Transversals in Trees

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Abstract: A transversal in a rooted tree is any set of nodes that meets every path from the root to a leaf. We let $c(T,k)$ denote the number of transversals of size k in a rooted tree T. We define a partial order on the set of all rooted trees with n nodes by saying that a tree T succeeds a tree T′ if $c(T,k)$ is at least $c(T',k)$ for all k and strictly greater than $c(T',k)$ for at least one k. We prove that, for every choice of positive integers d and n, the set of all rooted trees on n nodes where each node has at most d children has a unique minimal element with respect to this partial order and we describe this tree. In addition, we determine asymptotically the expected values of $c(T,k)$ in special families of trees. J. Graph Theory 73: 32–43, 2013

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1. FARLEY'S PROBLEM AND ITS SOLUTION

A *transversal* in a rooted tree is any set of nodes that meets every path from the root to a leaf. We let $c(T, k)$ denote the number of transversals of size k in a rooted tree T. If T has n nodes and $n \geq 2$, then

$$c(T, n) = 1,$$
$$c(T, n-1) = n,$$

and

$$\binom{n-1}{k-1} \leq c(T, k) \leq \binom{n}{k} \quad \text{for all } k = 1, 2, \ldots, n-2. \tag{1}$$

The $n - 2$ upper bounds in (1) are attained simultaneously if and only if T is the path (the tree with precisely one leaf); the $n - 2$ lower bounds in (1) are attained simultaneously if and only if T is the star (the tree where all leaves are children of the root). Jonathan David Farley asked how high can these lower bounds be raised if each node of T has at most two children; he offered a creative interpretation of this question in [8, 9]. In this section, we give an answer.

Harary and Schwenk [10] call a *caterpillar* a tree (an unrooted one) where removal of all nodes of degree one produces a path. Abusing this usage a little, we will call a caterpillar any rooted tree where removal of all leaves produces a rooted tree with precisely one leaf. Other than that, our terminology concerning rooted trees is the standard terminology presented in Appendices B.5.2 and B.5.3 of [4]. In particular, the *depth* of a node in a rooted tree means the length of the path from the root to this node (the root has depth 0, its children have depth 1, and so on); the *height* of the tree is the largest depth of its nodes. In a *positional tree*, the children of each node are labeled with distinct positive integers; a *d-ary tree* is a positional tree in which no children are labeled with integers greater than d; a *complete d-ary tree* is a d-ary tree where all leaves have the same depth and each node is a leaf or has exactly d children. A *binary tree* is a d-ary tree with $d = 2$ and a *full binary tree* is a binary tree where each node is a leaf or has exactly two children; by analogy, we say that a *full d-ary tree* is a d-ary tree where each node is a leaf or has exactly d children.

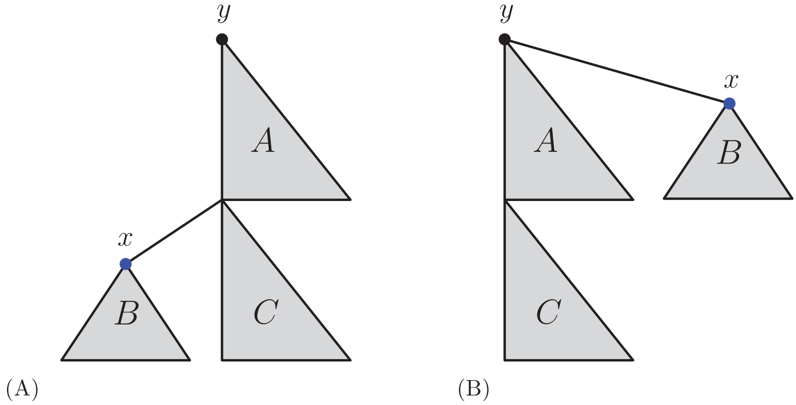

FIGURE 1. From a subtree of T to a subtree of T' in Lemma 1. (A) y is a proper ancestor of $p(x)$ in T. (B) y becomes the parent of x in T'.

We are interested in the values of $c(T, k)$ for d-ary trees T and positive integers k. Since these values are invariant under changes of labels on the children of each node of T, the labels are irrelevant to our interest. Accordingly, we will think of d-ary trees as ordinary, not positional, rooted trees where each node has at most d children.

Let $T_d^{\min}(n)$ denote the d-ary caterpillar where each internal node, except possibly the lowest one, has precisely d children. Farley proved that every binary tree on n nodes has $c(T, k) \geq c(T_2^{\min}(n), k)$ for all k ranging from 1 to essentially $1 + \lfloor \log_2 n \rfloor$. Using a different argument, we will extend this bound to all values of k and to all d-ary trees with $d \geq 2$.

Theorem 1. *Let n and d be positive integers such that $d < n$. If T is a d-ary tree on n nodes, then $c(T, k) \geq c(T_d^{\min}(n), k)$ for all $k = 1, 2, \ldots, n$.*

Given rooted trees T, T' on n nodes, we will write $T \succ T'$ to mean that $c(T, k) \geq c(T', k)$ for all $k = 1, 2, \ldots, n$ and that $c(T, k) > c(T', k)$ for at least one of these values of k. With this notation, a refinement of Theorem 1 can be stated as follows.

Theorem 2. *Let n and d be positive integers such that $d < n$. If T is a d-ary tree on n nodes, then $T \succ T_d^{\min}(n)$ or else $T = T_d^{\min}(n)$.*

Our proof of Theorem 2 relies on two ways of altering a rooted tree T so that the resulting tree succeeds T in the partial order $\succ$. We shall describe these alterations in terms of the *parent function* of a rooted tree that assigns to each node z of the tree its parent $p(z)$—except when z is the root, in which case $p(z)$ is undefined.

Lemma 1. *Let T be a rooted tree defined by parent function p. Let x and y be nodes of T such that y is a proper ancestor of $p(x)$. Let T' be the rooted tree defined by parent function p' such that*

$$p'(z) = \begin{cases} p(z) & \text{if } z \neq x, \\ y & \text{if } z = x. \end{cases}$$

Then $T \succ T'$.

This operation is illustrated in Figure 1.

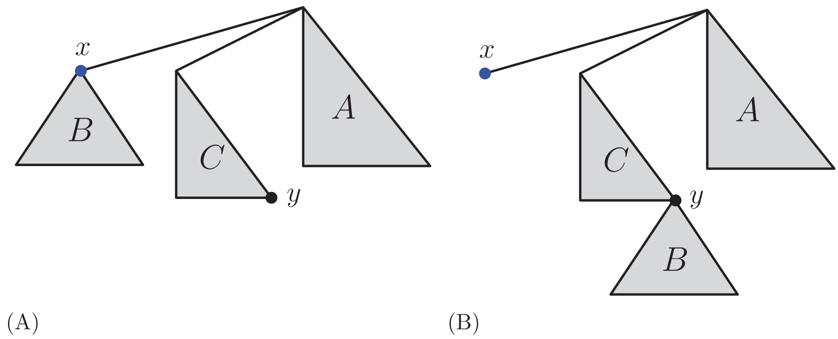

FIGURE 2. From a subtree of T to a subtree of T' in Lemma 2. (A) y is a proper descendant of a sibling of x in T. (B) Children of x in T become children of y in T'.

Proof. If z is a leaf of T, then z is a leaf of T' and every node on the path from the root to z in T' lies on the path from the root to z in T. It follows that every transversal in T' is a transversal in T, and so $c(T, k) \geq c(T', k)$ for all k. To see that $c(T, k) > c(T', k)$ for at least one k, consider the set that consists of $p(x)$ and all leaves of T that are not descendants of $p(x)$: this set is a transversal in T but not in T'. ■

Lemma 2. *Let T be a rooted tree defined by parent function p. Let x and y be nodes of T such that x is not a leaf and y is a leaf that is a proper descendant of a sibling of x. Let T' be the rooted tree defined by parent function p' such that*

$$p'(z) = \begin{cases} p(z) & \textit{if } p(z) \neq x, \\ y & \textit{if } p(z) = x. \end{cases}$$

Then $T \succ T'$.

This operation is illustrated in in Figure 2.

Proof. Given any set S' of nodes in T', define

$$f(S') = \begin{cases} S' & \text{if } S' \text{ meets the path from the root to } y, \\ S' - \{x\} \cup \{y\} & \text{otherwise.} \end{cases}$$

By this definition, $f(S')$ always meets the path from the root to y; this path is the same in T' and T. Every leaf of T distinct from y is a leaf of T'; unless this leaf is a descendant of x, the path from the root to it is again the same in the two trees and, in particular, x does not lie on this path; since $f(S') \supset S' - \{x\}$, it follows that

(i) if S' is a transversal in T', then $f(S')$ meets every path in T from the root to a leaf that is not a descendant of x.

To see that

(ii) if S' is a transversal in T', then S' meets every path in T from the root to a leaf that is a descendant of x,

note that S' meets the path from the root to x and that this path is the same in T' and T. Next, we claim that

(iii) if S' is a transversal in T' such that $f(S') \neq S'$, then $f(S')$ meets every path in T from a child of x to a leaf.

To justify (iii), note that S', being a transversal in T', meets every path in T' from the root to a leaf and that, since $f(S') \neq S'$, it does not meet the path from the root to y; it follows that S' meets every path in T' from a child of y to a leaf. Since every path in T from a child of x to a leaf is a path in T' from a child of y to a leaf, we conclude that S' meets every such path; now (iii) follows as $f(S') \supset S' - \{x\}$. In addition, we note that

(iv) if S' is a transversal in T' such that $f(S') \neq S'$, then $x \in S'$, $y \notin S'$

(since S' does not meet the path from the root to y, it does not include y and it does not meet the path from the root to the parent of x; but then, since it meets the path from the root to x, it must include x) and that

(v) if S' is a transversal in T' such that $f(S') \neq S'$, then $f(S')$ is not a transversal in T'

(because it does not meet the path from the root to x, which is a leaf of T').

From (i), (ii), (iii), (iv), we see that f maps every transversal in T' to a transversal of the same size in T; from (iv) and (v), we see that it maps distinct transversals in T' to distinct transversals in T. It follows that $c(T,k) \geq c(T',k)$ for all k. To see that $c(T,k) > c(T',k)$ for at least one k, consider the set that consists of $p(y)$ and all leaves of T that are not descendants of $p(y)$: this set S is a transversal in T, but there is no transversal S' in T' such that $f(S') = S$. ■

Proof of Theorem 2. Consider any d-ary tree T on n nodes. Assuming that there is no d-ary tree T' on n nodes such that $T \succ T'$, we shall prove that $T = T_d^{\min}(n)$. Lemma 2 guarantees that no two internal nodes of T are siblings, which means that T is a caterpillar; in turn, Lemma 1 guarantees that each internal node of T, except possibly the lowest one, has precisely d children. ■

Let us point out that the lower bound in Theorem 1 is easy to calculate: with $s = \lfloor (n-2)/d \rfloor$, we have

$$c(T_d^{\min}(n), k) = \begin{cases} \sum_{i=0}^{s} \binom{n-1-id}{k-1-i(d-1)} + 1 & \text{if } k = n-s-1, \\ \sum_{i=0}^{s} \binom{n-1-id}{k-1-i(d-1)} & \text{otherwise.} \end{cases}$$

To see this, let $r_0, r_1, \ldots, r_s$ denote the path produced when all leaves are deleted from $T_d^{\min}(n)$: r_0 is the root, each r_i with $0 \leq i < s$ has $d-1$ leaves and r_{i+1} as children, and r_s has between 1 and d leaves as children. Note that a transversal in $T_d^{\min}(n)$ includes no r_i at all if and only if it consists of all the leaves of $T_d^{\min}(n)$. Given a transversal S in $T_d^{\min}(n)$ that includes at least one r_i, consider the smallest subscript i such that $r_i \in S$; note that S must contain the $i(d-1)$ leaves that are children of $r_0, r_1, \ldots, r_{i-1}$ and that its remaining elements consist of r_i and an arbitrary set of its descendants.

2. VARIATIONS

Imposing an upper bound on the number of children of every node is a way of staying clear of the tree that attains simultaneously the $n-2$ lower bounds in (1), one where all children of the root are leaves. Another way to stay clear of this tree is to impose an upper bound on the number of leaves. There is a corresponding analogue of Theorem 2 and this analogue also follows directly from our two lemmas.

Theorem 3. *Let n and m be positive integers such that $m < n$ and let T_0 be the caterpillar with n nodes, where the root has m children and every node other than the root has at most one child. If T is a rooted tree with n nodes and at most m leaves, then $T \succ T_0$ or else $T = T_0$.*

Proof. Consider any rooted tree T with n nodes and at most m leaves. Assuming that there is no rooted tree T' with n nodes and at most m leaves such that $T \succ T'$, we shall prove that $T = T_0$. Lemma 2 guarantees that no two internal nodes of T are siblings, which means that T is a caterpillar; in turn, Lemma 1 guarantees that no internal node of T other than the root has two or more children. ■

The number of transversals of size k in the caterpillar T_0 featured in Theorem 3 is also easy to calculate.

$$c(T_0, k) = \begin{cases} \dbinom{n-1}{k-1} & \text{if } k < m, \\ \dbinom{n-1}{k-1} + \dbinom{n-m}{k-m+1} & \text{if } k \geq m. \end{cases}$$

To see this, note that precisely $\binom{n-1}{k-1}$ transversals of size k include the root and that every transversal that does not include the root consists of $m-1$ leaves that are children of the root and at least one node on the path from the root to the m-th leaf.

As T ranges through all d-ary trees on n nodes, the number $c(T, k)$ of transversals of size k attains its minimum at $T_d^{\min}(n)$. How close is this minimum to values of $c(T, k)$ that are typical for d-ary trees T on n nodes? This is the question that we tackle in the remainder of this article.

We will study $c(T, k)$ indirectly by focusing on the probability $\xi(T, p)$ of obtaining a transversal when we choose each node of T independently with probability p. These two quantities are related by the identity

$$\xi(T, p) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} \cdot \frac{c(T, i)}{\dbinom{n}{i}}. \tag{2}$$

Since the sequence of the probabilities $\binom{n}{i} p^i (1-p)^{n-i}$ has a sharp peak when i is around pn, identity (2) suggests that $\xi(T, k/n)$ is a good approximation of $c(T, k)/\binom{n}{k}$, which is the probability that a randomly chosen set of k of the n nodes of T is a transversal.

To elaborate on the last statement, we will bound $c(T, k)/\binom{n}{k}$ in terms of $\xi(T, p)$. For this purpose, note first that

$$c(T, i) \bigg/ \binom{n}{i} \leq c(T, i+1) \bigg/ \binom{n}{i+1} \quad \text{for all } i = 0, 1, \ldots, n-1. \tag{3}$$

This can be proved by counting in two different ways the number N of pairs (A, B) such that A is a transversal of size i, B is a transversal of size $i+1$, and $A \subset B$: choosing A first, we get $N = c(T, i) \cdot (n-i)$ and choosing B first, we get $N \leq c(T, i+1) \cdot (i+1)$.

Identity (2) and relation (3) combined imply

$$\xi(T, p) \leq \frac{c(T,k)}{\binom{n}{k}} + \sum_{i=k+1}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

and

$$\xi(T, p) \geq \frac{c(T,k)}{\binom{n}{k}} \cdot \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}.$$

for all k ranging from 0 to n. Using the well-known inequality [1–3, 5, 11]

$$\sum_{i \geq (p+t)n} \binom{n}{i} p^i (1-p)^{n-i} \leq \exp\left(-2t^2 n\right)$$

and its companion

$$\sum_{i \leq (p-t)n} \binom{n}{i} p^i (1-p)^{n-i} \leq \exp\left(-2t^2 n\right),$$

we conclude that

$$\xi(T, k/n - t) - \exp\left(-2t^2 n\right) \leq \frac{c(T,k)}{\binom{n}{k}} \leq \frac{\xi(T, k/n + t)}{1 - \exp(-2t^2 n)}. \tag{4}$$

for all k ranging from 0 to n and for all values of t that make $k/n - t$ nonnegative and $k/n + t$ at most 1. We will use (4) in the proofs of Corollary 1 and Corollary 2 in the next section.

We have already seen that the star minimizes all $c(T, k)$ and that the path maximizes all $c(T, k)$ with T ranging over all trees on n nodes. Similarly, we have

$$p + (1-p)p^{n-1} \leq \xi(T, p) \leq 1 - (1-p)^n \quad \text{whenever } n \geq 2$$

for all trees T on n nodes, with the lower bound attained by the star and the upper bound attained by the path (the lower bound follows from observing that one transversal consists of the root alone and another transversal consists of all the leaves; the upper bound follows from observing that the empty set is not a transversal). We are going to tighten these bounds under the assumption that, for a prescribed integer d greater than one, T is a full d-ary tree.

Theorem 4. *For every integer d such that $d \geq 2$, every real number p such that $0 \leq p \leq 1$, and every full d-ary tree T on n nodes, we have*

$$\xi\left(T_d^{\min}(n), p\right) \leq \xi(T, p) \leq \xi\left(C_d^n, p\right),$$

where C_d^n is the complete d-ary tree of height n.

Proof. The lower bound on $\xi(T, p)$ follows from our Theorem 1; the upper bound follows from the observation that every transversal in T is a transversal in C_d^n. ■

It may be worth emphasizing that the lower bound of Theorem 4 compares trees on n nodes to trees on n nodes, but the upper bound compares trees on n nodes to much larger trees.

3. ASYMPTOTICS

Given an integer d such that $d \geq 2$ and a real number p such that $0 \leq p \leq 1$, let us write

$$\ell_d(p) = \frac{p}{1-(1-p)p^{d-1}}$$

and let $u_d(p)$ denote the smallest nonnegative root of the polynomial

$$p - x + (1-p)x^d.$$

Theorem 5. *For every integer d such that $d \geq 2$ and every real number p such that $0 \leq p \leq 1$, we have*

$$\lim_{n\to\infty} \xi\big(T_d^{\min}(n), p\big) = \ell_d(p),$$
$$\lim_{n\to\infty} \xi\big(C_d^n, p\big) = u_d(p).$$

Proof. Since

$$\xi\big(T_d^{\min}(n), p\big) = p + (1-p)p^{n-1} \qquad \text{whenever } 2 \leq n \leq d+1$$

and

$$\xi\big(T_d^{\min}(n), p\big) = p + (1-p)p^{d-1}\xi\big(T_d^{\min}(n)_{n-d}, p\big) \qquad \text{whenever } n \geq d+2,$$

induction on n shows that, for all n greater than 1,

$$\xi\big(T_d^{\min}(n), p\big) = p\,\frac{1-((1-p)p^{d-1})^t}{1-(1-p)p^{d-1}} + p^{n-t}(1-p)^t \;\text{ with } t = \lfloor (n-2)/d \rfloor + 1.$$

We have $\xi(C_d^0, p) = p$ and

$$\xi\big(C_d^n, p\big) = p + (1-p)\xi(C_d^{n-1}, p)^d \qquad \text{whenever } n \geq 1.$$

Since $\xi(C_d^n, 0) = 0 = u_d(0)$ and $\xi(C_d^n, 1) = 1 = u_d(1)$ for all n, we may assume that $0 < p < 1$. Now, let us write $f(x) = p + (1-p)x^d$, so that $\xi(C_d^n, p) = f^n(p)$ for all n. By definition of $u_d(p)$ and since $f(0) > 0$, we have

$$0 < x < u_d(p) \Rightarrow x < f(x);$$

since the polynomial $f(x) - x$ is convex in the interval $[0, 1]$ and since $u_d(p)$ and 1 (possibly identical) are among its roots, we have

$$u_d(p) < x < 1 \Rightarrow f(x) < x;$$

since f is increasing in the interval [0, 1], we have

$$0 < x < u_d(p) \Rightarrow f(x) < f(u_d(p)) = u_d(p),$$
$$u_d(p) < x < 1 \Rightarrow u_d(p) = f(u_d(p)) < f(x).$$

It follows that the sequence $\xi(C_d^0, p), \xi(C_d^1, p), \xi(C_d^2, p), \ldots$ is monotone (increasing if $p < u_d(p)$ and decreasing if $u_d(p) < p$), and so it tends to a limit in (0, 1); this limit must be a fixed point of f, and so it equals $u_d(p)$. ■

We have noted that

$$c\big(T_d^{\min}(n), k\big) = \sum_{i=0}^{s} \binom{n-1-id}{k-1-i(d-1)} + \delta(k, n-s-1)$$

with $s = \lfloor (n-2)/d \rfloor$ and δ the Kronecker delta. A part of Theorem 5 yields a cleaner asymptotic formula:

Corollary 1. *Let $k : \mathbf{N} \to \mathbf{N}$ be a function such that $0 \le k(n) \le n$ for all n. If $\lim_{n\to\infty} k(n)/n = \lambda$ for some λ, then*

$$\lim_{n\to\infty} c\big(T_d^{\min}(n), k(n)\big) \Big/ \binom{n}{k(n)} = \ell_d(\lambda).$$

Proof. Set $T = T_d^{\min}(n)$ and (for instance) $t = n^{-1/3}$ in (4); then appeal to Theorem 5 for the value of $\lim_{n\to\infty} \xi(T_d^{\min}(n), k(n)/n)$. ■

A referee of earlier versions of this article asked for families of trees where the expected values of $c(T, k)$ could be determined, asymptotically or otherwise. The following result provides such families. These families consist exclusively of full d-ary trees; the number of nodes of every such tree equals $dm + 1$ for some nonnegative integer m.

Corollary 2. *Let $\mathcal{F}_d^{\omega}(n)$ denote the set of all full d-ary trees on n nodes that have height at least ω; let $\omega : \mathbf{N} \to \mathbf{N}$ and $k : \mathbf{N} \to \mathbf{N}$ be functions such that $\mathcal{F}_d^{\omega(dm+1)}(dm+1) \neq \emptyset$ and $0 \le k(dm+1) \le dm+1$ for all m. If $\lim_{n\to\infty} \omega(n) = +\infty$ and $\lim_{n\to\infty} k(n)/n = \lambda$ for some λ, then*

$$\begin{aligned}
&\lim_{m\to\infty} \min\{c(T, k(dm+1)) \Big/ \binom{dm+1}{k(dm+1)} : T \in \mathcal{F}_d^{\omega(dm+1)}(dm+1)\} \\
&= \lim_{m\to\infty} \max\{c(T, k(dm+1)) \Big/ \binom{dm+1}{k(dm+1)} : T \in \mathcal{F}_d^{\omega(dm+1)}(dm+1)\} \\
&= u_d(\lambda).
\end{aligned}$$

Proof. If $T \in \mathcal{F}_d^{\omega(n)}(n)$, then every transversal in $C_d^{\omega(n)}$ is a transversal in T and every transversal in T is a transversal in C_d^n. It follows that, for every real number p such that $0 \le p \le 1$,

$$\xi\big(C_d^{\omega(n)}, p\big) \le \xi(T, p) \le \xi\big(C_d^n, p\big);$$

by Theorem 5, both the lower and the upper bound tend to $u_d(p)$ as $n \to \infty$; we conclude that

$$\begin{aligned}&\lim_{m\to\infty} \min\{\xi(T,p) : T \in \mathcal{F}_d^{\omega(dm+1)}(dm+1)\}\\ &\quad= \lim_{m\to\infty} \max\{\xi(T,p) : T \in \mathcal{F}_d^{\omega(dm+1)}(dm+1)\}\\ &\quad= u_d(p).\end{aligned}$$

Then we set $n = dm + 1$ and (for instance) $t = n^{-1/3}$ in (4). ■

In a number of naturally arising classes of rooted trees, the minimum depth of a leaf grows logarithmically with the number of nodes. For instance, Devroye [6] proved that almost all random binary search trees have this property. These trees are not, in general, full binary trees; however, every binary tree on m nodes can be extended into a full binary tree on $2m + 1$ nodes by adding $m + 1$ leaves; in this sense, random binary search trees provide a class of full binary trees that can be taken for $\mathcal{F}_d^{\omega(2m+1)}(2m + 1)$ in Corollary 2 with $d = 2$. A similar comment, with varying values of d, applies to other classes of *split trees*, introduced by Devroye [7].

Theorem 1 and Corollary 1 combined assert that the minimum of the ratio $c(T, k(n))/\binom{n}{k(n)}$ with T ranging through all d-ary trees T on n nodes and with $\lim_{n\to\infty} k(n)/n = \lambda$ is asymptotic to $\ell_d(\lambda)$; Corollary 2 asserts that, when T is restricted to full d-ary trees of height at least $\omega(n)$ such that $\lim_{n\to\infty} \omega(n) = +\infty$, the ratio $c(T, k(n))/\binom{n}{k(n)}$ is asymptotic to $u_d(\lambda)$. How close are the two asymptotic values?

Since

$$u_2(\lambda) = \begin{cases} \dfrac{\lambda}{1-\lambda} & \text{if } 0 \le \lambda \le 1/2,\\[2mm] 1 & \text{if } 1/2 \le \lambda \le 1,\end{cases}$$

we have

$$u_2(\lambda)/\ell_2(\lambda) = \begin{cases} 1 + \dfrac{\lambda^2}{1-\lambda} & \text{if } \lambda \le 1/2,\\[4mm] 1 + \dfrac{(1-\lambda)^2}{\lambda} & \text{if } \lambda \ge 1/2.\end{cases}$$

in particular, the ratio $u_2(\lambda)/\ell_2(\lambda)$ increases in the interval $0 \le \lambda \le 1/2$ from 1 at $\lambda = 0$ to its maximum 3/2 at $\lambda = 1/2$ and then it decreases in the interval $1/2 \le \lambda \le 1$ until it reaches 1 again at $\lambda = 1$.

When $d \ge 3$, we have no explicit formula for $u_d(\lambda)$, but at least we can prove that

$$u_d(\lambda)/\ell_d(\lambda) < 1 + \frac{1}{d-1}\left(1 - \frac{1}{e}\right).$$

Actually, we have a finer result.

Theorem 6. *For every integer d such that $d \ge 2$, the ratio $u_d(\lambda)/\ell_d(\lambda)$ increases in the interval $[0, (d-1)/d]$ from its limit 1 at $\lambda = 0$ to its maximum*

$$1 + \frac{1}{d-1}\left(1 - \left(\frac{d-1}{d}\right)^{d-1}\right)$$

at $\lambda = (d-1)/d$ *and then it decreases in the interval* $[(d-1)/d, 1]$ *until it reaches* 1 *again at* $\lambda = 1$.

Proof. Write

$$\lambda^* = (d-1)/d$$

and consider the function $g : [\lambda^*, +\infty) \to \mathbf{R}$ defined by

$$g(x) = x^{d-1} - x^d.$$

Since $g'(x) = (d-1)x^{d-2} - dx^{d-1}$, this is a decreasing function; the inverse h of g is a decreasing function and it maps the interval $(-\infty, g(\lambda^*)]$ onto the interval $[\lambda^*, +\infty)$; for every y in the domain of h, the value of $h(y)$ is the largest real root of the polynomial $x^d - x^{d-1} + y$. Since a nonzero r is a root of the polynomial $x^d - x^{d-1} + (1-\lambda)\lambda^{d-1}$ if and only if λ/r is a root of the polynomial $\lambda - x + (1-\lambda)x^d$, it follows that

$$u_d(\lambda) = \frac{\lambda}{h((1-\lambda)\lambda^{d-1})},$$

and so

$$\frac{u_d(\lambda)}{\ell_d(\lambda)} = \frac{1-(1-\lambda)\lambda^{d-1}}{h((1-\lambda)\lambda^{d-1})}.$$

In particular, $\lim_{\lambda\to 0} u_d(\lambda)/\ell_d(\lambda) = 1/h(0) = 1$.

Since the function defined by $\lambda \mapsto (1-\lambda)\lambda^{d-1}$ increases in the interval $[0, \lambda^*]$ and maps it onto the interval $[0, g(\lambda^*)]$, proving that the ratio $u_d(\lambda)/\ell_d(\lambda)$ increases in the interval $[0, \lambda^*]$ reduces to proving that the ratio $(1-y)/h(y)$ increases in the interval $[0, g(\lambda^*)]$. For this purpose, note first that $g'(x)$ decreases in the interval $[\lambda^*, 1]$: more precisely, $g'(x)$ decreases in the interval $[(d-2)/d, +\infty)$ and we have $\lambda^* > (d-2)/d$. Since $h'(y) = 1/g'(h(y))$ and since $h(y)$ decreases in the interval $[0, g(\lambda^*)]$, we conclude that $h'(y)$ decreases in the interval $[0, g(\lambda^*)]$. In particular, as $h'(0) = 1/g'(1) = -1$, we have $h'(y) \le -1$ for all y in $[0, g(\lambda^*)]$; now it follows first that $h(y) \le 1-y$ for all y in $[0, g(\lambda^*)]$ and then that

$$\left(\frac{1-y}{h(y)}\right)' = \frac{-h(y)-(1-y)h'(y)}{h(y)^2} \ge \frac{-h(y)+(1-y)}{h(y)^2} \ge 0$$

for all y in $[0, g(\lambda^*)]$.

Finally, if $\lambda^* \le \lambda \le 1$, then λ belongs to the domain of g, and so

$$u_d(\lambda) = \frac{\lambda}{h((1-\lambda)\lambda^{d-1})} = \frac{\lambda}{h(g(\lambda))} = 1,$$

which implies

$$\frac{u_d(\lambda)}{\ell_d(\lambda)} = 1 + \frac{(1-\lambda)(1-\lambda^{d-1})}{\lambda}.$$

■

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