Electron. Commun. Probab. **30** (2025), article no. 14, 1–9.
https://doi.org/10.1214/25-ECP662
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Burning random trees

Luc Devroye* Austin Eide† Paweł Prałat‡

Abstract

Let $\mathcal{T}$ be a Galton—Watson tree with a given offspring distribution ξ, where ξ is a $\mathbb{Z}_{\geq 0}$-valued random variable with $\mathsf{E}[\xi] = 1$ and $0 < \sigma^2 := \mathsf{Var}[\xi] < \infty$. For $n \geq 1$, let T_n be the tree $\mathcal{T}$ conditioned to have n vertices. In this paper we investigate $b(T_n)$, the burning number of T_n. Our main result shows that asymptotically almost surely $b(T_n)$ is of the order of $n^{1/3}$.

Keywords: Galton—Watson trees; graph burning.
MSC2020 subject classifications: 60C05.
Submitted to ECP on April 2, 2024, final version accepted on January 31, 2025.

1 Introduction

Graph burning is a discrete-time process that models influence spreading in a network. Vertices are in one of two states: either *burning* or *unburned*. In each round, a burning vertex causes all of its neighbors to burn and a new *fire source* is chosen: a previously unburned vertex whose state is changed to burning. The updates repeat until all vertices are burning. The *burning number* of a graph G, denoted $b(G)$, is the minimum number of rounds required to burn all of the vertices of G.

Graph burning first appeared in print in a paper of Alon [2], motivated by a question of Brandenburg and Scott at Intel, and was formulated as a transmission problem involving a set of processors. It was then independently studied by Bonato, Janssen, and Roshanbin [5, 6] who, in addition to introducing the name *graph burning*, gave bounds and characterized the burning number for various graph classes. The problem has since received wide attention (e.g. [3, 4, 11, 14, 15, 16]), with particular focus the so-called Burning Number Conjecture that each connected graph on n vertices requires at most $\lceil\sqrt{n}\rceil$ turns to burn. This conjecture is best possible, as $b(P_n) = \lceil\sqrt{n}\rceil$ for a path on n vertices.

Clearly, $b(G) \leq b(T)$ for every spanning tree T of G. Hence, the Burning Number Conjecture can be stated as follows:

Conjecture 1.1. *For any tree T on n vertices, $b(T) \leq \lceil\sqrt{n}\rceil$.*

*School of Computer Science, McGill University, Montreal, Canada. E-mail: lucdevroye@gmail.com
†Department of Mathematics, Toronto Metropolitan University, Toronto, Canada. E-mail: austin.eide@torontomu.ca
‡Department of Mathematics, Toronto Metropolitan University, Toronto, Canada. E-mail: pralat@torontomu.ca

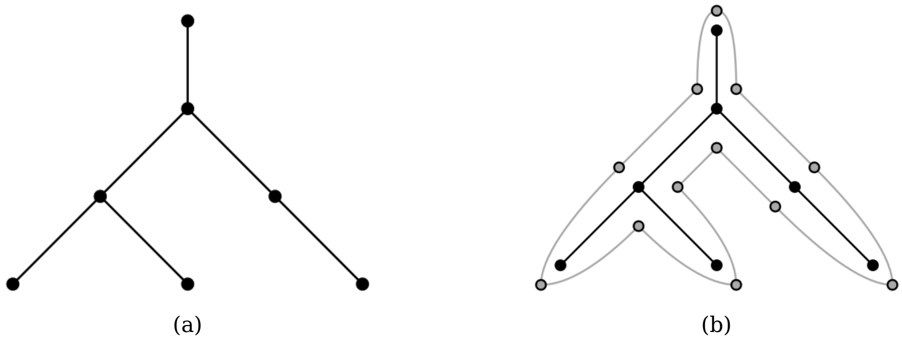

Figure 1: A tree T (a) and a cycle corresponding to a depth-first search of T (b). Any burning sequence for the cycle projects to a burning sequence for the tree.

Although the conjecture feels obvious, it has resisted attempts at its resolution. It is easy to couple the process on any tree T on n vertices ($n-1$ edges) with the process on $C_{2(n-1)}$, a cycle on twice as many edges. (See Figure 1.) This yields

$$b(T) \leq \left\lceil \sqrt{2(n-1)} \right\rceil = \sqrt{2n} + O(1) \qquad (\sqrt{2} \approx 1.41421).$$

In [4], Bessy, et. al. proved that

$$b(T) \leq \sqrt{\frac{12}{7}n} + 3 \qquad (\sqrt{12/7} \approx 1.30931).$$

This bound was consecutively improved by Land and Lu in [11] to

$$b(T) \leq \left\lceil \frac{\sqrt{24n+33}-3}{4} \right\rceil = \sqrt{\frac{3}{2}\,n} + O(1) \qquad (\sqrt{3/2} \approx 1.22475).$$

Currently, the best upper bound, due to Bastide, et. al. in [3], is

$$b(T) \leq \left\lceil \sqrt{\frac{4}{3}n} \right\rceil + 1 \qquad (\sqrt{4/3} \approx 1.15470).$$

Norin and Turcotte [16] proved arguably the strongest result in this direction:

$$b(T) \leq (1+o(1))\sqrt{n}.$$

That is, the conjecture holds asymptotically.

We intend to investigate the burning number of random trees. Let $\mathcal{T}$ be a Galton—Watson tree with a given offspring distribution ξ, where ξ is a $\mathbb{Z}_{\geq 0}$-valued random variable with

$$\mathsf{E}[\xi] = 1, \qquad \text{and} \qquad 0 < \sigma^2 := \mathsf{Var}[\xi] < \infty. \tag{1.1}$$

In other words, the Galton—Watson tree is critical, of finite variance, and satisfies $\mathsf{Pr}(\xi = 1) < 1$. In particular, it implies that $0 < \mathsf{Pr}(\xi = 0) < 1$.

For $n \geq 1$, let T_n be the tree $\mathcal{T}$ conditioned to have n vertices. (Implicitly, we assume $\mathsf{Pr}(|\mathcal{T}| = n) > 0$. This requires a straightforward assumption on n and ξ which we make explicit later; see Section 4.) The resulting random tree is an instance of the family of *simply generated trees* introduced by Meir and Moon [13]. This family contains many combinatorially interesting random trees such as uniformly chosen random plane trees, random unordered labelled trees (known as Cayley trees), and random d-ary trees. For more examples, see, Aldous [1] and Devroye [7]. For more on the relationship between conditioned Galton—Watson trees and simply generated trees, see Janson [9, Section 4].

Our main result shows that, with high probability, $b(T_n)$ is of the order of $n^{1/3}$.

Theorem 1.2. *Let T_n be a conditioned Galton—Watson tree of order n, subject to (1.1). For any $\epsilon = \epsilon(n)$ tending to 0 as $n \to \infty$, we have*

$$\Pr\Big((\epsilon n)^{1/3} \le b(T_n) \le (n/\epsilon)^{1/3}\Big) = 1 - O(\epsilon).$$

The paper is structured as follows. First, we make a simple observation that the burning number can be reduced to the problem of covering vertices of the graph with balls, a slightly easier problem; see Section 2. Section 3 is devoted to the lower bound for the burning number and the upper bound is provided in Section 4. We finish the paper with a few natural questions; see Section 5.

2 Covering a graph with balls

In this section, we show a simple but convenient observation that reduces the burning number to the problem of covering the graph's vertices with balls. Let $G = (V, E)$ be any graph. For any $r \in \mathbb{N}_0$ and vertex $v \in V$, we denote by $B_r(v)$ the ball of radius r centered at v, that is, $B_r(v) = \{u \in V : d(u, v) \le r\}$, where $d(u, v)$ denotes the distance between u and v.

First, note that since the burning process is deterministic, a fire source v makes all vertices in $B_t(v)$ burn after t rounds but only those vertices are affected. As a result, the burning number can be reformulated as follows:

$$b(G) = \min \left\{ k \in \mathbb{N} : \exists v_0, v_1, \dots, v_{k-1} \in V \text{ such that } \bigcup_{r=0}^{k-1} B_r(v_r) = V \right\}.$$

Dealing with balls of different radii is inconvenient so we will simplify the problem slightly by considering balls of the same radii. Let $\hat{b}(G)$ be the counterpart of $b(G)$ for this auxiliary covering problem, that is,

$$\hat{b}(G) = \min \left\{ k \in \mathbb{N} : \exists v_1, v_2, \dots, v_k \in V \text{ such that } \bigcup_{r=1}^{k} B_k(v_r) = V \right\}.$$

Covering with k balls of increasing radii (in particular, all of them of radii at most $k-1$) is not easier than covering with k balls of radius k. Hence, $\hat{b}(G) \le b(G)$. On the other hand, covering with $2k$ balls of increasing radii (in particular, k of them of radii at least k) is not more difficult than covering with k balls of radius k implying that $b(G) \le 2\hat{b}(G)$. We conclude that $\hat{b}(G)$ and $b(G)$ are of the same order:

Observation 2.1. *For any graph $G = (V, E)$,*

$$\hat{b}(G) \le b(G) \le 2\hat{b}(G).$$

In particular, we may prove the bounds in our main result (Theorem 1.2) for $\hat{b}(G)$ instead of $b(G)$, which will be slightly easier.

3 Lower bound

For an arbitrary tree τ and $i \in \mathbb{N}$, let

$$P_i(\tau) := \left| \Big\{ \{v, w\} : v, w \in V(\tau),\, d(v, w) = i \Big\} \right|.$$

In other words, $P_i(\tau)$ counts the number of unordered pairs of vertices which are distance i apart in τ. We have the following result from [8] that upper bounds the expected number of such pairs in the random tree T_n:

Theorem 3.1 ([8, Theorem 1.3]). *There exists a constant $c > 0$, dependent on the distribution of ξ, such that for all $n \in \mathbb{N}$ and $i \in \mathbb{N}$, $\mathsf{E}[P_i(T_n)] \le cni$.*

The essence of the lower bound on $b(T_n)$ is the following: Theorem 3.1 suggests that a typical ball of radius j in T_n contains $O(j^2)$ vertices; thus, covering T_n with j balls of radius at most j requires j to satisfy $j^3 = \Omega(n)$—that is, $j = \Omega(n^{1/3})$. We make this argument rigorous in the proof of Proposition 3.2.

Proposition 3.2. *Let $k = k(n) = (\epsilon n)^{1/3}$, where $\epsilon = \epsilon(n) \to 0$ as $n \to \infty$. With probability $1 - O(\epsilon)$, $\hat{b}(T_n) \ge k$, that is, there is no partition of the vertices of T_n into k disjoint sets $U_1, U_2, \ldots, U_k$ such that* $\operatorname{diam}(U_j) \le 2k$ for all $1 \le j \le k$.

Since $b(T_n) \ge \hat{b}(T_n)$ (Observation 2.1), Proposition 3.2 implies the corresponding lower bound in Theorem 1.2.

Proof. Let $Q_j(T_n) = \sum_{i=1}^{j} P_i(T_n)$, that is, $Q_j(T_n)$ counts pairs of vertices in T_n which are at most distance j apart. From Theorem 3.1, for all $n, j \in \mathbb{N}$,

$$\mathsf{E}[Q_j(T_n)] = \sum_{i=1}^{j} \mathsf{E}[P_i(T_n)] \le cn \sum_{i=1}^{j} i \le cnj^2.$$

For a contradiction, suppose there is a partition $U_1, U_2, \ldots, U_k$ of the vertices of T_n as described in the statement of the proposition. Since every pair of vertices in a given U_j is at most distance $2k$ apart, we must have

$$Q_{2k}(T_n) \ge \sum_{j=1}^{k} \binom{|U_j|}{2} = \sum_{j=1}^{k} \left(\frac{|U_j|^2}{2} - \frac{|U_j|}{2} \right) = \frac{1}{2} \sum_{j=1}^{k} |U_j|^2 - \frac{n}{2}.$$

By Jensen's inequality, we get

$$\frac{1}{k} \sum_{j=1}^{k} |U_j|^2 \ge \left(\frac{1}{k} \sum_{j=1}^{k} |U_j| \right)^2 = \left(\frac{n}{k} \right)^2$$

and therefore

$$Q_{2k}(T_n) \ge \frac{n^2}{2k} - \frac{n}{2} = \frac{n^2}{2k} \left(1 - O(k/n)\right) = \frac{n^2}{2k} \left(1 - o(n^{-2/3})\right).$$

On the other hand, $\mathsf{E}[Q_{2k}(T_n)] \le 4cnk^2$. Thus, by Markov's inequality,

$$\begin{aligned} \Pr\left(Q_{2k}(T_n) \ge \frac{n^2}{2k} - \frac{n}{2} \right) \quad &\le \quad \frac{\mathsf{E}[Q_{2k}(T_n)]}{\frac{n^2}{2k}} \left(1 + o(n^{-2/3})\right) \\ &= \quad O\left(\frac{k^3}{n} \right) = O(\epsilon). \end{aligned}$$

It follows that a partition of $V(T_n)$ with the stated properties exists with probability $O(\epsilon)$, which finishes the proof of the proposition. □

4 Upper bound

Here we prove the upper bound of Theorem 1.2. To do this, we first devise a deterministic strategy which covers any rooted tree τ with balls of radius $2k$. The balls are centered at the vertices of a particular subset $C \subset V(\tau)$; in the Galton—Watson setting, we are able to show that $|C|$ is less than or equal to $2k$ a.a.s. as long as k is at least of the order $n^{1/3}$, which yields the desired upper bound.

For any rooted tree τ with root r, for $i \in \mathbb{N}_0$ we write $\ell_i(\tau) := \{v \in V(\tau) : d_\tau(r,v) = i\}$ for the set of vertices at depth i. Let $h(\tau) \in \mathbb{N}_0 \cup \{\infty\}$ be the height of τ, that is, $h(\tau) := \sup\{i : \ell_i(\tau) \neq \emptyset\}$. For any $v \in V(\tau)$, let τ_v the (full) sub-tree of τ rooted at v. For $k \in \mathbb{N}$, and $j \in \{0, 1, \ldots, k-1\}$, let

$$\mathcal{C}_k^j(\tau) := \bigcup_{i=0}^{\infty} \Big\{ v \in \ell_{ik+j}(\tau) : h(\tau_v) \geq k \Big\}.$$

So $\mathcal{C}_k^j(\tau)$ consists of all vertices whose depth is j modulo k with subtrees of height at least k.

We first show that placing balls of radius $2k$ at the root and at each vertex in $\mathcal{C}_k^j(\tau)$ covers the vertices of τ.

Lemma 4.1. *Let τ be a tree rooted at r. For any $k \in \mathbb{N}$ and $j \in \{0, 1, \ldots, k-1\}$ we have*

$$V(\tau) = \bigcup_{v \in \mathcal{C}_k^j(\tau) \cup \{r\}} B_{2k}(v).$$

Proof. Fix $k \in \mathbb{N}$ and $j \in \{0, 1, \ldots, k-1\}$. Let $v \in V(\tau)$, let i be the smallest non-negative integer such that $d(r,v) < ik + j$, and let a be the unique ancestor of v in $\ell_{(i-2)k+j}(\tau)$, or $a = r$ if $i \in \{0, 1\}$. Then a is either the root r, or $d(r,a) \equiv j \pmod{k}$ and $h(\tau_a) \geq k$. In either case, $a \in \mathcal{C}_k^j(\tau) \cup \{r\}$. Since $d(a,v) \leq 2k$, we have $v \in B_{2k}(a)$. This finishes the proof of the lemma. □

Lemma 4.1 provides a scheme to cover a general rooted tree τ with balls of radius $2k$. Observe that for any j, $|\mathcal{C}_k^j(\tau)| \leq 2k - 1$ implies that $\hat{b}(\tau) \leq 2k$, and hence $b(\tau) \leq 4k$ by Observation 2.1. In particular, we conclude the following:

$$\text{if } \min_j |\mathcal{C}_k^j(\tau)| \leq 2k - 1, \text{ then } b(\tau) \leq 4k. \tag{4.1}$$

Our next lemma estimates the probability that a vertex selected uniformly at random from the random tree T_n has height at least k.

Lemma 4.2. *Consider the random tree $\tau = T_n$ on n vertices. Then there exists a constant $c > 0$, dependent on the distribution of ξ, such that the following property holds. Let u be a vertex selected uniformly at random from $V(\tau)$, and let its subtree be denoted by τ_u. Then, for all $k \geq 0$,*

$$\mathsf{Pr}\Big(h(\tau_u) \geq k\Big) \leq c \left(\frac{1}{k} + \frac{1}{\sqrt{n}} \right).$$

To prove Lemma 4.2 we require some standard tools for conditioned Galton—Watson trees, which we introduce now. Given a tree τ with s vertices, let $v_1, v_2, \ldots, v_s$ be the vertices of τ in depth-first search (dfs) order. We write d_i for the number of children of v_i and refer to $(d_1, d_2, \ldots, d_s)$ as the *preorder degree sequence* of τ. The preorder degree sequence gives rise to a representation of τ as a lattice path $(j, y_j)_{j=0}^s$ started from $(0, y_0) = (0, 0)$ with s steps, where the jth step is given by $y_j = y_{j-1} + (d_j - 1)$. See Figure 2 for an example.

Note that the lattice path corresponding to a tree with s vertices always ends at the point $(s, y_s) = (s, -1)$ and has a height strictly greater than -1 before then, evidenced by the fact that the height of the path at step j is one less than the number of vertices in the "queue" of the dfs of the tree after j vertices have been explored. Since vertex v_s

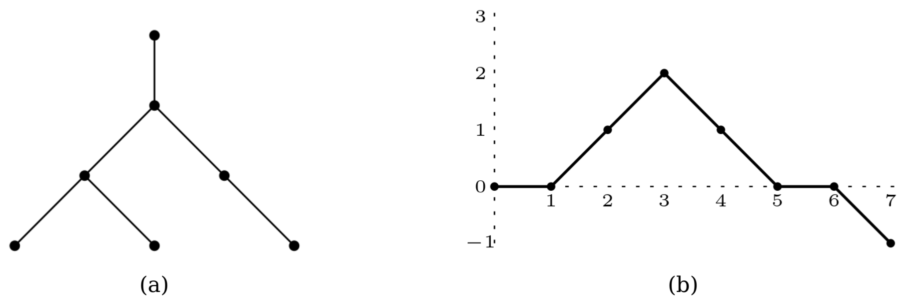

Figure 2: A rooted tree τ with preorder degree sequence $(1, 2, 2, 0, 0, 1, 0)$ (a) and its lattice path representation (b).

is always a leaf in the dfs order on τ, the point $(s-1, y_{s-1})$ must equal $(0, 0)$, meaning that the lattice path representation of τ can be identified with a *Łukasiewicz path* of length $s-1$. As the next lemma summarizes, there is a bijective correspondence between ordered trees with s vertices and lattice paths of this type.

Lemma 4.3 ([9, Lemma 15.2]). *A sequence $(d_1, d_2, \ldots, d_s) \in \mathbb{N}_0^s$ is the preorder degree sequence of a tree if and only if*

$$\sum_{i=1}^{j} d_i \geq j \quad \forall j \in \{1, 2, \ldots, s-1\}$$

and

$$\sum_{i=1}^{s} d_j = s - 1.$$

We also have the following useful property.

Lemma 4.4 ([9, Corollary 15.4]). *If $(d_1, d_2, \ldots, d_s) \in \mathbb{N}_0^s$ satisfies $\sum_{i=1}^{s} d_i = s-1$, then precisely one of the cyclic permutations of $(d_1, d_2, \ldots, d_s)$ is the preorder degree sequence of a tree.*

Let $(\xi_1^{(s)}, \xi_2^{(s)}, \ldots, \xi_s^{(s)})$ be the preorder degree sequence of a Galton—Watson tree T_s with offspring distribution ξ conditioned to have s vertices, and let $(\widetilde{\xi}_1^{(s)}, \widetilde{\xi}_2^{(s)}, \ldots, \widetilde{\xi}_s^{(s)})$ be a uniformly random cyclic permutation of $(\xi_1^{(s)}, \xi_2^{(s)}, \ldots, \xi_s^{(s)})$. Let $\xi_1, \xi_2, \ldots$ be a sequence of i.i.d. copies of ξ, and define, for any $j \geq 1$, $S_j = \sum_{i=1}^{j} \xi_i$. Lemmas 4.3 and 4.4 yield the following corollary:

Corollary 4.5. *The sequence $(\widetilde{\xi}_1^{(s)}, \widetilde{\xi}_2^{(s)}, \ldots, \widetilde{\xi}_s^{(s)})$ has the same distribution as the sequence $(\xi_1, \xi_2, \ldots, \xi_s)$ conditioned on $S_s = s-1$.*

Define the *span* of ξ as

$$h = \gcd\{i \geq 1 : \mathsf{Pr}(\xi = i) > 0\}.$$

We will use the following local limit theorems (see [10, Lemma 4.1 and (4.3)] and the sources referenced therein).

Lemma 4.6. *Suppose ξ satisfies (1.1) and has span h. Then, as $s \to \infty$, uniformly for $m \equiv 0 \pmod{h}$,*

$$\mathsf{Pr}(S_s = m) = \frac{h}{\sqrt{2\pi\sigma^2 s}} \left(e^{-(m-s)^2/2s\sigma^2} + o(1) \right).$$

If $\mathcal{T}$ is the Galton—Watson tree with offspring distribution ξ, then for $s \equiv 1 \pmod{h}$, as $s \to \infty$,

$$\mathsf{Pr}(|\mathcal{T}| = s) = \frac{h}{\sqrt{2\pi\sigma^2 s^3}}(1 + o(1)).$$

We are now ready to prove Lemma 4.2.

Proof of Lemma 4.2. Throughout the proof, we will use $c_1, c_2, \dots$ for non-explicit positive constants which do not depend on n (but may depend on the distribution of ξ). Implicitly, we will assume throughout that $n \equiv 1 \pmod h$, where h is the span of ξ, so that $\mathsf{Pr}(S_n = n-1) > 0$.

We identify the random vertex u with a random index of the dfs order on T_n. Consider $(\xi_u^{(n)}, \xi_{u+1}^{(n)}, \dots, \xi_n^{(n)}, \xi_1^{(n)}, \dots, \xi_{u-1}^{(n)})$. It is clear that this sequence has the same distribution as $(\widetilde{\xi}_1^{(n)}, \widetilde{\xi}_2^{(n)}, \dots, \widetilde{\xi}_n^{(n)})$, which, in turn, has the same distribution as $(\xi_1, \xi_2, \dots, \xi_n)$ conditioned on $S_n = n-1$ by Corollary 4.5.

For $k \in \mathbb{N}$ and $s \geq k$, let $\mathfrak{T}_s^k$ be the set of ordered trees with s vertices and height at least k. For a sequence $(d_1, d_2, \dots, d_s) \in \mathbb{N}_0^s$, we write $(d_1, d_2, \dots, d_s) \in \mathfrak{T}_s^k$ if $(d_1, d_2, \dots, d_s)$ is the preorder degree sequence of a tree in $\mathfrak{T}_s^k$. Note that for any $(d_1, d_2, \dots, d_s) \in \mathfrak{T}_s^k$, we have $\sum_{i=1}^s d_i = s-1$. Then,

$$\begin{aligned}
\mathsf{Pr}(h(\tau_u) \geq k) &= \sum_{s=k}^{n} \mathsf{Pr}((\widetilde{\xi}_1^{(n)}, \widetilde{\xi}_2^{(n)}, \dots, \widetilde{\xi}_s^{(n)}) \in \mathfrak{T}_s^k) \\
&= \sum_{s=k}^{n} \mathsf{Pr}((\xi_1, \xi_2, \dots, \xi_s) \in \mathfrak{T}_s^k \mid S_n = n-1) \\
&= \sum_{s=k}^{n} \frac{\mathsf{Pr}((\xi_1, \xi_2, \dots, \xi_s) \in \mathfrak{T}_s^k,\, S_n = n-1)}{\mathsf{Pr}(S_n = n-1)} \\
&= \sum_{s=k}^{n} \frac{\mathsf{Pr}((\xi_1, \xi_2, \dots, \xi_s) \in \mathfrak{T}_s^k) \cdot \mathsf{Pr}\left(\sum_{i=s+1}^{n} \xi_i = n-s\right)}{\mathsf{Pr}(S_n = n-1)} \\
&= \sum_{s=k}^{n} \frac{\mathsf{Pr}((\xi_1, \xi_2, \dots, \xi_s) \in \mathfrak{T}_s^k) \cdot \mathsf{Pr}(S_{n-s} = n-s)}{\mathsf{Pr}(S_n = n-1)}. \qquad (4.2)
\end{aligned}$$

By Lemma 4.6, there is a constant $c_1 > 0$ such that for $s \in \{1, 2, \dots, \lfloor n/2 \rfloor\}$, we have

$$\frac{\mathsf{Pr}(S_{n-s} = n-s)}{\mathsf{Pr}(S_n = n-1)} \leq c_1.$$

Thus,

$$\begin{aligned}
\sum_{s=k}^{\lfloor n/2 \rfloor} \frac{\mathsf{Pr}((\xi_1, \xi_2, \dots, \xi_s) \in \mathfrak{T}_s^k) \cdot \mathsf{Pr}(S_{n-s} = n-s)}{\mathsf{Pr}(S_n = n-1)} &\leq c_1 \sum_{s=k}^{\lfloor n/2 \rfloor} \mathsf{Pr}((\xi_1, \xi_2, \dots, \xi_s) \in \mathfrak{T}_s^k) \\
&\leq c_1 \sum_{s=k}^{\infty} \mathsf{Pr}((\xi_1, \xi_2, \dots, \xi_s) \in \mathfrak{T}_s^k) \\
&= c_1 \, \mathsf{Pr}(h(\mathcal{T}) \geq k),
\end{aligned}$$

where, recall, $\mathcal{T}$ is the unconditioned Galton—Watson tree with offspring distribution ξ. By Kolmogorov's Theorem [12, Theorem 12.7], there is a constant $c_2 > 0$ so that for any $k \geq 1$, we have $\mathsf{Pr}(h(\mathcal{T}) \geq k) \leq c_2/k$, and thus $c_1 \, \mathsf{Pr}(h(\mathcal{T}) \geq k) \leq c_3/k$ for some constant $c_3 > 0$. This bounds the partial sum of (4.2) where $k \leq s \leq n/2$.

For the other partial sum, we first observe that

$$\mathsf{Pr}((\xi_1, \xi_2, \dots, \xi_s) \in \mathfrak{T}_s^k) \leq \mathsf{Pr}(|\mathcal{T}| = s).$$

By Lemma 4.6, there is a constant $c_4 > 0$ so that

$$\frac{\mathsf{Pr}(|\mathcal{T}| = s)}{\mathsf{Pr}(S_n = n-1)} \leq \frac{c_4}{n}$$

for all $n \geq 1$ and $\lceil n/2 \rceil \leq s \leq n$. Using Lemma 4.6 again, we get that for all $j \geq 1$

$$\Pr(S_j = j) \leq \frac{c_5}{\sqrt{j}}$$

for some constant $c_5 > 0$. It follows that

$$\begin{aligned} \sum_{s=\lceil n/2 \rceil}^{n} \Pr(S_{n-s} = n-s) &= \sum_{j=0}^{n-\lceil n/2 \rceil} \Pr(S_j = j) \\ &\leq 1 + c_5 \sum_{j=1}^{n-\lceil n/2 \rceil} \frac{1}{\sqrt{j}} \\ &\leq c_6\sqrt{n} \end{aligned}$$

for some constant $c_6 > 0$, where the bound in the last line follows from a straightforward comparison of the sum with an integral. In all, we get that the partial sum of (4.2) with $n/2 \leq s \leq n$ is at most $c_7/\sqrt{n}$ (for some constant $c_7 > 0$) for all n.

Finally, combining the two bounds, we conclude that (4.2) is upper bounded by $c_8\left(\frac{1}{k} + \frac{1}{\sqrt{n}}\right)$ for some constant $c_8 > 0$, as desired. This finishes the proof of the lemma. □

We now have all the ingredients to finalize the upper bound. The next theorem, as discussed earlier (see (4.1)), implies that $b(\tau) \leq 4k = O((n/\epsilon)^{1/3})$ with probability $1 - O(\epsilon)$.

Theorem 4.7. *Let $\epsilon = \epsilon(n) \to 0$ as $n \to \infty$, and let $k = \lfloor \left(\frac{n}{\epsilon}\right)^{1/3} \rfloor$. Then, with probability $1 - O(\epsilon)$, we have*

$$\min_{j \in \{0,1,\ldots,k-1\}} |\mathcal{C}_k^j(T_n)| \leq 2k - 1.$$

Proof. Let $X_j = |\mathcal{C}_k^j(T_n)|$ for $j \in \{0, 1, \ldots, k-1\}$, and let Y be the number of vertices v in $\tau = T_n$ such that $h(\tau_v) \geq k$. Clearly, we have the identity $Y = \sum_{j=0}^{k-1} X_i$.

Now, observe that

$$\min_{j \in \{0,1,\ldots,k-1\}} X_j \leq \frac{1}{k}\sum_{j=0}^{k-1} X_j = \frac{Y}{k}.$$

By Lemma 4.2, $\mathsf{E}[Y] \leq c\left(\frac{n}{k} + \sqrt{n}\right)$. Therefore, by Markov's inequality,

$$\Pr\left(\frac{Y}{k} > 2k - 1\right) \leq \frac{\mathsf{E}[Y]}{(2k-1)k} \leq \frac{c}{2k-1}\left(\frac{n}{k^2} + \frac{\sqrt{n}}{k}\right) = O\left(\frac{n}{k^3}\right) = O(\epsilon).$$

This completes the proof of the theorem. □

5 Future directions

In this paper we showed that asymptotically almost surely (a.a.s.) $b(T_n)$ is close to $n^{1/3}$, that is, a.a.s. $\epsilon n^{1/3} \leq b(T_n) \leq n^{1/3}/\epsilon$, provided that $\epsilon = \epsilon(n) \to 0$ as $n \to \infty$. A more detailed analysis gives that $c_1\left(\frac{n}{\sigma^2}\right)^{1/3} \leq \mathsf{E}[b(T_n)] \leq c_2\left(\frac{n}{\sigma^2}\right)^{1/3}$, where c_1 and c_2 are universal constants, i.e., not dependent on the offspring distribution ξ. One could ask under what conditions there exists a universal constant c such that $\mathsf{E}[b(T_n)] = (c + o(1))\left(\frac{n}{\sigma^2}\right)^{1/3}$. A limit theory for $b(T_n)/n^{1/3}$ would also be of interest.

Achieving improvements with our techniques would require significant refinements of Theorem 3.1 and Lemma 4.2. For instance, Theorem 3.1 implies that the expected number of pairs of vertices within distance k of one another in T_n is $O(nk^2)$, with the implied constant dependent on ξ. If instead we had that a.a.s. the number of such pairs is $O(nk^2)$, the proof of Proposition 3.2 would (with minimal modification) give that $b(T_n) \geq \Omega(n^{1/3})$. Similarly, if we knew that a.a.s. the number of full subtrees of height at least k was at most $O(n/k)$, then the proof of Theorem 4.7 would imply that a.a.s. $b(T_n) = O(n^{1/3})$. At present, Lemma 4.2 only bounds the expected number of such subtrees. Refinements of this type would be interesting results in their own right.

References

[1] David Aldous, *The continuum random tree. ii. an overview*, Stochastic Analysis **167** (1991), 23–70. MR1166406

[2] Noga Alon, *Transmitting in the n-dimensional cube*, Discrete Applied Mathematics **37** (1992), 9–11. MR1176841

[3] Paul Bastide, Marthe Bonamy, Anthony Bonato, Pierre Charbit, Shahin Kamali, Théo Pierron, and Mikaël Rabie, *Improved pyrotechnics: Closer to the burning number conjecture*, The Electronic Journal of Combinatorics **30** (2023), P4—2. MR4650615

[4] Stéphane Bessy, Anthony Bonato, Jeannette Janssen, Dieter Rautenbach, and Elham Roshanbin, *Bounds on the burning number*, Discrete Applied Mathematics **235** (2018), 16–22. MR3732592

[5] Anthony Bonato, Jeannette Janssen, and Elham Roshanbin, *Burning a graph as a model of social contagion*, Algorithms and Models for the Web Graph: 11th International Workshop, WAW 2014, Beijing, China, December 17-18, 2014, Proceedings 11, Springer, 2014, pp. 13–22. MR3297344

[6] Anthony Bonato, Jeannette Janssen, and Elham Roshanbin, *How to burn a graph*, Internet Mathematics **12** (2016), no. 1-2, 85–100. MR3474055

[7] Luc Devroye, *Branching processes and their applications in the analysis of tree structures and tree algorithms*, Probabilistic Methods for Algorithmic Discrete Mathematics, Springer, 1998, pp. 249–314. MR1678582

[8] Luc Devroye and Svante Janson, *Distances between pairs of vertices and vertical profile in conditioned Galton–Watson trees*, Random Structures & Algorithms **38** (2011), no. 4, 381–395. MR2829308

[9] Svante Janson, *Simply generated trees, conditioned Galton–Watson trees, random allocations and condensation*, Probability Surveys **9** (2012), 103. MR2908619

[10] Svante Janson, *Asymptotic normality of fringe subtrees and additive functionals in conditioned Galton–Watson trees*, Random Structures & Algorithms **48** (2016), no. 1, 57–101. MR3432572

[11] Max R Land and Linyuan Lu, *An upper bound on the burning number of graphs*, Algorithms and Models for the Web Graph: 13th International Workshop, WAW 2016, Montreal, QC, Canada, December 14–15, 2016, Proceedings 13, Springer, 2016, pp. 1–8. MR3596443

[12] Russell Lyons and Yuval Peres, *Probability on trees and networks*, vol. 42, Cambridge University Press, 2017. MR3616205

[13] Amram Meir and John W Moon, *On the altitude of nodes in random trees*, Canadian Journal of Mathematics **30** (1978), no. 5, 997–1015. MR0506256

[14] Dieter Mitsche, Paweł Prałat, and Elham Roshanbin, *Burning graphs: a probabilistic perspective*, Graphs and Combinatorics **33** (2017), 449–471. MR3612204

[15] Dieter Mitsche, Paweł Prałat, and Elham Roshanbin, *Burning number of graph products*, Theoretical Computer Science **746** (2018), 124–135. MR3851414

[16] Sergey Norin and Jérémie Turcotte, *The burning number conjecture holds asymptotically*, Journal of Combinatorial Theory, Series B **168** (2024), 208–235. MR4752204

Acknowledgments. Part of this work was done during the 18th Annual Workshop on Probability and Combinatorics, McGill University's Bellairs Institute, Holetown, Barbados (March 22–29, 2024). The authors would like to thank the referees.