has density g. Using the transformation, $v = u^a$, it is easy to establish that W is distributed as

$$\left(C \sin\frac{\pi a}{2} - \cos\frac{\pi a}{2}\right)^{1/a} ,$$

where C is a Cauchy random variable restricted to $C \ge 1/\tan\frac{\pi a}{2}$. Equivalently, C is distributed as $\tan(\frac{\pi}{2}(1-aU))$, where U is uniformly distributed on $[0,1]$. In summary, we have

$$L_a \stackrel{\mathcal{L}}{=} \frac{SE}{\left(\sin\frac{\pi a}{2}\tan(\frac{\pi}{2}(1-aU)) - \cos\frac{\pi a}{2}\right)^{1/a}} .$$

Not in partcular that if E is exponential, S is a random sign, L is Laplace, C is Cauchy, and N_1, N_2 are i.i.d. normal random variates, then

$$L_1 \stackrel{\mathcal{L}}{=} \frac{EN_1}{N_2} \ ; \ L_1 \stackrel{\mathcal{L}}{=} EC \ ; \ L_1 \stackrel{\mathcal{L}}{=} \frac{E}{C} \ ; \ L_2 \stackrel{\mathcal{L}}{=} SE \stackrel{\mathcal{L}}{=} L .$$

THE STABLE LAWS. Ibragimov and Chernin (1952) and Zolotarev (1966, 1986) derived various useful representations for the stable laws. As an example, define $K(a) = a - 2I_{a>1}$, and let $G_{a,b}$ be the distribution function for $S_{a,b}$. Set $\theta = bK(a)/a$. Note that for $x > 0$, $b > 0$,

$$G_{a,b} = \begin{cases} 1 - \frac{1}{2}\int_{-\theta}^1 e^{-x^{\frac{a}{a-1}}U_a(z,\theta)}\,dz & (a>1) \\ \frac{1-\theta}{2} + \frac{1}{2}\int_{-\theta}^1 e^{-x^{\frac{a}{a-1}}U_a(z,\theta)}\,dz & (a<1) \\ \frac{1}{2}\int_{-1}^1 e^{-e^{-x/b}U_1(z,b)}\,dz & (a=1) \end{cases} .$$

The values for $x < 0$ are obtained by noting that $G_{a,b}(x) + G_{a,-b}(-x) \equiv 1$. The values for $b < 0$ are obtained by noting that $S_{a,b} \stackrel{\mathcal{L}}{=} -S_{a,-b}$. The functions U are defined as follows:

$$U_1(z,b) = \frac{\pi(1+bz)}{2\cos\left(\frac{\pi z}{2}\right)} e^{\frac{\pi}{2}(z+1/b)\tan(\frac{\pi z}{2})} ;$$

$$U_a(z,\theta) = \left(\frac{\sin\left(\frac{\pi a(z+\theta)}{2}\right)}{\cos\left(\frac{\pi z}{2}\right)}\right)^{\frac{a}{1-a}} \frac{\cos\left(\frac{\pi((a-1)z+a\theta)}{2}\right)}{\cos\left(\frac{\pi z}{2}\right)} .$$

All $S_{a,b}$'s are supported on the real line except $S_{a,1}$ for $a < 1$ (which is supported on the positive halfline) and $S_{a,-1}$ for $a < 1$ (which is supported on the negative halfline). Another representation is that the distribution of $S_{a,1}^{a/(a-1)}$ for $a < 1$ is given by

$$\frac{1}{\pi}\int_0^{\pi} e^{-xA(z)}dz ,$$

where

$$A(z) = \frac{\sin((1-a)z)}{\sin(az)}\left(\frac{\sin(az)}{\sin z}\right)^{\frac{1}{1-a}} .$$

In the integrals, we recognize exponential power mixtures, which lead to a variety of one-liners. Kanter (1975) used the last representation to suggest that

$$S_{a,1} \stackrel{\mathcal{L}}{=} \left(\frac{A(\pi U)}{E}\right)^{\frac{1-a}{a}} .$$

For $a < 1$, this would suffice for all values of $b \in [-1, 1]$ as

$$S_{a,b} \stackrel{\mathcal{L}}{=} \left(\frac{1+b}{2}\right)^{1/a} S_{a,1} + \left(\frac{1-b}{2}\right)^{1/a} S_{a,-1}$$

(Zolotarev, 1986, p. 61). Defining θ as above and

$$B_a(z) = \frac{\sin\left(\frac{\pi a(z+\theta)}{2}\right)}{\cos\left(\frac{\pi((a-1)z+a\theta)}{2}\right)}\left(\frac{\cos\left(\frac{\pi((a-1)z+a\theta)}{2}\right)}{\cos\left(\frac{\pi z}{2}\right)}\right)^{\frac{1}{a}} ,$$

Chambers, Mallows and Stuck (1976) (see also Zolotarev, 1986) suggest the extended one-liner

$$S_{a,b} \stackrel{\mathcal{L}}{=} B_a(U-1/2)E^{1-1/a}$$

valid for all $a \neq 1$ and $b \in [-1, 1]$. For $a = 1$, they obtain

$$S_{1,b} \stackrel{\mathcal{L}}{=} B_1(U-1/2) - \frac{2b}{\pi}\log E ,$$

where

$$B_1(z) = \frac{2b}{\pi}\log\left(\frac{1+bz}{\cos\left(\frac{\pi z}{2}\right)}\right) + (1+bz)\tan\left(\frac{\pi z}{2}\right) .$$

9 SCALE MIXTURES

We say that X is a scale mixture if $X = YZ$ can be decomposed as the product of two independent random variables Y and Z. Such mixtures are convenient ways of trying to discover one-liners. Famous scale mixtures occur when Y is uniform $[0, 1]$. In that case, the distribution of X is unimodal with a peak at the origin, and the mixture is called a Khinchine mixture. One should try replacing Y with all random variables for which one-liners are already known. The following are prime candidates: $Y = U^a$, Y is normal, Y is exponential, and Y is Cauchy. At this junction, it is impossible to be exhaustive. We will rather limit ourselves to a few nice examples.

KHINCHINE MIXTURES. The density of $X = U_1U_2$ (with U_1, U_2 i.i.d. uniform $[0, 1]$) is $-\log(x)$ on $[0, 1]$.

NORMAL SCALE MIXTURES. Let N be standard normal, and let X be a positive random variable with two-sided La-place transform $L(s) = \mathbf{E}e^{-sX}$. Then $Y = N\sqrt{X/2}$ has characteristic function $L(t^2)$. This is easily see by noting that

$$\begin{aligned}\mathbf{E}e^{itY} &= \mathbf{E}e^{itN\sqrt{X/2}} \\ &= \mathbf{E}e^{-t^2X} \quad \text{(condition on } X) \\ &= L(t^2) .\end{aligned}$$

Three main examples come to mind:

A. If X is exponential (thus, $L(s) = 1/(1+s)$), then $N\sqrt{X/2}$ is Laplace, as it has characteristic function $1/(1+t^2)$.

B. Assume that $0 < \alpha < 1$ and set $X = S_{\alpha,1}$. From Zo-lo-ta-rev (1986, p. 112), we know that for the positive stable distribution, $L(s) = e^{-s^\alpha}$ if $s \geq 0$. Therefore, $N\sqrt{S_{\alpha,1}/2}$ has characteristic function $e^{-|t|^{2\alpha}}$. That is, $N\sqrt{S_{\alpha,1}/2} \stackrel{\mathcal{L}}{=} S_{2\alpha,0}$. Symmetric stables can be built up from positive stables and normals. For the latter distributions, one-liners were exhibited aerlier in the paper.

C. If we take in the previous example $\alpha = 1$, then we note that $N\sqrt{S_{1,1}/2}$ has characteristic function

$$e^{-2t^2\log|t|} ,$$

where we used the fact that the two-sided La-place transform of $S_{1,1}$ is $e^{-s\log s}$ for $s > 0$ (Zo-lo-ta-rev, 1986, p. 112).

CAUCHY SCALE MIXTURES. Let C be standard Cauchy and let X be a positive random variable with two-sided La-place transform $L(s)$. Then $Y = CX$ has characteristic function $L(|t|)$. This is easily see by noting that

$$\begin{aligned}\mathbf{E}e^{itY} &= \mathbf{E}e^{itXN} \\ &= \mathbf{E}e^{-|t|X} \quad \text{(condition on } X) \\ &= L(|t|) .\end{aligned}$$

Three examples follow:

A. Since G_a has two-sided La-place transform $L(s) = 1/(1+s)^a$, valid for $\Re(s) \geq 0$, CG_a has characteristic function $1/(1+|t|)^a$.

B. Assume that $0 < \alpha < 1$. Arguing as in the previous subsection, we note that $S_{\alpha,1}C \stackrel{\mathcal{L}}{=} S_{\alpha,0}$. Thus, symmetric stables can also be obtained from Cauchy variables and positive extreme stables.

C. Finally, $S_{1,1}C$ has characteristic function $e^{-|t|\log|t|}$.

SYMMETRIC STABLE MIXTURES. Scale mixtures with stable distributions are best studied via characteristic functions. In particular, if X has density f, then the characteristic function of $X^c S_{a,0}$ is

$$\varphi(t) = \mathbf{E}e^{itX^c S_{a,0}} = \mathbf{E}e^{-|tX^c|^a} = \int f(x)e^{-|t|^a|x|^{ca}}\,dx\,.$$

This is particularly helpful if $f(x)$ has a factor $e^{-|x|^b}$. For example, if $f(x) = e^{-x^b}/\Gamma(1+1/b)$, $x > 0$, then, with $c = b/a$,

$$\varphi(t) = \int_0^\infty \frac{e^{-x^b(1+|t|^a)}}{\Gamma(1+1/b)}\,dx = \frac{1}{(1+|t|^a)^{1/b}}.$$

This is a generalized form of Linnik's distribution (Linnik (1962), Laha (1961), Lukacs (1970), pp. 96–97), which is obtained for $b = 1$ (and thus $c = 1/a$). The one-liner suggested by

$$L_a \stackrel{\mathcal{L}}{=} S_{a,0}E^{\frac{1}{a}}$$

is due to Devroye (1990). If $L_{a,b}$ denotes a generalized Linnik random variable, other consequences of the relationship given above include:

$$L_{a,b} \stackrel{\mathcal{L}}{=} S_{a,0}G_b^{1/a}\ ;$$
$$L_{1,1} \stackrel{\mathcal{L}}{=} CE\ ;$$
$$L_{2,1} \stackrel{\mathcal{L}}{=} N\sqrt{E/2} \stackrel{\mathcal{L}}{=} L\ ;$$
$$L_{a,1/2} \stackrel{\mathcal{L}}{=} S_{a,0}(|N|/\sqrt{2})^{2/a}\ ;$$
$$L \stackrel{\mathcal{L}}{=} N_1N_2 + N_3N_4\ .$$

The last statement involves four independent standard normals, and follows from the previous statement.

It is equally simple to verify that if $\{\alpha_j\}$ is a sequence of numbers from $(0,2]$, $\gamma_j \geq 0$, and $S_{\alpha_j,0}$ is a sequence of independent symmetric stable random variables with the given parameters, then $\sum_{j=1}^n S_{\alpha_j}(\gamma_j E)^{1/\alpha_j}$ has characteristic function

$$\frac{1}{1+\sum_{j=1}^n \gamma_j|t|^{\alpha_j}}.$$

POSITIVE EXTREME STABLE MIXTURES: THE MITTAG-LEFFLER DISTRIBUTION. The one-liner

$$X = S_{a,1}G_b^{1/a}$$

yields a random variable X with characteristic function

$$\varphi(t) = \left(\frac{1}{1+(-it)^a}\right)^b\ ,\ a \in (0,1], b > 0\ .$$

This distribution was studied by Pillai (1990). A related distribution was studied by Klebanov, Maniya and Melamed (1984). For $b = 1$, we obtain the Mittag-Leffler distribution with parameter a. Note that the stable $(a,1)$ random variate mentioned here has characteristic function

$$\psi(t) = \begin{cases} \exp(-t^a e^{-i\pi a/2}) & (t > 0) \\ \exp(-(-t)^a e^{i\pi a/2}) & (t < 0)\ . \end{cases}$$

Using Kanter's one-liner for $S_{a,1}$, we see that if E, E^* are i.i.d. exponential random variables and U is uniformly distributed on $[0,1]$, then

$$M_a \stackrel{\mathcal{L}}{=} \left(\frac{E\sin(a\pi U)}{\sin(\pi U)}\right)^{1/a}\left(\frac{E^*\sin(a\pi U)}{\sin((1-a)\pi U)}\right)^{(a-1)/a}$$

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