THE PEARSON IV DISTRIBUTION: RANDOM VARIATE GENERATION AND APPLICATIONS

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ABSTRACT. {Sentence requested by the journal: This is the first paper that shows how one can generate Pearson IV random variables uniformly fast over all possible values of both parameters.}

Official abstract: This paper presents the first uniformly fast random variate generators for the Pearson IV distribution that can be used over the entire range of both shape parameters. Additionally, we demonstrate their utility in a Bayesian setting.

1. INTRODUCTION

Undoubtedly, the most enigmatic member of Pearson's family (Pearson, [42]) is the Pearson type IV distribution. It is indexed by two parameters, a shape parameter $a > \frac{1}{2}$ and a skewness parameter $s \in \mathbb{R}$, and it includes several classical laws as special or limiting cases.

Its probability density on the real line is given by

$$f(x) = \frac{\gamma\, e^{s \arctan(x)}}{(1+x^2)^a}, \qquad x \in \mathbb{R}, \tag{1}$$

where the normalizing constant γ can be expressed, using Legendre's duplication formula, as

$$\gamma \stackrel{\text{def}}{=} \frac{\left|\Gamma\left(a - \frac{is}{2}\right)\right|^2}{\Gamma(a)\,\Gamma\left(a-\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}\right)} = \frac{4^{a-1}\left|\Gamma\left(a - \frac{is}{2}\right)\right|^2}{\pi\,\Gamma(2a-1)}.$$

Here Γ denotes Euler's gamma function, defined for complex numbers $z = x + iy$ with $x > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

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We write $P_{a,s}$ for a random variable having the Pearson type IV distribution with parameters (a, s). Since

$$P_{a,s} \stackrel{d}{=} -P_{a,-s},$$

we may assume without loss of generality that $s \geq 0$. Throughout the paper, we will use the notation U for a uniform $[0, 1]$ random variable, N for a standard normal random variable, E for an exponential random variable, T_a for a Student-t random variable with parameter $a > 0$, and G_a for a gamma random variable with shape parameter a and scale factor 1.

The purpose of this paper is to develop random variate generation algorithms for the Pearson type IV family that are *uniformly fast* over wide ranges of the parameters.

Devroye [10, 11] mentions that $\arctan(P_{a,s})$ has a log-concave density. Via an inequality due to Devroye [12], this would yield a rejection algorithm taking an expected number of iterations equal to 4, but it would require access to the normalization constant γ, which depends on the complex gamma function. For implementations, see Heinrich [24] and Nagahara [39, 40]. If, however, the exact computation of γ is impossible, then the question arises whether one can still generate $P_{a,s}$. The present paper answers this in the affirmative. In addition, the new methods have uniformly bounded expected complexity over the entire range of the parameters. Furthermore, we present simpler rejection algorithms that avoid any evaluation of γ altogether and are designed to be used for certain regions of the parameter space.

We emphasize the following two guiding principles underlying all algorithms in this work:

(i) The generators are *exact*: no numerical approximation or discretization error is permitted.
(ii) The expected running time per sample is uniformly bounded over all admissible parameter values.

In view of (i), we will not consider approximate methods or iterative algorithms like the inversion method, which tries to solve $F(X) = U$ with U uniform $[0, 1]$, where F is the distribution function. Furthermore, the numerical evaluation of F is complicated by the presence of the complex-valued normalizing constant γ and the lack of elementary expressions for F. Early approaches focused on direct analytical and numerical evaluation of the density and related quantities using special functions. In particular, Nagahara [39] developed explicit representations of the density and characteristic function in terms of the complex gamma function, enabling direct computation of the normalizing constant. Standard references such as Johnson, Kotz and Balakrishnan [29] summarize these formulas within the Pearson system framework. However, the direct special-function approach suffers from numerical instability when parameters are extreme. A second class of methods is based on numerical integration of the density to compute the distribution function. Modern high-precision approaches rely on double-exponential transformations for numerical integration, introduced by Takahasi and Mori [47] and Mori [36]. These methods evaluate distribution functions accurately but can be computationally expensive. A third approach consists of approximation techniques, either through asymptotics or moment matching within the Pearson system. Early approximations for related distributions appear in Shenton and Carpenter [45], while Johnson, Kotz and Balakrishnan discuss broader system-level approximations [29]. Early computational foundations for Pearson system evaluation and approximation can be found in Woodward's work [49].

We now summarize the main algorithmic contributions of this paper. All methods below are uniformly fast over the indicated parameter regions. In particular, Algorithms 1 and 3 together cover the full parameter space.

(i) A UNIVERSAL LOG-CONCAVE GENERATOR. Algorithm 1 is uniformly fast for $a \geq 1$ and all $s \in \mathbb{R}$.
(ii) REJECTION FROM A GAUSSIAN DENSITY. For $a > 1$ and $|s| \leq \frac{3(a-1)}{2\pi}$, Algorithm 2 provides a simple rejection sampler using a normal envelope.
(iii) REJECTION FROM A GAMMA DENSITY. Algorithm 3 is uniformly fast for $a \in (\frac{1}{2}, 1]$.
(iv) REJECTION FROM AN EXPONENTIAL DENSITY. When $|s| \leq 3$ and $1 \leq a \leq 3$, Algorithm 4 yields an efficient exponential-envelope method.
(v) REJECTION FROM THE STUDENT–t DENSITY. Algorithm 5 is uniformly fast for bounded $|s|$, and is practically useful for $|s| \leq 5$.
(vi) THE SKEWED CAUCHY DENSITY. When $a = 1$, a direct one-line sampler is available, see (20).
(vii) THE STUDENT–t LAW. When $s = 0$, the distribution reduces to the Student–t family, admitting the simple sampler (19).

2. The Pearson IV density

The Pearson IV density (1) is unimodal and has mean

$$\mu = \frac{s}{2(a-1)}$$

for $a > 1$, and variance

$$\sigma^2 = \frac{s^2 + 4(a-1)^2}{4(a-1)^2(2a-3)}$$

for $a > 3/2$. It has a unique mode at

$$m = \frac{s}{2a}.$$

The log-density is

$$\log f(x) = \log(\gamma) + s\arctan(x) - a\log(1+x^2), x \in \mathbb{R}.$$

The derivatives of $\log f$ are

$$(\log f)'(x) = \frac{s - 2ax}{1+x^2}, \tag{2}$$

$$(\log f)''(x) = \frac{2ax^2 - 2sx - 2a}{(1+x^2)^2}. \tag{3}$$

This shows that the Pearson IV density is log-concave on the interval defined by

$$|x - m| \le D \stackrel{\text{def}}{=} \sqrt{1+m^2},$$

and log-convex outside that interval. For random variate generation, it is more convenient to work with densities that are log-concave over their entire range. For this reason, we propose to look at the arctan-mapped Pearson IV random variable

$$\arctan(P_{a,s}).$$

For $a \ge 1$, Exercise 1 on page 308 in Devroye [10] notes that $\arctan(P_{a,s})$ has a log-concave density supported on $[-\pi/2, \pi/2]$ and given by

$$h(y) = \begin{cases} \gamma e^{sy}(\cos^2(y))^{a-1} & \text{if } |y| \le \frac{\pi}{2}, \\ 0 & \text{else.} \end{cases} \tag{4}$$

The function $g = \log(h)$ has the following derivatives, all decreasing in y on $[0, \pi/2]$:

$$g'(y) = s - 2(a-1)\tan(y), \tag{5}$$

$$g''(y) = -\frac{2(a-1)}{\cos^2(y)}, \tag{6}$$

$$g'''(y) = -\frac{4(a-1)\tan(y)}{\cos^2(y)}, \tag{7}$$

$$g''''(y) = -\frac{4(a-1)(1+2\sin^2(y))}{\cos^4(y)}. \tag{8}$$

The modes of h and g occur at

$$m = \arctan(\beta),$$

where we set $\beta = s/(2(a-1))$, noting that for $a = 1$, we have $m = \pi/2$. Also,

$$h(m) = \gamma e^{s\arctan(\beta)} \left(\frac{1}{1+\beta^2}\right)^{a-1} = \gamma \left(\frac{e^{2\beta\arctan(\beta)}}{1+\beta^2}\right)^{a-1}.$$

The mean and variance $\mathbb{V}\{X\}$ of a random variable X with density (4) can be expressed as a function of the complex digamma function and the trigamma function ψ_1, e.g.,

$$\mathbb{V}\{X\} = \frac{1}{4}\left(\psi_1(a+is/2) + \psi_1(a-is/2)\right) \in \left[\frac{2a}{s^2+4a^2}, \frac{2(a+1)}{s^2+4a^2}\right] \tag{9}$$

[3]. For any random variable X with a unimodal density h with mode at m, Dharmadhikari and Joag-Dev [15][16] and Abu-Bakut, Bobkov and Madiman [1] showed that

$$\frac{1}{2e} \le h(m)\sqrt{\mathbb{V}\{X\}}.$$

For log-concave densities, Fradelizi [17] improved this:

$$\frac{1}{\sqrt{12}} \le h(m)\sqrt{\mathbb{V}\{X\}} \le \frac{1}{\sqrt{2}}. \tag{10}$$

The upper bound improves over an earlier result by Ibragimov [28], who showed that $h(m)\sqrt{\mathbb{V}\{X\}} \le 1$. Therefore,

$$h(m) \ge \sqrt{\frac{4a^2+s^2}{24(a+1)}}. \tag{11}$$

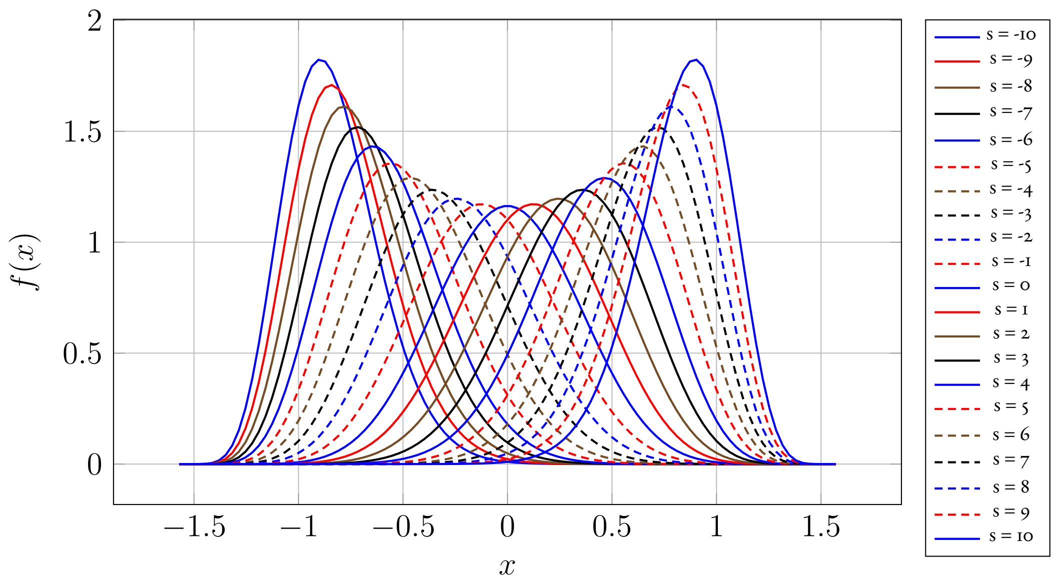

FIGURE 1. The arctan-mapped Pearson IV density $\gamma e^{sx}\cos^{2a-2}(x)$ for $a = 5$ and various values of s.

3. REJECTION BASED ON LOG-CONCAVITY OF THE ARCTAN-MAPPED PEARSON IV DENSITY

In this section, we assume that $a \geq 1$. As the arctan-mapped Pearson IV density is log-concave for $a \geq 1$, we can design a generator as a special case of the universal method of Devroye [12], wich is valid for all log-concave densities. However, the application of this turns around access to the normalization constant γ:

(i) By an inequality due to Devroye [12], we have

$$h(x) \leq \min\left(h(m), h(m)\exp(1 - h(m)|x-m|)\right). \tag{12}$$

Using this would yield an algorithm taking an expected number of iterations equal to 4, but it would require access to γ. For implementations, see Heinrich [24] and Nagahara [39, 40].

(ii) Let L be such that $h(m) \geq L$, which we apply in (12) to obtain

$$h(x) \leq \min\left(h(m), h(m)\exp(1 - L|x-m|)\right). \tag{13}$$

This avoids computing γ, and is new. The expected number of iterations becomes

$$4\frac{h(m)}{L}.$$

Candidates for L include $(\gamma^-/\gamma)h(m)$ (see (25) in the appendix for the definitions of γ^- and γ^+), (11), which is based on Fradelizi's inequality [17]), and a custom-designed lower bound for $h(m)$. The former replacement pushes the expected numbers of iterations up slightly to $4\gamma/\gamma^- \leq 4\gamma^+/\gamma^-$, which is uniformly bounded over all $a \geq 1$ and $s \in \mathbb{R}$.

The universal method for log-concave densities from Devroye [9], adapted here for use with the inequality based on (13) is given below.

Algorithm 1 Universal log-concave method: $a \geq 1$

let m be the location of the mode of the log-concave density h (4):
    $m = \arctan(s/(2(a-1)))$ if $a \geq 1$, $m = \pi/2$ if $a = 1$
set L so that $h(m) \geq L$
repeat
    let V be uniform on $[-2, 2]$
    if $V < -1$ **then**
        replace V by $-1 + \log(V + 2)$
    else if $V > 1$ **then**
        replace V by $1 - \log(V - 1)$
    end if
    $Y \leftarrow m + V/L$
    let U be uniform on $[0, 1]$
until $U \min(1, \exp(1 - L|Y - m|)) \leq h(Y)/h(m)$
return Y ▷ Y has density h
    ▷ $\tan(Y) \stackrel{\mathcal{L}}{=} P_{a,s}$ if h is as in (4)

Remark 1. *ADAPTIVE METHODS. The method given here has a uniformly bounded time and is useful when the parameters vary in an application. For fixed parameters, several adaptive methods make the method more efficient as more random variates are generated. See, e.g., Gilks [18], Gilks and Wild [19, 20] and Gilks, Best and Tan [21].* □

Remark 2. *REFERENCES. Additional references on random variate generation for log-concave laws include Hörmann, Leydold and Derflinger [27], Leydold and Hörmann [31, 32] and Devroye [10].* □

Remark 3. *THE UNIVERSAL ALGORITHM WITHOUT ACCESS TO THE NORMALIZATION CONSTANT. With $L = h(m)$, we know that the expected number of iterations in algorithm 1 is 4. In Lemma 4 in the appendix, we show that*

$$\gamma^- \leq \gamma \leq \gamma^+,$$

where γ^- and γ^+ are explicit functions of the parameters, a and s. Thus, setting

$$\Delta = \frac{h(m)}{\gamma} = e^{sm}(\cos^2(m))^{a-1},$$

we have $h(m) \geq L \stackrel{\text{def}}{=} \gamma^- \Delta$. If this value of L is used in algorithm 1, then the expected number of iterations is the integral of the bounding functions, or

$$4\frac{\gamma}{\gamma^-} \leq 4\frac{\gamma^+}{\gamma^-}.$$

With the choices given in Lemma 4 in the appendix, we see that the expected number of iterations is

$$\leq 4 \times \left(\frac{1 + \frac{3}{2\pi^2\sqrt{a^2+(s/2)^2}}}{1 - \frac{3}{2\pi^2\sqrt{a^2+(s/2)^2}}} \right)^2 \times \sqrt{\frac{(a+0.177)(a+0.677)}{(a+1/6)(a+2/3)}}.$$

Uniformly over all $a \geq 1$ and $s \geq 0$, this does not exceed

$$4 \times \left(\frac{2\pi^2+3}{2\pi^2-3} \right)^2 \times \sqrt{\frac{1.177 \times 1.677 \times 18}{35}} \approx 5.34.$$

Note, though, that as $a \to \infty$, the expected number of iterations tends to 4, since the inequalities for the gamma function get tighter.

4. Gaussian domination for the arctan-mapped density

In this section, we custom-design a rejection algorithm based on a normal envelope that is uniformly fast for all $a \geq 1$ and for sufficiently small s such that $m \leq 0.2344$. It outperforms the universal log-concave method in this range in simulations.

Upper bounds for the arctan-mapped density h (see (4)) can be based on the derivatives of $g = \log h$ (see (5)) and standard Taylor:

$$g(y) \leq \begin{cases} g(m) + \frac{(y-m)^2}{2} g''(m), & y \geq m, \\ g(m) + \frac{(y-m)^2}{2} g''(m) - \frac{(y-m)^3}{6} g'''(m), & y \leq m, \end{cases} \tag{14}$$

$$\leq \begin{cases} g(m) + \frac{(y-m)^2}{2} g''(m), & y \geq m, \\ g(m) + \frac{(y-m)^2}{4} g''(m), & m - D \leq y \leq m, \end{cases} \tag{15}$$

whenever

$$\frac{D|g'''(m)|}{6} \leq \frac{|g''(m)|}{4}.$$

This is satisfied if we pick D such that $D\beta \leq 3/4$. For example, if we set $D = \pi$, then for all m satisfying

$$\tan(m) \leq \frac{3}{4\pi} \approx 0.2387324, \tag{16}$$

we have

$$h(y) \leq h(m) e^{-\tau^2 \frac{(y-m)^2}{2}}, y \in \mathbb{R}, \tag{17}$$

where

$$\tau \stackrel{\text{def}}{=} \sqrt{|g''(m)|/2} = \sqrt{(a-1)(1+\beta^2)}.$$

A condition equivalent to (16) is $|m| \leq \arctan\left(\frac{3}{4\pi}\right) \approx 0.2344$. Algorithm 2 uses rejection either from the Gaussian implied by (17) or rejection from the uniform density on $[-\pi/2, \pi/2]$.

Lemma 1. *The expected number of iterations taken by algorithm 2 is not more than*

$$\sqrt{2\pi + \pi^2} \approx 4.02,$$

uniformly over all values of the parameters with $a \geq 1$ *and* $|m| \leq \arctan\left(\frac{3}{4\pi}\right) \approx 0.2344.$

Proof. If we were to use (15), then the expected number of iterations before halting would be

$$\frac{\sqrt{2\pi} h(m)}{\tau}.$$

However, if we bound h by $h(m)$ and use rejection from the uniform density on $[-\pi/2, \pi/2]$, then the expected number of iterations before halting would be $\pi h(m)$. Taking the best of both leads to the cut-off value at $\tau = \sqrt{2/\pi}$. Using (9) and Fradelizi's inequality (10), we have

$$h(m) \leq \sqrt{a + \frac{s^2}{4a}}.$$

Assume first $\tau \leq \sqrt{2/\pi}$. Then $a - 1 \leq 2/\pi$ and $s^2/(4(a-1)) \leq 2/\pi$, which implies that

$$\pi h(m) \leq \pi \sqrt{1 + 4/\pi} = \sqrt{\pi^2 + 4}.$$

If, on the other hand, $\tau \geq \sqrt{2/\pi}$, then

$$\begin{aligned}
\frac{\sqrt{2\pi} h(m)}{\tau} &\leq \frac{\sqrt{2\pi}\sqrt{a + \frac{s^2}{4a}}}{\tau} \leq \frac{\sqrt{2\pi}\sqrt{1 + (a-1) + \frac{s^2}{4(a-1)}}}{\tau} \\
&= \frac{\sqrt{2\pi}\sqrt{1+\tau^2}}{\tau} = \sqrt{2\pi}\sqrt{1 + \frac{1}{\tau^2}} \leq \sqrt{2\pi}\sqrt{1 + \frac{\pi}{2}} = \sqrt{2\pi + \pi^2}.
\end{aligned}$$

□

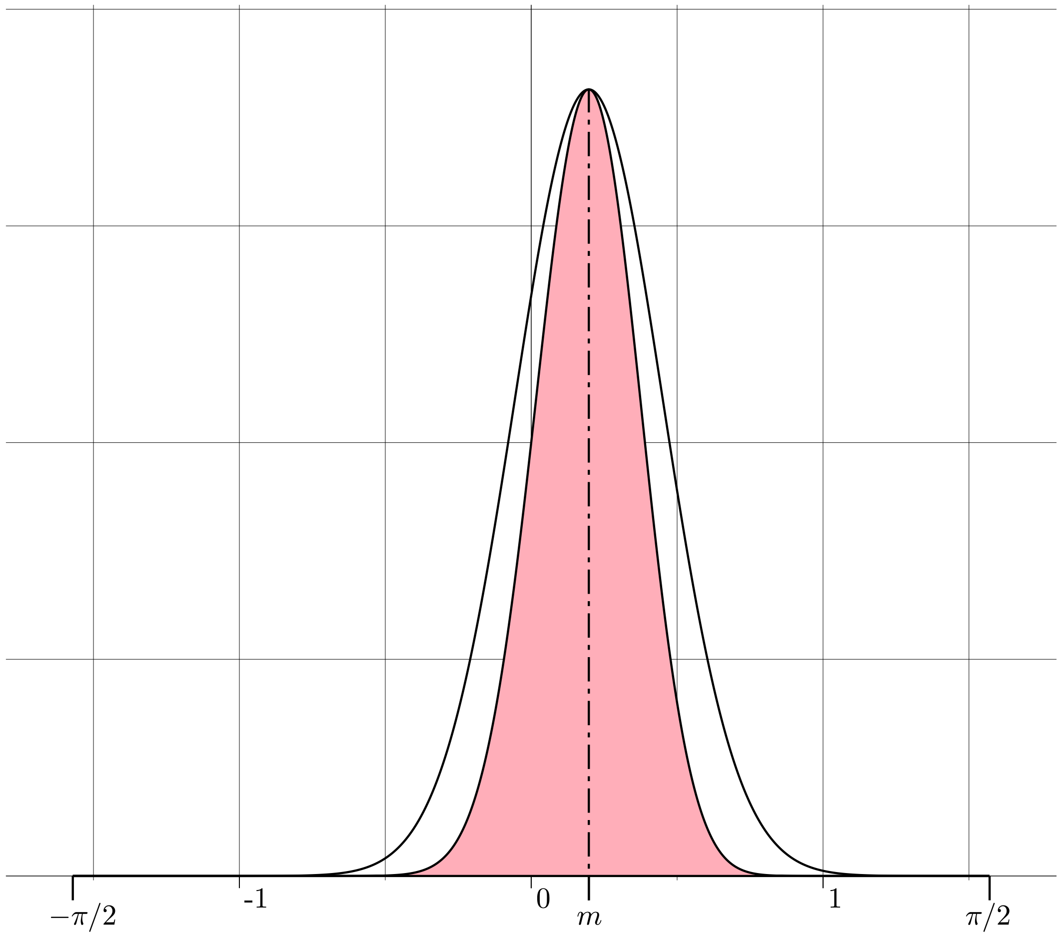

FIGURE 2. The arctan-mapped Pearson IV density is shown in pink, together with the normal envelopes for $s = 6, a = 16$.

Algorithm 2 PearsonIV generator for $a > 1$ by rejection from the normal density when $\tan(m) \leq 3/(4\pi) \approx 0.2387324$, or equivalently, when $m \leq \arctan(3/(4\pi)) \approx 0.2344$.

let $\beta = s/(2(a-1)), m = \arctan(\beta), \tau = \sqrt{(a-1)(1+\beta^2)}$
if $\tau \geq \sqrt{2/\pi}$ **then**
  repeat
    let N be a standard Gaussian random variable
    let U be uniform on $[0,1]$
    set $Y \leftarrow m + N/\tau$
  until $Uh(m)e^{-\tau^2 \frac{(Y-m)^2}{2}} \leq h(Y)$
    ▷ the acceptance condition is equivalent to $Ue^{-N^2/2} \leq h(Y)/h(m)$
else
  repeat
    let Y be uniform on $[-\pi/2, \pi/2]$
    let U be uniform on $[0,1]$
  until $U \leq h(Y)/h(m)$
end if
return $X \leftarrow \tan(Y)$ ▷ $X \stackrel{\mathcal{L}}{=} P_{a,s}$

5. Symmetrization for parameter values $a \leq 1$

In this section, we develop a generator that is uniformly fast for $a \in (1/2, 1], s \in \mathbb{R}$. As $P_{a,s} \stackrel{\mathcal{L}}{=} -P_{a,-s}$, symmetrization may be helpful. Define the symmetric density

$$g(x) = \frac{f(x) + f(-x)}{2} = \frac{\gamma \cosh(s \arctan(x))}{(1+x^2)^a}, x \in \mathbb{R}.$$

Setting $Y = \arctan(X)$, where X has density g on $\mathbb{R}$ yields the following symmetric density on $(-\pi/2, \pi/2)$:

$$h(y) = \gamma \cosh(sy)(\cos^2(y))^{a-1}.$$

Consider next the random variable $Z = \pi/2 - |Y|$ with density

$$\eta(z) = 2\gamma \cosh(s(\pi/2 - z))(\sin(z))^{2(a-1)}, 0 \leq z \leq \pi/2. \tag{18}$$

For $a \in (1/2, 1]$, the density (18) is decreasing and has an infinite peak at the origin unless $a = 1$. Most of its mass is near zero, and thus, we will attempt rejection using the bound

$$\begin{aligned}\eta(z) &\leq 2\gamma e^{s\pi/2} e^{-sz} (2z/\pi)^{2(a-1)} \\ &\leq 2\gamma (2/\pi)^{2(a-1)} e^{s\pi/2} e^{-sz} z^{2(a-1)}, 0 < z \leq \pi/2.\end{aligned}$$

Assume first that $s \geq 1$. Introduce a uniform $[0, 1]$ random variate U and an independent gamma $(2a - 1)$ random variate G_{2a-1}. We apply rejection from the gamma distribution by generating independent pairs $(Z, U) = (G_{2a-1}/s, U)$ until for the first time, $Z \leq \pi/2$ and

$$U(2Z/\pi)^{2(a-1)} \leq (\sin(Z))^{2(a-1)},$$

or, equivalently,

$$U \leq \left(\frac{2Z}{\pi \sin(Z)} \right)^{2(1-a)}.$$

The returned random variable Z has density η given in (18). The probability of acceptance is thus at least

$$\begin{aligned}&\mathbb{E}\left\{ \left(\frac{2G_{2a-1}/s}{\pi \sin(G_{2a-1}/s)} \right)^{2(1-a)} \mathbb{1}_{G_{2a-1}/s \leq \pi/2} \right\} \\ &\geq \mathbb{E}\left\{ \frac{2G_{2a-1}/s}{\pi \sin(G_{2a-1}/s)} \mathbb{1}_{G_{2a-1}/s \leq \pi/2} \right\} \\ &\geq \frac{2}{\pi} \mathbb{P}\left\{ G_{2a-1}/s \leq \pi/2 \right\},\end{aligned}$$

where $\mathbb{1}$ is the indicator function. By Markov's inequality, the probability in this expression is at least

$$1 - \frac{2\mathbb{E}\{\frac{G_{2a-1}}{s}\}}{\pi} = 1 - \frac{2(2a-1)}{s\pi} \geq 1 - \frac{2}{s\pi} \geq 1 - \frac{2}{\pi}$$

uniformly for all $s \geq 1, a \leq 1$. In this range, the rejection algorithm's expected number of iterations is at most

$$\frac{\pi^2}{2\pi - 4}.$$

Having generated Z with density (18), we need to set $X = \tan(S(\pi/2 - Z))$, where S is a random sign to obtain a random variate with the symmetrized Student-t density g. Finally, a random variate with the Pearson IV density f is obtained as

$$\begin{cases} X & \text{with probability } \frac{f(X)}{f(X)+f(-X)} = \frac{e^{s\arctan(X)}}{e^{s\arctan(X)}+e^{-s\arctan(X)}}, \\ -X & \text{else.} \end{cases}$$

Next, assume that $0 \leq s \leq 1$. Then

$$\eta(z) \leq 2\gamma(2/\pi)^{2(a-1)} e^{s\pi/2} z^{2(a-1)}, 0 < z \leq \pi/2.$$

Thus, we can generate random pairs

$$(Z, U) = \left(V^{\frac{1}{2a-1}} \frac{\pi}{2}, U \right)$$

where U, V are independent uniform random variates, until for the first time

$$U \leq e^{-sZ}.$$

The random variate Z has density (18). The expected number of iterations is

$$\mathbb{E}\{e^{sZ}\} \leq e^{s\pi/2} \leq e^{\pi/2}.$$

We combine the algorithms below.

Algorithm 3 PearsonIV generator for parameter $a \in (1/2, 1]$.

if $s \geq 1$ **then**
  repeat
    generate $Z = G_{2a-1}/s$ and U uniformly on $[0,1]$
  until $Z < \pi/2$ and $U \leq \left(\frac{2Z}{\pi \sin(Z)}\right)^{2(1-a)}$
else
  repeat
    generate V and U uniformly on $[0,1]$
    set $Z \leftarrow V^{\frac{1}{2a-1}} \frac{\pi}{2}$
  until $U \leq e^{-sZ}$
end if
$Y \leftarrow S(\pi/2 - Z)$, where S is a random sign
$X \leftarrow \tan(Y)$
with probability $e^{-sY}/(e^{-sY} + e^{sY})$, $X \leftarrow -X$
return X $\triangleright X \overset{\mathcal{L}}{=} P_{a,s}$

Remark 4. *GAMMA RANDOM VARIATES. For uniformly fast gamma random variates, we refer to the surveys in Devroye [10] and Luengo [33]. In terms of the rejection constant, the method of Marsaglia and Tsang [35] is highly recommended. Many simulation studies confirm that the method of Schmeiser and Lal [44] is quite competitive if the gamma parameter is at least one. Xi, Tan and Liu [50] suggested generating* $\log(G_a)$ *instead. As* $\log(G_a)$ *has a log-concave density for all values of* $a > 0$*, a uniformly fast generator is easily obtained either by the universal method of Devroye [9] or a specialized algorithm as developed, e.g., in Devroye [13]. For algorithm 3, with a gamma parameter less than one, we recommend the one-liner recently developed by Greaves [23].* □

6. Rejection from an exponential for the arctan-mapped density

In this section, we introduce the new algorithm 4, which has uniformly bounded expected complexity for $1 \leq a \leq a^*$, $0 \leq s \leq s^*$, where a^* and s^* are small constants. It is simple and useful in the limited range $1 \leq a \leq 3$, $0 \leq s \leq 3$.

For $a \geq 1$, we have $h(y) \leq \gamma e^{sy}$ on $[-\pi/2, \pi/2]$. This leads directly to the following rejection algorithm:

Algorithm 4 PearsonIV generator by rejection from the exponential density when $a \geq 1$.

1: **repeat**
2: let V be uniform on $[0, 1]$
3: let $Y \leftarrow \frac{1}{s} \log \left(V e^{s\pi/2} + (1 - V) e^{-s\pi/2} \right)$
4: let U be uniform on $[0, 1]$
5: **until** $U \leq (\cos(Y))^{2a-2}$
6: **return** $X \leftarrow \tan(Y)$ $\triangleright X \stackrel{\mathcal{L}}{=} P_{a,s}$

If we write $\gamma = \gamma(a, s)$ to make the dependence of the normalization constant on the parameters explicit, then it is easily seen that the expected number of iterations of the algorithm 4 is

$$\frac{\gamma(a, s)}{\gamma(1, s)},$$

which, by Lemma 25, for fixed $a \geq 1$, increases in proportion to s^{2a-2} as $s \uparrow \infty$.

7. Special case $s = 0$: The Student-t distribution.

We note the special case

$$P_{a,0} = \frac{T_{2a-1}}{\sqrt{2a-1}},$$

where the random variable T_a with parameter $a > 0$ is a Student-t(a) random variable with density

$$\frac{1}{B(a/2, 1/2)\sqrt{a}(1 + x^2/a)^{\frac{a+1}{2}}},$$

and B is the beta function. First derived by Helmert [25, 26] and Lüroth [34] and later by Pearson [42], it was named after Gosset (William S. Gosset [22]) by Ronald Fisher. We recall that

$$T_a \stackrel{\mathcal{L}}{=} \frac{N}{\sqrt{\frac{G_{a/2}}{a/2}}},$$

where N is standard normal, and G_a denotes an independent gamma (a) random variable. Let $H_{a,b} = G_a/G_b$ be the ratio of two independent gamma random variables (also called the beta prime distribution or beta distribution of the second kind with parameters a and b) and let $B_{a,b}$

be a beta random variable. From the definition of the Student-t distribution,

$$\frac{T_a^2}{a} = \frac{(1/2)N^2}{G_{a/2}} \stackrel{\mathcal{L}}{=} \frac{G_{1/2}}{G_{a/2}} \stackrel{\mathcal{L}}{=} H_{1/2,a/2} \stackrel{\mathcal{L}}{=} B_{1/2,1/2}H_{1,a/2} \stackrel{\mathcal{L}}{=} \sin^2(\pi U')\left(\frac{1}{U^{\frac{2}{a}}} - 1\right),$$

where U and U' are i.i.d. (independent identically distributed) uniform $[0,1]$ random variables. This yields a one-liner for the Student-t distribution due to Bailey [2], also called the polar method for Student-t distribution:

$$T_a \stackrel{\mathcal{L}}{=} \sqrt{a}\sin(\pi U')\sqrt{\frac{1}{U^{\frac{2}{a}}} - 1}. \tag{19}$$

See also Devroye [11] for variations on this polar method. Earlier methods for the Student-t distribution include algorithms by Best [5] and Ulrich [48]. Very simple special cases, obtainable by the inversion method, include the Cauchy law (obtained for $a = 1$), for which we have

$$T_1 \stackrel{\mathcal{L}}{=} \tan(\pi(U - 1/2)),$$

and the t_2 law, for which we have

$$T_2 \stackrel{\mathcal{L}}{=} \frac{2U - 1}{\sqrt{2U(1 - U)}}$$

(see, e.g., Jones [30]).

8. Rejection from the Student-t density

As the Pearson IV density is the Student-t density multiplied by a factor that biases the distribution to one side, one could try rejection from the Student-t distribution. This yields an algorithm with expected complexity uniformly bounded over all $a > 1/2$ and $|s| < c$ for some fixed constant c. We do not recommend this algorithm for $s > 5$. Using

$$f(x) \le \frac{\gamma e^{s\pi/2}}{(1 + x^2)^a},$$

it suffices to generate i.i.d. pairs (X, U), where $X = T_{2a-1}/\sqrt{2a-1}$ is a random variable with density proportional to $(1 + x^2)^{-a}$ and U is uniform on $[0,1]$, until

$$Ue^{s\pi/2} \le e^{s\arctan(X)},$$

or, equivalently, to generate i.i.d. pairs (X, E), where E is standard exponential, until

$$-E + s\pi/2 \leq s \arctan(X).$$

Algorithm 5 PearsonIV generator by rejection from Student-t law

1: **repeat**
2: let E be an exponential random variable
3: let T_{2a-1} is a Student-t random variable with parameter $2a-1$
4: set $X \leftarrow T_{2a-1}/\sqrt{2a-1}$
5: **until** $E \geq s(\pi/2 - \arctan(X))$
6: **return** X $\triangleright X \stackrel{\mathcal{L}}{=} P_{a,s}$

As

$$\frac{\gamma e^{-s\pi/2}}{(1+x^2)^a} \leq f(x) \leq \frac{\gamma e^{s\pi/2}}{(1+x^2)^a},$$

it is easy to see that the expected number of iterations is at least $(1/2)e^{\pi s/2}$ and at most $e^{\pi s}$, which is uniformly bounded for all $a > 1/2$ and $s < c$ for any fixed constant c.

9. Special case $a = 1$: the skewed Cauchy distribution.

When $a = 1$, we discover the skewed Cauchy density $h(y) = \gamma \exp(sy)$. Thus, a random variate with density h simply is

$$W = \begin{cases} \frac{1}{s} \log\left(e^{-\frac{\pi s}{2}} + U\left(e^{\frac{\pi s}{2}} - e^{-\frac{\pi s}{2}}\right)\right) & \text{if } s > 0, \\ \pi(U - 1/2) & \text{if } s = 0, \end{cases} \tag{20}$$

where U is uniform on $[0, 1]$. Furthermore,

$$P_{1,s} \stackrel{\mathcal{L}}{=} \tan(W).$$

We also rediscover the standard method for generating Cauchy random variables:

$$P_{1,0} \stackrel{\mathcal{L}}{=} \tan(\pi(U - 1/2)).$$

10. Simulations.

We verified by simulation that all proposed algorithms generate variates consistent with the Pearson IV distribution. All timings were obtained on an Intel Xeon Gold 6234 CPU @ 3.30 GHz (64-bit, x86-64 GNU/Linux) and are reported in microseconds per variate, averaged over 10^5 samples. No constants were precomputed for any of the algorithms. For calibration, generating an exponential variate via $\ln(1/U)$, where $U \stackrel{\mathcal{L}}{=} \mathrm{Uniform}(0,1)$, required $0.17\,\mu$s. A Student-t variate using Bailey's method required $0.74\,\mu$s, while the skewed Cauchy method (valid for $P_{1,s}$, any s) required $2.11\,\mu$s per variate. Algorithm 3, for $a \in (1/2, 1]$, achieved timings between $5.3\,\mu$s and $6.3\,\mu$s across all admissible (a, s). The table below compares Algorithms 1, 2, 4 and 5 for various parameter combinations (a, s), where Algorithm 1 uses L as in (11). The universal Algorithm 1 demonstrates stable performance across the entire parameter space, albeit slower than Algorithms 2, 4 and 5 for restricted subsets of the parameter domain. The latter algorithms, on the other hand, are not applicable for many parameter choices.

	$a = 1$	$a = 3$	$a = 9$
$s = 1$	5.9 10.9 2.8	7.7 14 6.3 7.6	8.1 15.9 11.4 7.5
$s = 3$	16.1 23.1 2.8	79.6 12 20 7.5	143.1 15 7.4
$s = 9$	18.1 2.8	7.1	9.6

Table 1. Timing in microseconds per random variate. The color scheme: algorithm 1 (universal log-concave generator), algorithm 2 (rejection from a gaussian), algorithm 4 (rejection from an exponential), algorithm 5 (rejection from a Student-t).

II. A statistical model involving the Pearson IV family

In this section, we describe a Bayesian statistical model involving the Pearson IV family of distributions. For convenience, we define the following notation. If X is a Pearson IV random variable with parameters $a = m_0/2 + 1$ and $s = m_0\,\mu_0$, with density

$$f_X(x) = K(\mu_0, m_0)\exp\{m_0\,\mu_0\arctan(x) - (m_0/2+1)\log(x^2+1)\},\ x \in \mathbb{R},$$

having normalizing constant

$$K(\mu_0, m_0) = \frac{4^{m_0/2}\,\left|\,\Gamma\left(\frac{m_0}{2}+1-i\,\frac{m_0\,\mu_0}{2}\right)\,\right|^2}{\pi\,\Gamma(m_0)},$$

then we write

$$X \sim \text{Pearson IV}(\mu_0, m_0),$$

which has mean and variance

$$\mathbb{E}\{X\} = \mu_0 \ \text{ and } \ \mathbb{V}\{X\} = (\mu_0^2+1)/(m_0-1).$$

If X is a random variable having density belonging to the natural exponential family generated by the convolved hyperbolic secant distribution with parameters μ and n, with density

$$f_X(x) = H(x,n)\ \exp\{x\ \arctan(\mu) - \frac{n}{2}\ \log(\mu^2+1)\},\ x \in \mathbb{R},$$

where $H(x,n)$ is the density of a convolved hyperbolic secant distribution,

$$H(x,n) = \frac{2^{n-2}}{\pi\,\Gamma(n)}\left|\,\Gamma\left(\frac{n}{2}+i\,\frac{x}{2}\right)\,\right|^2,$$

then we write

$$X \sim \text{NEF-CHS}(\mu, n),$$

which has mean and variance

$$\mathbb{E}\{X\} = n\,\mu \ \text{ and } \ \mathbb{V}\{X\} = n\,(\mu^2+1).$$

Note that if $\bar{X} = X/n$, then $\mathbb{E}\{\bar{X}\} = \mu$ and $\mathbb{V}\{\bar{X}\} = (\mu^2+1)/n$. See Morris [37, 38] for additional properties of the NEF-CHS and the other five natural exponential families with quadratic variance functions. Devroye [14] defines a uniformly fast and exact algorithm for generating NEF-CHS variates.

Given these definitions, we assume that a current observation Y and a future observation Z are conditionally independent given an unknown mean parameter $\mu \in \mathbf{R}$ and a known sample size $n \geq 1$, with common density belonging to the NEF-CHS sampling family,

$$Y, Z \mid \mu \overset{ind}{\sim} \textsc{nef-chs}(\mu, n).$$

Further, we assume μ has a Pearson IV prior distribution with parameters μ_0 and m_0,

$$\mu \sim \text{Pearson IV}(\mu_0, m_0).$$

Note that μ is a parameter in the sampling family and a random variable in the prior distribution. This is a standard pattern for Bayesian statistical models.

Letting $\mu_1 = (m_0\,\mu_0 + y)/(m_0 + n)$ and $m_1 = m_0 + n$, Bayes' Theorem enables us to compute the posterior distribution of μ given $Y = y$,

$$(\mu \mid Y = y) \sim \text{Pearson IV}(\mu_1, m_1),$$

with mean $\mathbb{E}\{\mu \mid Y = y\} = \mu_1$ and variance $\mathbb{VAR}(\mu \mid Y = y) = (\mu_1^2 + 1)/(m_1 - 1)$. Because the prior and posterior both belong to the Pearson IV family, it is called the conjugate family for NEF-CHS sampling.

The prior predictive distribution of Y has density

$$f_Y(y) = H(y, n)\,\frac{K(\mu_0, m_0)}{K(\mu_1, m_1)},$$

with mean and variance

$$\mathbb{E}\{Y\} = n\,\mu_0 \;\text{ and } \mathbb{V}\{Y\} = n\,(\mu_0^2 + 1)\,\frac{m_0 + n}{m_0 - 1}.$$

We call this the "Pearson IV–NEF-CHS(n, μ_0, m_0)" distribution, in analogy with the name beta-binomial for a beta mixture of binomials, and we write

$$Y \sim \text{Pearson IV–}\textsc{nef-chs}(n, \mu_0, m_0).$$

The posterior distribution is the reference distribution for estimating μ. The prior predictive distribution is the reference distribution for model checking. See, for example, Box [6, 7]. The posterior predictive distribution of Z given $Y = y$ is

$$(Z \mid Y = y) \sim \text{Pearson IV–}\textsc{nef-chs}(n, \mu_1, m_1).$$

This is the reference distribution used to predict future observations.

Given this setup, values of μ from either the prior distribution or the posterior distribution can be generated using the algorithms developed in the earlier sections of this article. To generate values of Y from the prior predictive distribution, we use a two-step process: first, (i) generate $\mu \sim$ Pearson IV(μ_0, m_0), then (ii) generate $Y \mid \mu \sim$ NEF-CHS(μ, n). To generate values of Z from the posterior predictive distribution given $Y = y$, we use a similar two step process: first, (i) generate $\mu \sim$ Pearson IV(μ_1, m_1), then (ii) generate $Z \mid \mu \sim$ NEF-CHS(μ, n); note the change from (μ_0, m_0) to (μ_1, m_1).

12. Appendix: Bounding the normalization constant

The gamma function is defined for complex $z = x + iy$ with $x > 0$ by Euler's integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

Explicit inequalities for Euler's gamma function with real argument are often tied to Stirling's approximation (Stirling [46]). A prime example is Robbins's upper and lower bound (Robbins [43]). Olver et al. [41] summarize most of the well-known bounds.

Lemma 2. *(Batir [4]).*

$$\sqrt{2e}\left(\frac{x+1/2}{e}\right)^{x+1/2} \le \Gamma(1+x) \le \sqrt{2\pi}\left(\frac{x+1/2}{e}\right)^{x+1/2}, x > 0, \tag{21}$$

and

$$\sqrt{2\pi(x+1/6)}\left(\frac{x}{e}\right)^{x} \le \Gamma(1+x) \le \sqrt{2\pi\left(x+\frac{e^2}{2\pi}-1\right)}\left(\frac{x}{e}\right)^{x}, x \ge 1. \tag{22}$$

Lemma 3. *(Boyd [8]; see also (5.11.11) in Olver et al. [41]). For $\Gamma(z)$ with $z = x + iy$, $x > 0$,*

$$|\Gamma(z)| = \left| \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z \right| \times |1 + R(z)|\,, \tag{23}$$

where

$$|R(z)| \leq \frac{3}{2\pi^2 |z|}. \tag{24}$$

Lemma 4. *Let $a \geq 1$ and $s \geq 0$. We have $\gamma^- \leq \gamma \leq \gamma^+$, where*

$$\begin{aligned}
\gamma^* &= \frac{(a - 1/2)\left(1 + (s/2a)^2\right)^{a-1/2} e^{-s \arctan(s/2a)}}{\sqrt{\pi/e}\,(1 + 1/2a)^a \sqrt{a}}, \\
\gamma^+ &= \gamma^* \times \frac{\left(1 + \frac{3}{2\pi^2 \sqrt{a^2 + (s/2)^2}}\right)^2}{\sqrt{1 + \frac{1}{6a}}\sqrt{1 + \frac{1}{6(a+1/2)}}}, \\
\gamma^- &= \gamma^* \times \frac{\left(1 - \frac{3}{2\pi^2 \sqrt{a^2 + (s/2)^2}}\right)^2}{\sqrt{1 + \frac{0.177}{a}}\sqrt{1 + \frac{0.177}{a+1/2}}}.
\end{aligned} \tag{25}$$

Proof. Combine Batir's inequality (22) with Boyd's bounds (23) (24) after rewriting γ as follows:

$$\gamma \stackrel{\text{def}}{=} \frac{|\Gamma(a - is/2)|^2}{\Gamma(a)\Gamma(a - 1/2)\Gamma(1/2)} = \frac{(a^2 - 1/4)a\,|\Gamma(a - is/2)|^2}{\Gamma(a+1)\Gamma(a + 3/2)\sqrt{\pi}},$$

and noting that $e^2/(2\pi) - 1 < 0.177$ and

$$\left| \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z \right| = \sqrt{\frac{2\pi}{\sqrt{x^2 + y^2}}} \left(\frac{\sqrt{x^2 + y^2}}{e}\right)^x e^{-y \arctan(y/x)}. \tag{26}$$

□

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