

# A note on the Horton–Strahler number for random trees

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## Abstract

We consider the Horton–Strahler number  $S_n$  for random equiprobable binary trees with  $n$  nodes. We give a simple probabilistic proof of the well-known result that  $ES_n = \log_4 n + O(1)$  and show that for every  $x > 0$ ,  $P\{|S_n - \log_4 n| \geq x\} \leq D/4^x$ , for some constant  $D > 0$ .

**Keywords:** Analysis of algorithms; Probabilistic analysis; Horton–Strahler number; Random binary trees

## 1. Introduction

Originally used to classify river systems [4,12], the Horton–Strahler number has also been applied to binary trees. Let  $T$  be a binary tree with  $n$  nodes such that each node has at most one left and one right node. For example, with  $n = 3$  there are exactly five different trees. Let  $|T|$  be the number of nodes in  $T$ . Similarly, let  $|u|$  be the number of nodes in the subtree rooted at node  $u$  in  $T$ . For a node  $u$  in the binary tree  $T$ , let the Horton–Strahler number  $S(u)$  be defined as

$$S(u) = \begin{cases} 0 & \text{if } |u| = 0, \\ \max(S(v), S(w)) + I_{[S(v)=S(w)]} & \text{if } |u| \geq 1 \text{ and } u \text{ has children } v \text{ and } w, \end{cases}$$

where  $I_A$  is the indicator of the event  $A$ . We define  $S(T)$  as the Horton–Strahler number of the root of tree  $T$ . For example, Fig. 1 shows a tree with Horton–Strahler number three. At times, we use  $S(u)$  and  $S(T)$  interchangeably, even though  $u$  is a node and  $T$  is a tree.

The two extreme values for the Horton–Strahler number are immediately apparent. At the one extreme is a single chain of  $n$  nodes and Horton–Strahler number one (see Fig. 2). This is sometimes called a “*gourmand de la vigne*” by Viennot [15], because when viewed with its exter-

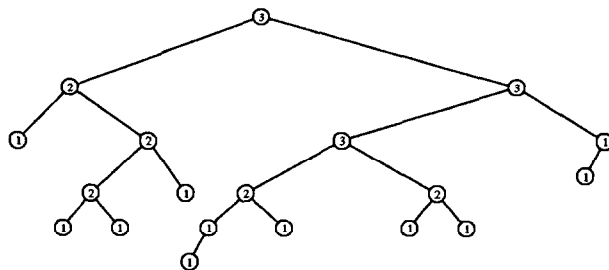


Fig. 1. A binary tree with Horton–Strahler number three.

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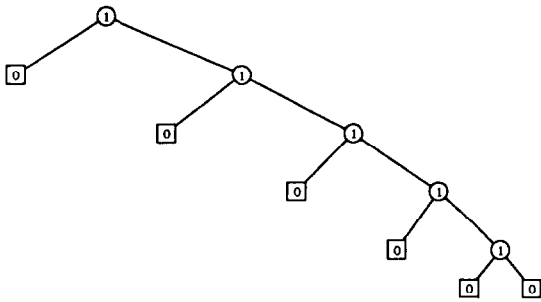


Fig. 2. A “gourmand de la vigne” with five nodes and Horton–Strahler number one.

nal nodes ( $\square$ ), the chain resembles the bottom part of a vine which is cut to improve the quality and quantity of the wine.

At the other extreme is the complete tree with  $k$  levels,  $2^k - 1$  nodes and Horton–Strahler number  $k$  (see Fig. 3). Generalizing this, it is clear that, for each binary tree  $T$  with  $n$  nodes,  $S(T) \leq \log_2 n + 1$  [6].

The Horton–Strahler number arises in computer science because of its relationship to expression evaluation. Often in a computer, an arithmetic expression is evaluated by micro-operations using registers. To facilitate this process, the expression is stored as an expression tree with the operators in the internal nodes and the operands in the external nodes. The arithmetic expression is evaluated by traversing the corresponding tree. In 1958, Ershov [1] showed that by always traversing the child node with the lower Horton–Strahler number first, the corresponding register use is minimal. Furthermore, the minimum number of registers required to evaluate an expression tree  $T$  is exactly  $S(T) + 1$ . As expression evaluation is a specialized type of postorder traversal, this can be generalized so that the minimum stack size required for a postorder traversal of binary tree  $T$  is  $S(T) + 1$  [3]. In fact,

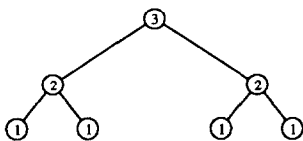


Fig. 3. A complete tree with seven nodes and Horton–Strahler number three.

the Horton–Strahler number occurs in almost every field involving some kind of natural branching pattern. More recently, the Horton–Strahler number has been used to draw trees [6,16]. Viennot [15] provides a thorough overview. See also [13], [14] and [16].

## 2. The Horton–Strahler number for equiprobable binary trees

Let an equiprobable binary tree (EBT) with  $n$  nodes be a binary tree with  $n$  nodes drawn uniformly and at random from all possible binary trees with  $n$  nodes. Let  $S_n$  be the Horton–Strahler number of a random EBT with  $n$  nodes so that  $ES_n$  and  $\text{Var } S_n$  are the corresponding expected value and variance.

The result is well-known. Under the assumption that the corresponding expression trees with  $n$  internal nodes and  $n + 1$  external nodes are equiprobable, the expected minimum number of registers needed to evaluate an arithmetic expression with  $n$  operators is  $ES_n + 1$ .

Based on exact computations of  $ES_n$  up to  $n = 100$ , Shreve [11] conjectured that  $ES_n \sim \log_4 n$ . Flajolet, Raoult and Vuillemin [2], Kemp [5], and Meir, Moon and Pounder [7–9] independently analysed  $S_n$  via recurrences and generating functions. Flajolet, Raoult and Vuillemin [2] showed that

$$ES_n = \log_4 n + D(\log_4 n) + o(1)$$

where  $|D(x)| \leq 1$  for  $x > 0$ . Kemp [5] showed that for all  $\varepsilon > 0$ ,

$$ES_n = \log_4 n + C + F(n) + O(n^{-0.5+\varepsilon})$$

where  $C = 0.82574 \dots$  is a constant and  $F(n)$  is a function with  $F(n) = F(4n)$  for all  $n > 0$  and  $-0.574 < F(n) < -0.492$ . Meir, Moon and Pounder [8] showed that  $S_n$  is very highly concentrated about  $\log_4 n$ . In fact, for any  $s > 0$ ,

$$E |S_n - \log_4 n|^s = O(1).$$

The latter result implies that

$$ES_n \sim \log_4 n \quad \text{and} \quad \text{Var } S_n = O(1).$$

### 3. A probabilistic analysis

Almost everything with respect to the Horton–Strahler number for EBTs is known. Furthermore by Chebyshev's inequality, the Meir, Moon and Pounder result [8] implies that if  $a_n$  is a sequence tending to infinity, then

$$P\{|S_n - \log_4 n| > a_n\} \rightarrow 0,$$

as  $n \rightarrow \infty$ . Using probabilistic analysis, we present a stronger result.

Let  $T$  be a binary tree with  $n$  nodes. Let  $r$  be the reduction function from binary trees to binary trees defined recursively as

$$r(\square) = () \quad (1)$$

$$r\left(\begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \square \quad \square \end{array}\right) = \square \quad (2)$$

$$r\left(\begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \square \quad T \end{array}\right) = r\left(\begin{array}{c} \circ \\ \swarrow \quad \searrow \\ T \quad \square \end{array}\right) = r(T) \quad (3)$$

$$r\left(\begin{array}{c} \circ \\ \swarrow \quad \searrow \\ T_L \quad T_R \end{array}\right) = r(T_L) \begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \square \quad \square \end{array} r(T_R) \quad (4)$$

where  $()$  is the empty tree,  $\square$  is an external node,  $\circ$  is an internal node, and  $T$ ,  $T_L$  and  $T_R$  are binary trees with at least one internal node each.

We note that

$$S(T) = S(r(T)) + 1.$$

We will show that each reduction reduces the size of the tree by a factor of about four and increases the Horton–Strahler number by one. This observation explains why  $ES_n$  is close to  $\log_4 n$ .

Let  $T' = r(T)$ . The number of external nodes in  $T'$  is equal to  $l(T)$ , the number of leaves in  $T$ . The number of (internal) nodes in  $T'$  is equal to the number of external nodes in  $T'$  minus one. Thus,  $|T'| = l(T) - 1$ . We note the following fact for reductions on EBTs.

**Fact 1.** *If each binary tree  $T$  with  $n$  nodes is equally likely, then given  $|T'| = k < n$ , each tree  $T'$  is equally likely.*

**Proof.** For any tree  $T'$ , we examine the “expansion” of  $T'$  back to  $T$  so that  $|T| = n$  and  $r(T) = T'$ . The internal nodes of  $T'$  result from Case 4 of  $r$ . The external nodes of  $T'$  result from Case 2. Therefore, in any “expansion” each external node in  $T'$  must expand to a parent node of two external nodes

$$\left(\text{i.e. } \square \rightarrow \begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \square \quad \square \end{array}\right)$$

The remaining  $n - (k + k + 1)$  internal nodes of  $T$  result from Case 3. These pairs of single-parents with only-children (external nodes)

$$\left(\begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \square \quad \square \end{array} \text{ or } \begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \square \quad \square \end{array}\right)$$

can be re-inserted anywhere in the expansion except below the leaves of  $T$ . Each combination of insertions results in a different tree  $T$ . As this argument is identical for all  $T'$ , we note that for all  $T'$  with  $k$  nodes there is an equal number of expansions to trees with  $n$  nodes.  $\square$

Before we use reductions to derive the upper and lower converging bounds for  $ES_n$ , we need the mean and variance of  $L_n$ , the number of leaves in a random binary tree. By mimicking the argument in [10] for the average internal path length of a random equiprobable binary tree, we set up the following double generating function

$$Q(w, z) = \sum_{n \geq 0} \sum_{k \geq 0} Q_{nk} w^n z^k,$$

where  $Q_{nk}$  is the number of trees with  $n$  nodes and  $k$  leaves. This, in turn, may be expressed equivalently as

$$\begin{aligned} Q(w, z) &= \sum_{\text{all trees } T} w^{|T|} z^{l(z)} \\ &= \frac{1 - \sqrt{1 - 4w(wz - w + 1)}}{2w}. \end{aligned}$$

From this, it is straightforward to derive

$$EL_n = \frac{n(n+1)}{2(2n-1)} \sim \frac{n}{4}$$

and

$$\text{Var } L_n = \frac{n(n+1)(n^2-3n+2)}{2(2n-1)^2(2n-3)} \leq \frac{n}{8},$$

for  $n \geq 3$  [6,17].

We now can start with the upper bound.

**Theorem 2.** For a random EBT with  $n$  nodes and for every  $x > 0$ ,

$$P\{S_n > \lfloor \log_4 n + x \rfloor\} \leq 1/4^x.$$

**Proof.** Let  $T_0$  be a random EBT with  $n$  nodes. Let  $T_1 = r(T_0)$ , let  $T_2 = r(T_1)$ , etc. Then,

$$\begin{aligned} E|T_{k+1}| &= E\{(I(T_k) - 1)I_{\{|T_k| \geq 1\}}\} \\ &= E\left\{\left(\frac{|T_k|(|T_k| + 1)}{2(|T_k| - 1)} - 1\right)I_{\{|T_k| \geq 1\}}\right\} \quad (\text{by [6]}) \\ &\leq E\left\{\frac{|T_k|}{4}\right\}. \end{aligned}$$

Therefore by this inequality and Fact 1,  $E\{|T_k|\} \leq E|T_0|/4^k = n/4^k$ . So by Markov's inequality,  $P\{|T_k| \geq 1\} \leq E|T_k| \leq n/4^k$ . Thus since  $[S_n - k > 0] = [|T_k| > 0]$ , we have  $P\{S_n > k\} = P\{|T_k| \geq 1\}$ . Consequently, if  $k = \lfloor \log_4 n + x \rfloor$  then  $P\{S_n > k\} \leq n/4^k \leq 1/4^x$ .  $\square$

**Theorem 3.** For a random EBT with  $n$  nodes and for every  $x \geq 1$ ,

$$P\{S_n < \lfloor \log_4 n - x \rfloor\} \leq C/4^x,$$

where  $C > 0$  is a suitable constant.

**Proof.** Let  $T_0, T_1, T_2, \dots$  be a sequence of random binary trees obtained by successive reductions and  $|T_0| = n$ . Then by Fact 1 and the bound on the variance of the leaves,  $\text{Var}\{|T_{k+1}| | T_k\} \leq c|T_k|$ , where  $c = 1/8$ . Also,  $E\{|T_{k+1}| | T_k\} \leq |T_k|/4$ . Therefore,

$$\begin{aligned} \text{Var } |T_{k+1}| &\leq E\{c|T_k|\} + \text{Var}\left\{\frac{|T_k|}{4}\right\} \\ &\leq \frac{cn}{4^k} + \frac{1}{16} \text{Var } |T_k|. \end{aligned}$$

Iterating the preceding inequality, we have

$$\begin{aligned} \text{Var } |T_{k+1}| &\leq \frac{cn}{4^k} + \frac{1}{16} \left( \frac{cn}{4^{k-1}} + \frac{1}{16} \text{Var } |T_{k-1}| \right) \\ &= \frac{cn}{4^k} \left( 1 + \frac{1}{4} \right) + \frac{1}{16^2} \text{Var } |T_{k-1}| \\ &\vdots \\ &\leq \frac{cn}{4^k} \left( 1 + \frac{1}{4} + \frac{1}{4^2} + \dots + \frac{1}{4^k} \right) \\ &\quad + \frac{1}{16^{k+1}} \text{Var } |T_0| \\ &= \frac{4}{3} \cdot \frac{cn}{4^k} \quad (\text{since } \text{Var } |T_0| = 0). \end{aligned}$$

We note by inspection  $E|T_{k+1}| \geq E|T_k|/4 - 1$ . Iterating this, we obtain

$$\begin{aligned} E|T_{k+1}| &\geq \frac{E|T_0|}{4^{k+1}} - 1 - \frac{1}{4} - \dots - \frac{1}{4^k} \\ &\geq \frac{n}{4^{k+1}} - \sum_{j=0}^{\infty} \frac{1}{4^j} = \frac{n}{4^{k+1}} - \frac{4}{3}. \end{aligned}$$

We have

$$\begin{aligned} P\{S_n \leq k\} &= P\{|T_k| = 0\} \\ &= P\{|T_k| - E|T_k| \leq -E|T_k|\} \\ &\leq \frac{\text{Var } |T_k|}{E^2 |T_k|} \quad (\text{by Chebyshev's inequality}) \\ &\leq \frac{4}{3} \cdot \frac{cn}{4^k} \cdot \frac{1}{(n/4^k - 4/3)^2}. \end{aligned}$$

If  $k = \lfloor \log_4 n - x \rfloor$  then

$$P\{S_n \leq k\} \leq \frac{c4^{x+1}}{(4^{x-1} - 4/3)^2} \leq \frac{8c}{9 \cdot 4^x} = \frac{1}{9 \cdot 4^x}$$

when  $x \geq 2$ .  $\square$

We combine the upper and lower bounds.

**Theorem 4.** For a random EBT with  $n$  nodes and for every  $x > 0$

$$P\{|\log_4 n - S_n| \geq x\} \leq D/4^x,$$

for some constant  $D > 0$ .

**Proof.** This follows directly from Theorems 2 and 3.  $\square$

From this theorem, we have the following corollaries.

**Corollary 5.** For a random EBT with  $n$  nodes and for all  $s > 0$ ,

$$E\{|S_n - \log_4 n|^s\} = O(1).$$

**Corollary 6.** For a random EBT with  $n$  nodes and for all  $\lambda \in (0, \log 4)$ ,

$$E\{e^{\lambda |S_n - \log_4 n|}\} < \infty.$$

Furthermore, Corollary 5 implies that  $\text{Var } S_n = O(1)$ . In conclusion, we remark that the results from Theorems 2, 3 and 4, and Corollaries 5 and 6 are all non-asymptotic in nature. That is, the results hold for *all*  $n$ . Finally, we see that while the trivial upper bound  $S_n \leq \log_2 n + 1$  assumed that *every* node in the tree successfully contributed to the Horton–Strahler number, Theorem 4 implies that in EBTs only approximately half the nodes actually do contribute.

Generalizations of the present results to suitably defined  $m$ -ary Horton–Strahler numbers for random  $m$ -ary trees would be interesting. For tree-drawing purposes, it would also be of interest to introduce new classes of random trees indexed by a real number  $c \in (0, 1]$ , such that  $ES_n \sim c \log_2 n$ . The EBT just corresponds to  $c = 1/2$ .

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