SIMULATING RANDOM VARIATES FROM THE PEARSON IV AND BETAIZED MEIXNER-MORRIS DISTRIBUTIONS

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ABSTRACT. We develop uniformly fast random variate generators for the Pearson IV distribution that can be used over the entire range of both shape parameters. Additionally, we derive an efficient algorithm for sampling from the betaized Meixner-Morris density, which is proportional to the product of two generalized hyperbolic secant densities.

1. INTRODUCTION

The following statistical setting motivates the methods described in this paper; see Hill and Morris (2023). Let $Y_1, \ldots, Y_k$ be a sample of independent random variables,

$$Y_i \mid \lambda \stackrel{\mathcal{L}}{=} \text{NEF-GHS}(n_i, \lambda),\ i = 1, \ldots, k,$$

with known sample sizes $n_i \geq 1$ and common unknown parameter $\lambda \in \mathbb{R}$, where NEF-GHS is short for the Meixner-Morris natural exponential family generated by the generalized hyperbolic secant distribution. The sampling mean and variance of Y_i given λ are

$$\mathbb{E}\{Y_i \mid \lambda\} = n_i \lambda$$

and

$$\mathbb{V}\{Y_i \mid \lambda\} = n_i(\lambda^2 + 1).$$

Morris (1982) proved that the NEF-GHS distribution is one of only six natural exponential families with quadratic variance functions (i.e., the variance is a quadratic function of the mean); the other five are the normal, Poisson, gamma, binomial, and negative binomial families. For the NEF-GHS distribution, the variance function is $V(\lambda) = \lambda^2 + 1$.

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Let $Y. = \sum_{i=1}^{k} Y_i$, $n. = \sum_{i=1}^{k} n_i$, and $\bar{Y}. = Y./n.$. The sampling distribution of $Y.$ given λ is

$$Y. \mid \lambda \stackrel{\mathcal{L}}{=} \text{NEF-GHS}(n., \lambda),$$

with mean and variance

$$\mathbb{E}\{Y. \mid \lambda\} = n.\lambda$$

and

$$\mathbb{V}\{Y. \mid \lambda\} = n.(\lambda^2 + 1).$$

The statistic $Y.$ is complete sufficient for λ and the sample average $\bar{Y}.$ is the maximum likelihood estimator and uniform minimum variance unbiased estimator of λ.

The conditional distribution of $Y_1, \ldots, Y_k$ given $Y.$ does not depend on λ. It is the reference distribution for checking the NEF-GHS sampling model. The conditional means, variances, and covariances are

$$\mathbb{E}\{Y_i \mid Y.\} = n_i \bar{Y}.,$$

$$\mathbb{V}\{Y_i \mid Y.\} = \frac{n_i(n. - n_i)(\bar{Y}.^2 + 1)}{n. + 1},$$

and

$$\text{Cov}\{Y_i, Y_j \mid Y.\} = -\frac{n_i n_j(\bar{Y}.^2 + 1)}{n. + 1}.$$

We call the conditional distribution of Y_i given $Y.$ the betaized Meixner-Morris distribution. The name was suggested by analogy to the conditional distribution of a gamma variable given the sum of independent gamma variables being a beta distribution.

The conjugate prior for NEF-GHS sampling is the Pearson IV family of distributions, with two parameters, a prior mean $\mu_0 \in \mathbb{R}$ and a prior sample size $m_0 \geq 1$, say. The prior mean and variance are

$$\mathbb{E}\{\lambda\} = \mu_0$$

and

$$\mathbb{V}\{\lambda\} = \frac{\mu_0^2 + 1}{m_0 - 1}.$$

The posterior distribution of λ given $Y.$ is also a member of the Pearson IV family, with parameters $\mu_1 = (m_0\mu_0 + Y.)/(m_0 + n.)$ and $m_1 = m_0 + n.$. The posterior mean and variance are

$$\mathbb{E}\{\lambda \mid Y.\} = \mu_1$$

and

$$\mathbb{V}\{\lambda \mid Y.\} = \frac{\mu_1^2 + 1}{m_1 - 1}.$$

The prior predictive marginal distribution of $Y_.$ is the mixture of the NEF-GHS sampling model averaged with respect to the Pearson IV prior distribution. The prior predictive mean and variance of $Y_.$ are

$$\mathbb{E}\{Y_.\} = n_.\mu_0$$

and

$$\mathbb{V}\{Y_.\} = n_.(\mu_0^2 + 1)\frac{m_0 + n_.}{m_0 - 1}.$$

This distribution is the reference distribution for checking the prior distribution.

The posterior predictive marginal distribution of a future observation $Y_.'$ is the mixture of the NEF-GHS sampling model averaged with respect to the Pearson IV posterior distribution. The posterior predictive mean and variance of $Y_.'$ given $Y_.$ are

$$\mathbb{E}\{Y_.' \mid Y_.\} = n_.\mu_1$$

and

$$\mathbb{V}\{Y_.' \mid Y_.\} = n_.(\mu_1^2 + 1)\frac{m_1 + n_.}{m_1 - 1}.$$

This distribution is used to predict future observations.

Given this setup, we need to be able to simulate random variables from the following distributions:

(i) the NEF-GHS distribution; this is covered in Devroye (1993).
(ii) the betaized Meixner-Morris conditional distribution; this is covered in the second part of this note.
(iii) the Pearson IV distribution for prior and posterior samples; this is covered in the first part of this paper.
(iv) Pearson IV mixtures of NEF-GHS densities for prior and posterior predictive marginal distributions; this can be done in two stages using (iii) followed by (i).

2. The Pearson IV distribution

Undoubtedly, the most enigmatic member of Pearson's family of distributions (Pearson, 1895) is the Pearson IV distribution, which is characterized by two shape parameters, $a > 1/2$ and $s \in \mathbb{R}$. Its density on the real line is given by

$$f(x) = \frac{\gamma\, e^{s \arctan(x)}}{(1+x^2)^a},$$

where, by Legendre's duplication formula,

$$\gamma \stackrel{\text{def}}{=} \frac{|\Gamma(a - is/2)|^2}{\Gamma(a)\Gamma(a-1/2)\Gamma(1/2)} = \frac{4^{a-1}\,|\Gamma(a - is/2)|^2}{\pi\Gamma(2a-1)},$$

and Γ is the complex gamma function. We write $P_{a,s}$ to denote a Pearson type IV random variable with the given parameters. As $P_{a,s} \stackrel{\mathcal{L}}{=} -P_{a,-s}$, we will assume without loss of generality that $s \geq 0$.

The purpose of this paper is to propose random variate generation algorithms that are uniformly fast over all choices of the parameters. As far as we know, no explicit uniformly fast methods are known for this important distribution. As described above in the introduction, we are personally motivated by random variates from this law being necessary in a Bayesian framework, as the natural exponential families with quadratic variance functions derived by Morris (1982, 1983) have Pearson conjugate families (Hill and Morris, 2023). For example, in the notation of the introduction, we would have $s = m_0\mu_0$, $a = m_0/2 + 1$, and $x = \lambda$.

In Section 3, we recall some facts about $P_{a,0}$, the Student-t distribution. In the subsequent sections, we will develop several generators for the Pearson IV distribution. Some of these require access to the normalization constant γ, which depends upon the complex gamma function. One of the novelties in this paper is a simple method that does not require explicit knowledge of γ. We recall the two design principles for the methods given below:

(i) The generators have to be theoretically exact; no approximation of any kind is allowed.
(ii) The expected time per random variate should be uniformly bounded over all parameter choices.

In the second part of the paper, we deal with a new distribution whose density is proportional to the product of two densities for the GHS (generalized hyperbolic secant) distribution. Crucial for simulation in a statistical setup (Hill and Morris, 2023), it will be called the betaized Meixner-Morris distribution.

3. Student-t distribution

The random variable T_a with parameter $a > 0$ is a Student-t(a) random variable if it has density

$$\frac{1}{B(a/2, 1/2)\sqrt{a}(1 + x^2/a)^{\frac{a+1}{2}}},$$

where B denotes the beta function. First derived by Helmert (1875, 1876) and Lüroth (1876) and later by Pearson (1895), it was named after Gosset (William S. Gosset, 1908) by Ronald Fisher. It is in the Pearson IV family, as

$$\frac{T_{2a-1}}{\sqrt{2a-1}} \stackrel{\mathcal{L}}{=} P_{a,0}.$$

We recall that

$$T_a \stackrel{\mathcal{L}}{=} \frac{N}{\sqrt{\frac{G_{a/2}}{a/2}}},$$

where N is standard normal, and G_a denotes an independent gamma (a) random variable. Let $H_{a,b} = G_a/G_b$ be the ratio of two independent gamma random variables and let $B_{a,b}$ be a beta random variable. From the definition of the Student-t distribution,

$$\frac{T_a^2}{a} = \frac{(1/2)N^2}{G_{a/2}} \stackrel{\mathcal{L}}{=} \frac{G_{1/2}}{G_{a/2}} \stackrel{\mathcal{L}}{=} H_{1/2,a/2} \stackrel{\mathcal{L}}{=} B_{1/2,1/2}H_{1,a/2} \stackrel{\mathcal{L}}{=} \sin^2(\pi U')\left(\frac{1}{U^{\frac{2}{a}}} - 1\right),$$

where U and U' are i.i.d. uniform $[0,1]$ random variables. This yields a one-liner for the Student-t distribution due to Bailey (1994), also called the polar method for Student-t distribution:

$$T_a \stackrel{\mathcal{L}}{=} \sqrt{a}\sin(2\pi U')\sqrt{\frac{1}{U^{\frac{2}{a}}} - 1}.$$

See also Devroye (1996) for variations on this polar method. Earlier methods for the Student-t distribution include algorithms by Best (1978) and Ulrich (1984). Very simple special cases, obtainable by the inversion method, include the Cauchy law (obtained for $a = 1$), for which we have $T_1 \stackrel{\mathcal{L}}{=} \tan(\pi(U - 1/2))$, and the t_2 law, for which we have

$$T_2 \stackrel{\mathcal{L}}{=} \frac{2U - 1}{\sqrt{2U(1-U)}}$$

(see, e.g. Jones, 2002).

4. Rejection from Student-t distribution

The obvious thing to try is to use the rejection method from Student-t distribution, for which many uniformly fast algorithms are known. Using

$$f(x) \le \frac{\gamma e^{s\pi/2}}{(1+x^2)^a},$$

it suffices to generate i.i.d. pairs (X, U), where $X = T_{2a-1}/\sqrt{2a-1}$ is a random variable with density proportional to $(1+x^2)^{-a}$ and U is uniform on $[0,1]$, until

$$Ue^{s\pi/2} \le e^{s\arctan(X)},$$

or, equivalently, to generate i.i.d. pairs (X, E), where E is standard exponential, until

$$-E + s\pi/2 \le s\arctan(X).$$

As

$$\frac{\gamma e^{-s\pi/2}}{(1+x^2)^a} \le f(x) \le \frac{\gamma e^{s\pi/2}}{(1+x^2)^a},$$

it is easy to see that the expected number of iterations is at least $(1/2)e^{\pi s/2}$ (and at most $e^{\pi s}$), which is uniformly bounded over all values of $a > 1/2$ and s smaller than any fixed constant. As soon as $s \geq 10$, or something of that order of magnitude, this simple method is unfeasible.

5. Log-concavity

As noted in Exercise 1 on page 308 in Devroye (1986), when $a \geq 1$, $\arctan(P_{a,s})$ has a log-concave density on $[-\pi/2, \pi/2]$ given by

$$h(y) = \begin{cases} \gamma e^{sy}(\cos^2(y))^{a-1} & \text{if } |y| \leq \frac{\pi}{2}, \\ 0 & \text{else.} \end{cases} \tag{1}$$

We note that $\log(h)$ has first derivative $s - 2(a-1)\tan(y)$ and second derivative $-2(a-1)/\cos^2(y)$, which is negative.

Remark 1. THE SKEWED CAUCHY FAMILY. *When* $a = 1$, $h(y) = \gamma \exp(sy)$, *so that a random variate with density* h *simply is*

$$W = \begin{cases} \frac{1}{s}\log\left(e^{-\frac{\pi s}{2}} + U\left(e^{\frac{\pi s}{2}} - e^{-\frac{\pi s}{2}}\right)\right) & \textit{if } s > 0, \\ \pi(U - 1/2) & \textit{if } s = 0, \end{cases}$$

where U *is uniform on* $[0,1]$. *Therefore,* $P_{1,s} \stackrel{\mathcal{L}}{=} \tan(W)$. *We also rediscover the standard method for generating Cauchy random variables:* $P_{1,0} \stackrel{\mathcal{L}}{=} \tan(\pi(U - 1/2))$. *The family of distributions* $P_{1,s}$ *will be called the skewed Cauchy family.*

In the remainder of this section, we assume that $a > 1$. The mode of h occurs at

$$y^* = \arctan(\beta),$$

where we set $\beta = s/(2(a-1))$. Also,

$$h(y^*) = \gamma e^{s \arctan(\beta)} \left(\frac{1}{1+\beta^2}\right)^{a-1} = \gamma \left(\frac{e^{2\beta \arctan(\beta)}}{1+\beta^2}\right)^{a-1}.$$

Assuming that we have constant time access to the value of γ—which requires the complex gamma function—, we can apply the universal method for log-concave densities from Devroye (1984), which is repeated here for the sake of completeness.

Algorithm 1 Log-concave method

let m be the location of the mode of log-concave density h
$M \leftarrow h(m)$
repeat
  let V be uniform on $[-2, 2]$
  if $V < -1$ **then**
    replace V by $-1 + \log(V + 2)$
  else if $V > 1$ **then**
    replace V by $1 - \log(V - 1)$
  end if
  $Y \leftarrow m + V/M$
  let U be uniform on $[0, 1]$
until $UM \min(1, \exp(1 - M|Y - m|)) \leq h(Y)$
return Y $\triangleright$ Y has density h
$\triangleright$ $\tan(Y) \stackrel{\mathcal{L}}{=} P_{a,s}$ if h is as in (1)

Remark 2. ADAPTIVE METHODS. *The method given here has a uniformly bounded time and is useful when the parameters vary in an application. For fixed parameters, several adaptive methods make the method more efficient as more random variates are generated. See, e.g., Gilks (1992), Gilks and Wild (1992, 1993) and Gilks, Best and Tan (1995).* $\square$

Remark 3. REFERENCES. *Additional references on random variate generation for log-concave laws include Hörmann, Leydold and Derflinger (2004), Leydold and Hörmann (2000, 2001) and Devroye (1986). For an implementation of the algorithm in this section, see Heinrich (2004).* $\square$

The expected number of iterations of this algorithm is four, as it is based on the inequality

$$h(y) \leq h(m) \min\left(1, e^{1-|y-m|h(m)}\right). \tag{2}$$

We can apply this method here with the values $m = y^*$ and $M = h(y^*)$ given above. As the algorithm returns Y with density h, $\tan(Y) \stackrel{\mathcal{L}}{=} P_{a,s}$. Note that the algorithm needs access to the exact value of γ, which in turn depends upon the complex gamma function.

6. LOG-CONCAVITY WITHOUT ACCESS TO THE NORMALIZATION CONSTANT

Assume that we have a log-concave density as in (1) but have no computational access to the normalization constant. That situation was dealt with in all generality in Devroye (2012), but since we have a special situation here, there is a simpler solution. The constant γ in (1) is easily explicitly bounded, as we will see below. Assume that

$$\gamma^- \leq \gamma \leq \gamma^+,$$

where γ^- and γ^+ are explicit functions of the parameters, a and s. In (1), we have the following value at the mode $m = y^*$ of h:

$$h(m) = \gamma e^{sm} (\cos^2(m))^{a-1} \stackrel{\text{def}}{=} \gamma\Delta.$$

Therefore, in our case, inequality (2) implies

$$h(y) \leq \gamma^+ \Delta \min\left(1, e^{1-|y-m|\gamma^- \Delta}\right). \tag{3}$$

Furthermore,

$$h(y) \geq \gamma^- e^{sy} (\cos^2(y))^{a-1}, |y| \leq \frac{\pi}{2}. \tag{4}$$

If we use the bounds (3) and (4) in the rejection method, then the expected number of iterations is the ratio of the integrals of the bounding functions, or

$$4 \times \frac{\gamma^+}{\gamma^-} \times \frac{\gamma}{\gamma^-} \leq 4 \left(\frac{\gamma^+}{\gamma^-}\right)^2.$$

As we will see in the next section, there is a nice supply of good bounds for γ that makes rejection based on the given inequalities uniformly fast over all values of s and all values $a \geq 1$. With the choice (7) given below, we see that the expected number of iterations is

$$\leq 4 \times \left(\frac{1 + \frac{3}{2\pi^2 \sqrt{a^2 + (s/2)^2}}}{1 - \frac{3}{2\pi^2 \sqrt{a^2 + (s/2)^2}}} \right)^4 \times \frac{(a + 0.177)(a + 0.677)}{(a + 1/6)(a + 2/3)}.$$

Uniformly over all $a \geq 1$ and $s \geq 0$, this does not exceed

$$4 \times \left(\frac{2\pi^2 + 3}{2\pi^2 - 3}\right)^4 \times \frac{1.177 \times 1.677 \times 18}{35} \approx 7.15.$$

Note though that as $a \to \infty$, the expected number of iterations tends to 4, since the inequalities for the gamma function get tighter.

Here is the complete algorithm for generating a random variate $Y \stackrel{\mathcal{L}}{=} \arctan P_{a,s}$ for $s \in \mathbb{R}, a \geq 1$:

Algorithm 2 PearsonIV generator, parameter $a \geq 1$

compute γ^+ and γ^- $\triangleright$ see (7) in the next section
$\beta \leftarrow s/(2(a-1))$
$m \leftarrow \arctan(\beta)$ $\triangleright$ the mode of log-concave density h
$\Delta \leftarrow e^{sm}\left(\frac{1}{1+\beta^2}\right)^{a-1}$
repeat
    let V be uniform on $[-2,2]$
    if $V < -1$ **then**
        replace V by $-1+\log(V+2)$
    else if $V > 1$ **then**
        replace V by $1-\log(V-1)$
    end if
    $Y \leftarrow m + V/(\gamma^- \Delta)$
    let U be uniform on $[0,1]$
until $|Y| \leq \frac{\pi}{2}$ and $U\gamma^+\Delta \min(1, \exp(1-|Y-m|\gamma^-\Delta)) \leq \gamma^- e^{sY}(\cos^2(Y))^{a-1}$
return $\tan(Y)$ $\triangleright$ $\tan(Y) \overset{\mathcal{L}}{=} P_{a,s}$

7. Bounding the normalization constant

The gamma function is defined for complex $z = x + iy$ with $x > 0$ by Euler's integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

Explicit inequalities for Euler's gamma function with real argument are often tied to Stirling's approximation (Stirling, 1730). A prime example is Robbins's upper and lower bound (Robbins, 1955). Olver et al. (2023) summarize most of the well-known bounds. Batir (2008) showed the bounds

$$\sqrt{2e}\left(\frac{x+1/2}{e}\right)^{x+1/2} \leq \Gamma(1+x) \leq \sqrt{2\pi}\left(\frac{x+1/2}{e}\right)^{x+1/2}, x > 0,$$

and

$$\sqrt{2\pi(x+1/6)}\left(\frac{x}{e}\right)^x \leq \Gamma(1+x) \leq \sqrt{2\pi\left(x + \frac{e^2}{2\pi} - 1\right)}\left(\frac{x}{e}\right)^x, x \geq 1.$$

Note that $e^2/(2\pi) - 1 < 0.177$. It is the latter inequality that we will employ.

For $\Gamma(z)$ with $z = x+iy$, $x > 0$, we will use Boyd's inequality (Boyd, 1994; see also (5.11.11) in Olver et al., 2023) which implies that

$$|\Gamma(z)| = \left|\frac{\sqrt{2\pi}}{z}\left(\frac{z}{e}\right)^z\right| \times |1 + R(z)|,$$

where
$$|R(z)| \leq \frac{3}{2\pi^2|z|}. \tag{5}$$
Note that
$$\left|\frac{\sqrt{2\pi}}{z}\left(\frac{z}{e}\right)^z\right| = \sqrt{\frac{2\pi}{\sqrt{x^2+y^2}}}\left(\frac{\sqrt{x^2+y^2}}{e}\right)^x e^{-y\arctan(y/x)}. \tag{6}$$
Combining Batir's last inequality with Boyd's bounds, we can derive suitable values for γ^+ and γ^-, recalling that $a \geq 1$, $s \geq 0$, and
$$\gamma \stackrel{\text{def}}{=} \frac{|\Gamma(a-is/2)|^2}{\Gamma(a)\Gamma(a-1/2)\Gamma(1/2)} = \frac{(a^2-1/4)a\,|\Gamma(a-is/2)|^2}{\Gamma(a+1)\Gamma(a+3/2)\sqrt{\pi}}.$$
First we define the key constant (itself an approximation of γ),
$$\gamma^* = \frac{(a-1/2)\left(1+(s/2a)^2\right)^{a-1/2} e^{-s\arctan(s/2a)}}{\sqrt{\pi/e}\,(1+1/2a)^a\sqrt{a}}.$$
Then, for the algorithm of the previous section, we suggest
$$\gamma^+ = \gamma^* \times \frac{\left(1+\frac{3}{2\pi^2\sqrt{a^2+(s/2)^2}}\right)^2}{\sqrt{1+\frac{1}{6a}}\sqrt{1+\frac{1}{6(a+1/2)}}},$$
$$\gamma^- = \gamma^* \times \frac{\left(1-\frac{3}{2\pi^2\sqrt{a^2+(s/2)^2}}\right)^2}{\sqrt{1+\frac{0.177}{a}}\sqrt{1+\frac{0.177}{a+1/2}}}. \tag{7}$$

8. Symmetrization for parameter values $a \leq 1$

Finally, we develop a generator that is uniformly fast for $a \in (1/2,1]$, $s \in \mathbb{R}$. As $P_{a,s} \stackrel{\mathcal{L}}{=} -P_{a,-s}$, symmetrization may be helpful. Define the symmetric density
$$g(x) = \frac{f(x)+f(-x)}{2} = \frac{\gamma\cosh(s\arctan(x))}{(1+x^2)^a}.$$
Setting $Y = \arctan(X)$, where X has density g on $\mathbb{R}$ yields the following symmetric density on $(-\pi/2,\pi/2)$:
$$h(y) = \gamma\cosh(sy)(\cos^2(y))^{a-1}.$$
Consider next the random variable $Z = \pi/2 - |Y|$ with density
$$\eta(z) = 2\gamma\cosh(s(\pi/2-z))(\sin(z))^{2(a-1)}, 0 \leq z \leq \pi/2. \tag{8}$$

For $a \in (1/2,1]$, the density (8) is decreasing and has an infinite peak at the origin unless $a=1$. Most of its mass is near zero, and thus, we will attempt rejection using the bound

$$\begin{aligned}\eta(z) &\le 2\gamma e^{s\pi/2} e^{-sz} (2z/\pi)^{2(a-1)} \\ &\le 2\gamma (2/\pi)^{2(a-1)} e^{s\pi/2} e^{-sz} z^{2(a-1)}, 0 < z \le \pi/2.\end{aligned}$$

We can apply rejection from the gamma distribution by generating independent pairs $(Z, U) = (G_{2a-1}/s, U)$ until for the first time, $Z \le \pi/2$ and

$$U(2Z/\pi)^{2(a-1)} \le (\sin(Z))^{2(a-1)},$$

or, equivalently,

$$U \le \left(\frac{2Z}{\pi \sin(Z)} \right)^{2(1-a)}.$$

The returned random variable Z has density η given in (8). The probability of acceptance is thus at least

$$\begin{aligned}&\mathbb{E} \left\{ \left(\frac{2G_{2a-1}/s}{\pi \sin(G_{2a-1}/s)} \right)^{2(1-a)} \mathbb{1}_{G_{2a-1}/s \le \pi/2} \right\} \\ &\ge \mathbb{E} \left\{ \frac{2G_{2a-1}/s}{\pi \sin(G_{2a-1}/s)} \mathbb{1}_{G_{2a-1}/s \le \pi/2} \right\} \\ &\ge \frac{2}{\pi} \mathbb{P} \left\{ G_{2a-1}/s \le \pi/2 \right\},\end{aligned}$$

where $\mathbb{1}$ is the indicator function. By Markov's inequality, the probability in this expression is at least

$$1 - \frac{2\mathbb{E}\{\frac{G_{2a-1}}{s}\}}{\pi} = 1 - \frac{2(2a-1)}{s\pi} \ge 1 - \frac{2}{s\pi} \ge 1 - \frac{2}{\pi}$$

uniformly for all $s \ge 1, a \le 1$. In this range, the expected number of iterations of the rejection algorithm is at most

$$\frac{\pi^2}{2\pi - 4}.$$

Having generated Z with density (8), we need to set $X = \tan(S(\pi/2 - Z))$, where S is a random sign to obtain a random variate with the symmetrized Student-t density g. Finally, a random variate with the Pearson IV density f is obtained as

$$\begin{cases} X & \text{with probability} \frac{f(X)}{f(X)+f(-X)} = \frac{e^{s \arctan(X)}}{e^{s \arctan(X)} + e^{-s \arctan(X)}} \\ -X & \text{else.} \end{cases}$$

We summarize the algorithm below.

Algorithm 3 PearsonIV generator, parameter $1/2 < a \le 1$

repeat
    generate $Z = G_{2a-1}/s$ and U uniformly on $[0,1]$
until $Z < \pi/2$ and $U \le \left(\frac{2Z}{\pi \sin(Z)}\right)^{2(1-a)}$
$Y \leftarrow S(\pi/2 - Z)$, where S is a random sign
$X \leftarrow \tan(Y)$
with probability $e^{-sY}/(e^{-sY} + e^{sY})$, $X \leftarrow -X$
return X $\qquad\triangleright\ X \overset{\mathcal{L}}{=} P_{a,s}$

Remark 4. GAMMA RANDOM VARIATES. *For uniformly fast gamma random variates, we refer to the surveys in Devroye (1986) and Luengo (2022). Many simulation studies confirm that the method of Schmeiser and Lal (1980) is quite competitive if the gamma parameter is at least one. Xi, Tan and Liu (2013) suggested generating* $\log(G_a)$ *instead. As* $\log(G_a)$ *has a log-concave density for all values of* $a > 0$*, a uniformly fast generator is quite easily obtained either by the universal method of Devroye (1984) or a specialized algorithm as developed, e.g., in Devroye (2014).* $\square$

9. PUTTING THINGS TOGETHER FOR THE PEARSON IV DISTRIBUTION.

We conclude this first part by giving an overview of methods developed above, which are all uniformly fast over the ranges of the parameters specified below:

(i) For $a \ge 1$ and all s, one can use the universal log-concave generator.
(ii) For $s \le 5$ and all a, one can use rejection from the Student-t density.
(iii) For $a \in (1/2, 1], s \ge 1$, one can use rejection from the gamma density after symmetrizing the Pearson distribution.
(iv) For $a = 1$, the skewed Cauchy density, there is a simple one-liner.
(v) For $s = 0$, the Student-t law, there is a simple one-liner.

10. THE GHS DISTRIBUTION REVISITED

Natural exponential families of distributions have probability density functions of the form $e^{\theta x}\mu(dx)$ where μ is a given measure, and $\theta \in \mathbb{R}$ is a natural parameter. When we compute the mean and the variance, and force the variance to be a quadratic function of the mean as θ is varied, the number of families becomes severely restricted. Morris (1982) proved that there are in fact only six natural exponential families with this property: the normal, Poisson, gamma, binomial, negative binomial and NEF–GHS families, where GHS is an abbreviation for generalized hyperbolic secant distribution, which goes back to Harkness and Harkness (1968). The NEF–GHS distribution with parameters $\rho > 0$ and $\lambda \in \mathbb{R}$, also called Meixner's distribution and named after physicist Meixner (1934), has density

$$f(x) = (1+\lambda^2)^{-\rho/2} e^{x \arctan \lambda} f_\rho(x) \ , \tag{9}$$

where f_ρ is the density of the (GHS) distribution with parameter ρ. While the five other families play crucial roles in statistics, the NEF–GHS distribution became prominent as a model in the finance literature, where it is known as Meixner's distribution. However, Morris independently introduced and named the NEF–GHS distribution in statistics. In the present paper, we will call it the Meixner-Morris distribution.

Schoutens and Teugels (1998) and Grigelionis (1999, 2000) first introduced the Meixner-Morris distribution and process. Schoutens (2001, 2002, 2003) highlighted its uses in the financial modelling of stock markets. Further theoretical properties were developed by Ferreira-Castilla and Schoutens (2012) and surveyed by Mazzola and Muliere (2011).

The GHS density f_ρ is not explicitly known in any standard way. Its shortest description is as a product of two gamma functions with imaginary argument,

$$\begin{aligned} f_\rho(x) &= \frac{2^{\rho-2}}{\pi\Gamma(\rho)} \Gamma\left(\frac{\rho+ix}{2}\right)\Gamma\left(\frac{\rho-ix}{2}\right) \\ &= \frac{2^{\rho-2}}{\pi\Gamma(\rho)} \left|\Gamma\left(\frac{\rho+ix}{2}\right)\right|^2 \\ &= \frac{2^{\rho-2}(\Gamma(\rho/2))^2}{\pi\Gamma(\rho)} \left(\frac{\left|\Gamma\left(\frac{\rho+ix}{2}\right)\right|}{\Gamma(\rho/2)}\right)^2 \end{aligned} \tag{10}$$

(Harkness and Harkness, 1968). It is easy to see that f_ρ is symmetric about 0.

Devroye (1993) gives various uniformly fast random variate generators for the GHS and Meixner-Morris distributions based on the unimodality of f_ρ. For a method that requires first finding the supremum of a ratio of two functions numerically; see Grigoletto and Provasi (2008).

Let us derive a simple representation of f_ρ:

$$f_\rho(x) = \frac{2^{\rho-2}(\Gamma(\rho/2))^2}{\pi\Gamma(\rho)} \times \mathbb{E}\left\{\cos(xZ/2)\right\}, \tag{11}$$

where $Z = \log(G_{\rho/2}) - \log(G'_{\rho/2})$ and $G_{\rho/2}$ and $G'_{\rho/2}$ are independent gamma $(\rho/2)$ random variables. It is noteworthy that both $\log(G_{\rho/2})$ and Z have log-concave densities. Still, it is not immediately clear how this would aid in random variate generation, or even to show that f_ρ is log-concave. As we will see in the next section, f_ρ can be bounded from both sides by suitable log-concave functions.

PROOF OF (11). Let G_ρ be a gamma (ρ) random variable with $\rho > 0$. Then $W_\rho \stackrel{\text{def}}{=} \log(G_\rho)$ has a generalized Gumbel distribution with density

$$\frac{e^{\rho y}\, e^{-e^y}}{\Gamma(\rho)}, y \in \mathbb{R}.$$

Recalling the definition of the complex gamma function, we have for $\rho > 0, x \in \mathbb{R}$,

$$\Gamma(\rho + ix) = \int_0^\infty t^{\rho+ix-1} e^{-t}\, dt$$

(see, e.g., Whittaker and Watson, 1927). Rewritten, we define the complex-valued function

$$\begin{aligned}
h(x) &\stackrel{\text{def}}{=} \frac{\Gamma(\rho+ix)}{\Gamma(\rho)} \\
&= \int_0^\infty \frac{t^{\rho-1}e^{-t}}{\Gamma(\rho)} \cos(x \log t)\, dt + i \int_0^\infty \frac{t^{\rho-1}e^{-t}}{\Gamma(\rho)} \sin(x \log t)\, dt \\
&= \mathbb{E}\left\{\cos(x \log G_\rho) + i \sin(x \log G_\rho)\right\} \\
&= \mathbb{E}\left\{\cos(xW_\rho)\right\} + i\mathbb{E}\left\{\sin(xW_\rho)\right\}.
\end{aligned}$$

Define $H(x) \stackrel{\text{def}}{=} |h(x)|^2$. Then

$$\begin{aligned}
H(x) &= \left(\mathbb{E}\left\{\cos(xW_\rho)\right\}\right)^2 + \left(\mathbb{E}\left\{\sin(xW_\rho)\right\}\right)^2 \\
&= \mathbb{E}\left\{\cos(xW_\rho)\cos(xW'_\rho) + \sin(xW_\rho)\sin(xW'_\rho)\right\} \text{ (where } W_\rho \text{ and } W'_\rho \text{ are i.i.d.)} \\
&= \mathbb{E}\left\{\cos(xZ)\right\},
\end{aligned}$$

where we define the symmetric random variable $Z \stackrel{\text{def}}{=} W_\rho - W'_\rho$. Now replace x by $x/2$ and ρ by $\rho/2$ throughout to get the desired representation. □

Remark 5. THE MEIXNER-MORRIS DISTRIBUTION. *The Meixner-Morris distribution is infinitely divisible since its characteristic function has a power representation (Schoutens, 2002; see also Devianto, Safatri, Herli and Maiyastri, 2018),*

$$\varphi(t) = \left(\frac{\cos(\arctan(\lambda))}{\cosh(t - i\arctan(\lambda))}\right)^{2\rho}.$$

This implies, for example, that the sum of independent GHS *random variables with parameters $\rho_1, \ldots, \rho_n$ is* GHS *with parameter $\sum_{i=1}^n \rho_i$. We can associate with it a Lévy process commonly called the Meixner process (see, e.g., Schoutens, 2002). This Lévy process has no Brownian part and a pure jump part governed by the absolutely continuous Lévy measure*

$$\nu(dx) = \frac{\rho}{x \sinh(\pi x)}\, dx$$

for the case $\lambda = 0$ (see, Eberlein (2009), or Ferreira-Castillo and Schoutens (2012)).

11. The betaized Meixner-Morris distribution.

As shown in Hill and Morris (2023), and summarized in our introduction, the conjugate prior for λ in the Meixner-Morris distribution is the Pearson IV distribution. Hence it is natural to deal with both distributions at the same time. Consider next the independent random variables $X_1, \ldots, X_n$, where X_i is Meixner-Morris distributed with parameters ρ_i and λ in the notation of (9) and (10). Let $S = \sum_{i=1}^n X_i$. In a statistical setting, one would like to generate $X_1, \ldots, X_n$ given their sum S. This can be done sequentially if we know how to generate X_1 given the sum S. The distribution of X_1 given S does not depend on λ and has density

$$\frac{f_{\rho_1}(x) f_{\sum_{i=2}^n \rho_i}(S-x)}{f_{\sum_{i=1}^n \rho_i}(S)}, x \in \mathbb{R}.$$

For a smoother treatment, let us define the betaized Meixner-Morris density with shape parameters $a, b > 0$ and $s \in \mathbb{R}$ and denote it by f in the present section:

$$f(x) \stackrel{\text{def}}{=} \frac{f_a(x) f_b(s-x)}{f_{a+b}(s)}, x \in \mathbb{R}; \tag{12}$$

for example, in the notation of the introduction, we would take $a = n_i$, $b = n. - n_i$, $x = Y_i$, and $s = Y.$. Using (10), the density becomes

$$\begin{aligned} f(x) &= \frac{\frac{2^{a-2}}{\pi\Gamma(a)} \left|\Gamma\left(\frac{a+ix}{2}\right)\right|^2 \frac{2^{b-2}}{\pi\Gamma(b)} \left|\Gamma\left(\frac{b+i(s-x)}{2}\right)\right|^2}{\frac{2^{a+b-2}}{\pi\Gamma(a+b)} \left|\Gamma\left(\frac{a+b+is}{2}\right)\right|^2} \\ &= \frac{\Gamma(a+b)}{4\pi\Gamma(a)\Gamma(b)} \frac{\left|\Gamma\left(\frac{a+ix}{2}\right)\right|^2 \left|\Gamma\left(\frac{b+i(s-x)}{2}\right)\right|^2}{\left|\Gamma\left(\frac{a+b+is}{2}\right)\right|^2}. \end{aligned}$$

It is well-known that the conditional distribution of a gamma variable given the sum of independent gamma variables is a beta distribution. By analogy, we call the conditional distribution of a Meixner-Morris random variable given the sum of independent Meixner-Morris variables a <u>betaized</u> <u>Meixner-Morris</u> random variable, to avoid the acronym-based name “betaized GHS”.

The mean and variance of this distribution are given by

$$\begin{aligned} \mu &= \frac{a}{a+b} s, \\ \sigma^2 &= \frac{ab}{(a+b)^2} \frac{s^2 + (a+b)^2}{1+a+b}. \end{aligned} \tag{13}$$

While f itself is the product of two unimodal densities, the resulting density is not necessarily unimodal. We are able, however, to tightly sandwich f between two log-concave functions, which will enable us to develop a uniformly fast algorithm for sampling.

Recall from (5) and (6) that with $z = x + iy$, $r = |z| = \sqrt{x^2 + y^2}$, $x \geq 1/2, y \in \mathbb{R}$, we have

$$|\Gamma(z)|^2 = \theta \, \frac{2\pi}{r} \, \left(\frac{r}{e}\right)^{2x} e^{-2y \arctan(y/x)},$$

where θ is some number bounded as follows

$$\left(1 - \frac{3}{2x\pi^2}\right)^2 \stackrel{\text{def}}{=} \theta^-(x) \leq \theta \leq \theta^+(x) \stackrel{\text{def}}{=} \left(1 + \frac{3}{2x\pi^2}\right)^2.$$

With the proper substitutions, assuming $\min(a, b) \geq 1$,

$$f(x) = \theta g(x),$$

where now

$$\theta^-(a/2)\theta^-(b/2) \leq \theta \leq \theta^+(a/2)\theta^+(b/2),$$

$$g(x) = \gamma \left(\left(\frac{a}{2}\right)^2 + \left(\frac{x}{2}\right)^2\right)^{\frac{a-1}{2}} \left(\left(\frac{b}{2}\right)^2 + \left(\frac{s-x}{2}\right)^2\right)^{\frac{b-1}{2}} e^{-x \arctan\left(\frac{x}{a}\right) - (s-x) \arctan\left(\frac{s-x}{b}\right)},$$

and

$$\gamma = \frac{\Gamma(a+b)\pi}{e^{a+b}\Gamma(a)\Gamma(b) \left|\Gamma\left(\frac{a+b+is}{2}\right)\right|^2}.$$

Lemma 1. *The function g above is log-concave when* $\min(a, b) \geq 1$.

PROOF. As g consists of a product of two functions, it suffices to establish the log-concavity of each, to conclude that g is log-concave. So, let us prove that

$$((a/2)^2 + (x/2)^2)^{(a-1)/2} \, e^{-x \arctan\left(\frac{x}{a}\right)},$$

is log-concave, or, equivalently, that

$$(a^2 + x^2)^{a-1} \, e^{-2x \arctan\left(\frac{x}{a}\right)}$$

is log-concave. Define the logarithm of the latter function as

$$h(x) = (a-1) \log\left(a^2 + x^2\right) - 2x \arctan\left(\frac{x}{a}\right).$$

We have

$$h'(x) = \frac{2(a-1)x}{a^2 + x^2} - 2 \arctan\left(\frac{x}{a}\right) - \frac{2x}{a(1 + \left(\frac{x}{a}\right)^2)} = \frac{-2x}{a^2 + x^2} - 2 \arctan\left(\frac{x}{a}\right).$$

Thus,

$$
\begin{aligned}
h''(x) &= -\frac{2}{a^2+x^2} + \frac{4x^2}{(a^2+x^2)^2} - \frac{2}{a(1+\left(\frac{x}{a}\right)^2)} \\
&= \frac{-2(a^2+x^2)+4x^2-2a(a^2+x^2)}{(a^2+x^2)^2} \\
&= \frac{-2a^2+2x^2-2a^3-2ax^2}{(a^2+x^2)^2} \\
&= -\frac{2(a-1)x^2+2(a+1)a^2}{(a^2+x^2)^2},
\end{aligned}
$$

which is nonpositive when $a \geq 1$. $\square$

In summary, we have a random variate generation problem on our hands where the target density f can be bounded as follows:

$$\alpha g(x) \leq f(x) \leq \beta g(x),$$

where g is proportional to a log-concave density,

$$\alpha = (1-3/(a\pi^2))^2(1-3/(b\pi^2))^2, \tag{14}$$

and

$$\beta = (1+3/(a\pi^2))^2(1+3/(b\pi^2))^2. \tag{15}$$

We do not know the mode and mean of that density proportional to g, but are given the mean and variance of the target density f. This situation leads to the algorithms developed below. The betaized Meixner-Morris distribution is but a special case. We have the following inequality:

Lemma 2. *The following inequality holds when* $\min(a,b) \geq 1$*:*

$$f(x) \leq \frac{A}{\sigma} \min\left(1, e^{B-\frac{C|x-\mu|}{\sigma}}\right), x \in \mathbb{R},$$

where μ *and* σ^2 *are the mean and variance of* f *(see (13)),*

$$A = \beta^2,$$

$$B = 1 + \frac{1+\sqrt{3(1+1/\alpha^2)}}{\sqrt{12+12/\alpha^2}} = \frac{3}{2} + \frac{1}{\sqrt{12+12/\alpha^2}},$$

$$C = \frac{\alpha}{\sqrt{12+12/\alpha^2}},$$

and α *and* β *are as in (14) and (15).*

PROOF. Let us introduce the notation c_g, m_g, M_g, σ_g^2 and μ_g for the integral of g, mode of g, the value of the density g/c_g at that mode, the variance of g/c_g and the mean of g/c_g. Without subscripts, these quantities would be $1, m, M, \sigma^2$ and μ for the target density f. We begin with the inequality

$$f(x) \le \beta g(x) = \beta c_g \frac{g(x)}{c_g} \le \beta c_g\, M_g \min\left(1, e^{1-c_g M_g |x-m_g|}\right), x \in \mathbb{R}.$$

We will upper bound $c_g M_g$ and $|m_g - \mu_g|$ and lower bound $c_g M_g$ to prove the Lemma. First of all, the Johnson-Rogers inequality (1951) for unimodal densities implies that

$$|m_g - \mu_g| \le \sqrt{3}\sigma_g;$$

see Dharmadhikari and Joag-Dev (1988) and Bottomley (2004). By taking integrals, we also have

$$\alpha \le c_g \le \beta.$$

To relate μ and μ_g, we note that

$$|\mu - \mu_g| = \left| \int (x-\mu) \frac{g}{c_g} \right| \le \sqrt{\int (x-\mu)^2 \frac{g}{c_g}} \le \sqrt{\int (x-\mu)^2 \frac{f}{\alpha c_g}} \le \sqrt{\int (x-\mu)^2 \frac{f}{\alpha^2}} = \frac{\sigma}{\alpha}.$$

Furthermore,

$$\begin{aligned}
\sigma_g^2 &= \int (x-\mu_g)^2 \frac{g}{c_g} \\
&\le \int (x-\mu_g)^2 \frac{f}{c_g \alpha} \\
&\le \int (x-\mu_g)^2 \frac{f}{\alpha^2} \\
&= \int (x-\mu)^2 \frac{f}{\alpha^2} + \frac{(\mu-\mu_g)^2}{\alpha^2} \\
&= \frac{\sigma^2 + (\mu-\mu_g)^2}{\alpha^2} \\
&\le \frac{\sigma^2(1+1/\alpha^2)}{\alpha^2}.
\end{aligned}$$

Combining this and using the triangle inequality, we see that

$$|m_g - \mu| \le \frac{\sqrt{3}\sigma}{\alpha}\sqrt{1+1/\alpha^2} + \frac{\sigma}{\alpha} = \frac{\sigma}{\alpha}\left(1 + \sqrt{3(1+1/\alpha^2)}\right).$$

Furthermore,

$$\sigma_g^2 = \int (x-\mu_g)^2 \frac{g}{c_g} \ge \int (x-\mu_g)^2 \frac{f}{c_g \beta} \ge \int (x-\mu)^2 \frac{f}{\beta^2} = \frac{\sigma^2}{\beta^2}.$$

By Bobkov and Ledoux (2019, proposition B2), we have for any log-concave density h with mode m and variance σ^2,

$$\frac{1}{\sqrt{12}} \le \sigma h(m) \le 1.$$

The left-hand side of this inequality is valid for all densities, not just the log-concave densities (see Statulevicius (1965) and Hensley (1980)). The right-hand side of the inequality is due to Fradelizi, Guédon and Pajor (2014; see also Bobkov and Chistyakov, 2015). Thus,

$$\frac{\alpha}{\sigma\sqrt{12+12/\alpha^2}} \le \frac{1}{\sqrt{12}\,\sigma_g} \le c_g M_g \le \frac{1}{\sigma_g} \le \frac{\beta}{\sigma}.$$

Therefore,

$$\begin{aligned}
f(x) &\le \beta c_g\, M_g \min\left(1, e^{1-c_g M_g|x-m_g|}\right) \\
&\le \frac{\beta^2}{\sigma}\min\left(1, e^{1+|m_g-\mu|\frac{\alpha}{\sigma\sqrt{12+12/\alpha^2}}-\frac{\alpha}{\sigma\sqrt{12+12/\alpha^2}}|x-\mu|}\right) \\
&\le \frac{\beta^2}{\sigma}\min\left(1, e^{1+\frac{1+\sqrt{3(1+1/\alpha^2)}}{\sqrt{12+12/\alpha^2}}-\frac{\alpha}{\sigma\sqrt{12+12/\alpha^2}}|x-\mu|}\right).
\end{aligned}$$

The proof follows by inspection of all the coefficients. $\square$

The bound on f consists of a constant piece of value A/σ centered at μ of width $2B\sigma/C$, and two exponential tails, to the right of $\mu+B\sigma/C$ and to the left of $\mu-B\sigma/C$. The total area under the constant piece is $2AB/C$ and under both exponential tails, $2A/C$. Therefore, a rejection method based on Lemma 2, has an expected number of iterations equal to

$$2(1+B)\frac{A}{C},$$

which only depends upon α and β, and thus remains bounded uniformly over all parameter choices $a, b \ge 1$. As $a, b \to \infty$, we see that $\alpha, \beta \to 1$, $A \to 1$, $B \to 1+(1+\sqrt{6})/\sqrt{24}$, $C \to 1/\sqrt{24}$, so that the expected number of iterations tends to

$$2(2\sqrt{24}+1+\sqrt{6}) = 26.49\ldots.$$

Recall that these are uniform bounds of a one-size-fits-all generic algorithm. We now have all the ingredients for the first algorithm for the betaized Meixner-Morris distribution.

Algorithm 4 Betaized Meixner-Morris generator, parameters $a, b \geq 1$

$\alpha \leftarrow (1 - 3/(a\pi^2))^2(1 - 3/(b\pi^2))^2$
$\beta \leftarrow (1 + 3/(a\pi^2))^2(1 + 3/(b\pi^2))^2$
$A \leftarrow \beta^2$
$B \leftarrow \frac{3}{2} + \frac{1}{\sqrt{12+12/\alpha^2}}$
$C \leftarrow \frac{\alpha}{\sqrt{12+12/\alpha^2}}$
$\mu \leftarrow \frac{a}{a+b} s$
$\sigma \leftarrow \sqrt{\frac{ab}{(a+b)^2} \frac{s^2+(a+b)^2}{1+a+b}}$
repeat
  let V be uniform on $[0, 1]$
  if $V \leq B/(1+B)$ **then**
    $X \leftarrow \mu + BV'\sigma/C$ where V' is uniform on $[-1, 1]$
  else
    $X \leftarrow \mu + S(B+E)\sigma/C$ where E is exponential,
      and S is a random sign
  end if
  let U be uniform on $[0, 1]$
until $U(A/\sigma)\min(1, \exp(B - C|X-\mu|/\sigma)) \leq f(X)$
return X ▷ X has density f

We can tighten the inequality of Lemma 2 by a more delicate argument, resulting in the following Lemma.

Lemma 3. *Let μ and σ^2 be the mean and variance of f (see (13)). Let α and β be as in (14) and (15). Define*

$$\eta = \frac{\sigma}{\alpha}\left(1 + \sqrt{3(1+1/\alpha^2)}\right),$$

$$\tau = \frac{\alpha}{\sigma\sqrt{12+12/\alpha^2}},$$

and

$$\tau' = \frac{\beta}{\sigma}.$$

The following inequality holds when $\min(a, b) \geq 1$*:*

$$f(x) \leq \begin{cases} \beta\tau' & \text{if } |x-\mu| \leq \eta + \frac{1}{\tau'} \\ \frac{\beta}{|x-\mu|-\eta} & \text{if } \eta + \frac{1}{\tau'} \leq |x-\mu| \leq \eta + \frac{1}{\tau} \\ \beta\tau\min\left(1, e^{1+\eta\tau-\tau|x-\mu|}\right) & \text{if } |x-\mu| \geq \eta + \frac{1}{\tau}. \end{cases}$$

PROOF. We keep the notation of the proof of Lemma 2. We begin with inequality

$$f(x) \leq \beta g(x) = \beta c_g \frac{g(x)}{c_g} \leq \beta c_g M_g \min\left(1, e^{1-c_g M_g|x-m_g|}\right), x \in \mathbb{R}.$$

Recalling

$$|m_g - \mu| \le \frac{\sigma}{\alpha}\left(1 + \sqrt{3(1+1/\alpha^2)}\right),$$

and

$$\tau \stackrel{\text{def}}{=} \frac{\alpha}{\sqrt{12+12/\alpha^2}\sigma} \le c_g M_g \le \frac{\beta}{\sigma} \stackrel{\text{def}}{=} \tau',$$

we obtain

$$f(x) \le \beta c_g\, M_g \min\left(1, e^{1+\eta c_g M_g - c_g M_g |x-\mu|}\right) \le \max_{\tau \le t \le \tau'} \beta t \min\left(1, e^{1+\eta t - t|x-\mu|}\right).$$

For $|x-\mu| \le \eta$, the best we can hope for is the bound $f(x) \le \beta\tau'$. Assume next that $|x-\mu| > \eta$. Then define the following threshold $t^* = 1/(|x-\mu|-\eta)$, which makes the exponent in the upper bound zero. If $\tau' \le t^*$, then $f(x) \le \beta\tau'$. If $\tau \ge t^*$, then

$$f(x) \le \beta\tau \min\left(1, e^{1+\eta\tau-\tau|x-\mu|}\right).$$

Finally, if $\tau \le t^* \le \tau'$, we bound by

$$f(x) \le \beta t^* = \frac{\beta}{|x-\mu|-\eta}.$$

Combining this yields the bound

$$f(x) \le \begin{cases} \beta\tau' & \text{if } \tau' \le t^* = \frac{1}{|x-\mu|-\eta} \\ \beta t^* = \frac{\beta}{|x-\mu|-\eta} & \text{if } \tau \le t^* \le \tau' \\ \beta\tau \min\left(1, e^{1+\eta\tau-\tau|x-\mu|}\right) & \text{if } \tau \ge \frac{1}{|x-\mu|-\eta}. \end{cases}$$

This translates into the bound given in Lemma 3. □

We can develop a slightly more complex rejection algorithm as we have a tripartite dominating function. The only novelty is the central portion of the upper bound. In this respect, we observe that a random variate with density proportional to $1/x$ on an interval $[x', x'']$ can be generated as $1/(x''^{1-U} x'^U)$, where U is uniform on $[0,1]$. We begin with the calculation of the area under the dominating curve, which is, for the central part,

$$2\beta\tau'\left(\eta + \frac{1}{\tau'}\right),$$

for the middle portion,

$$2\beta \log\left(\frac{\tau'}{\tau}\right),$$

and for the exponential tails,

$$2\beta.$$

Again, the sum is a continuous function of α and β, and thus uniformly bounded away from infinity over all choices of $a, b \ge 1$ and $s \in \mathbb{R}$. For comparison with the bound of Lemma 2, we compute the limiting value of the area as $a, b \to \infty$ as

$$2(3 + \sqrt{6} + (1/2)\log(24)) = 14.07\ldots.$$

Thus, Lemma 3 increases the efficiency by almost 60% for large values of the parameters. The algorithm is given below.

Algorithm 5 Betaized Meixner-Morris generator, parameters $a, b \geq 1$

$\mu \leftarrow \frac{a}{a+b} s$
$\sigma \leftarrow \sqrt{\frac{ab}{(a+b)^2} \frac{s^2+(a+b)^2}{1+a+b}}$
$\alpha \leftarrow (1 - 3/(a\pi^2))^2 (1 - 3/(b\pi^2))^2$
$\beta \leftarrow (1 + 3/(a\pi^2))^2 (1 + 3/(b\pi^2))^2$
$\eta \leftarrow \frac{\sigma}{\alpha} \left(1 + \sqrt{3(1 + 1/\alpha^2)}\right)$
$\tau \leftarrow \frac{\alpha}{\sigma \sqrt{12 + 12/\alpha^2}}$
$\tau' \leftarrow \frac{\beta}{\sigma}$
$q_1 \leftarrow \beta(1 + \tau'\eta)$
$q_2 \leftarrow \beta \log(\tau'/\tau)$
$q_3 \leftarrow \beta$
$q = q_1 + q_2 + q_3$
repeat
    let U, V be uniform on $[0, 1]$
    if $V \leq q_1/q$ **then**
        $X \leftarrow \mu + V'(\eta + 1/\tau')$ where V' is uniform on $[-1, 1]$
        Accept $\leftarrow [U\beta\tau' \leq f(X)]$
    else if $V \leq q_2/q$ **then**
        $Y \leftarrow 1/({\tau'}^{1-V'} \tau^{V'})$ where V' is uniform on $[0, 1]$
        $X \leftarrow \mu + S(\eta + Y)$ where S is a random sign
        Accept $\leftarrow [U\beta/Y \leq f(X)]$
    else
        $X \leftarrow \mu + S(\eta + (1 + E)/\tau)$ where E is exponential
            and S is a random sign
        Accept $\leftarrow [U\beta\tau e^{-E} \leq f(X)]$
    end if
until Accept
return X $\triangleright$ X has density f

12. Conclusion.

We presented uniformly fast algorithms for the Pearson IV distribution over the entire range of parameters. We also introduced the betaized Meixner-Morris distribution and developed uniformly fast algorithms for that three-parameter family, provided that the first two parameters (a and b) have values at least equal to one.

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