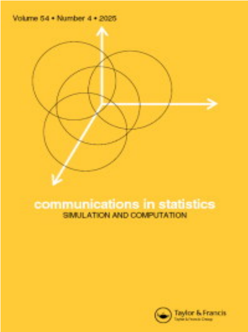

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Inequalities and simulation methods for univariate log-concave densities

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ABSTRACT

We present explicit inequalities and uniformly efficient random variate generation methods for univariate log-concave densities, even in cases where only a subset of the distribution's parameters – such as the mode, mean, or variance – are known. Additionally, we address scenarios where the density is only computable up to a normalization constant. These methods are applied to develop generators for gamma and beta densities that are uniformly fast across all parameter choices. Furthermore, we extend these techniques to other distributions, including the polynomially tilted secant and cosecant distributions, the Pearson IV distribution, Chernoff's density, Losev's distribution, and Sitenko's distribution.

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1. Introduction

In 1984, the author of this note published a black-box style rejection algorithm that can be used for any log-concave density f on the real line for which the location of a mode m is known. Defining $M = f(m)$, it is based on the inequality

$$f(x) \le M \min\left(1, e^{1-|x-m|M}\right).$$

An extension to discrete distributions on the integers followed in 1987. As the integral over the bounding curve is precisely 4, this implies that one can develop a simple von Neumann rejection algorithm (von Neumann 1963) for generating random variates from f such that the expected time is uniformly bounded over the entire family of distributions. However, one has to know m and be able to compute $f(m)$. In some cases, we do not know where m is, but have information in another form, such as the value of the mean μ or the variance σ^2. In other cases, we only know f up to an unknown normalization constant.

The purpose of this note is to fill in the gaps, and provide uniformly fast random variate generation algorithms when only partial information is available. The primary goal in this work is to develop simulation algorithms that are uniformly fast over all log-concave densities for which we have access to one or more parameters, such as m, μ, or σ^2.

We will introduce a logical nomenclature for our algorithms, as listed in Table 1, where LC refers to "log-concave".

For all these cases, we derive appropriate inequalities that will permit to develop simple black box "off-the-shelf" rejection algorithms that are uniformly fast over the class of all log-concave

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Table 1. Nomenclature for the algorithms in this note.

Name	f Available	m Known	μ Known	σ^2 Known
LC $-f-m$	Yes	Yes		
LC $-f-\mu-\sigma$	Yes	No	Yes	Yes
LC $-f-\mu$	Yes	No	Yes	
LC $-g-m$	Up to normalization constant	Yes		
LC $-g-m-\sigma$	Up to normalization constant	Yes		Yes
LC $-g-\mu-\sigma$	Up to normalization constant	No	Yes	Yes

densities. The expected number of iterations in von Neumann's rejection method is equal to the area under the bounding curve. This enables us to see what price we must pay for having only partial information. All algorithms can be adapted to discrete log-concave distributions but that will be done elsewhere.

Past work on random variate generation for log-concave densities includes Devroye (1984 and 1986, Sec. 8.2), where the first uniformly fast algorithms were developed. An extension to discrete log-concave distributions followed in 1987. Hörmann, Leydold, and Derflinger (2004) (see also Leydold and Hörmann 2000, 2001) designed black-box automated random variate generators. In some of that work, they assume the availability of the derivative of f in black-box format. For example, they automate the method of finding a threshold beyond which an exponentially decreasing cap is used, and perform an optimization along the way. Their methods are fast and efficient, but a uniform performance guarantee is still missing. Further black box attempts include Devroye (2012). If many random variates are needed for a fixed distribution, then one could use table methods – adaptive or not – to break the line up into intervals on which we can do a good job bounding f. See, e.g. the book of Devroye (1986, Chapter 8), Hörmann's table method (Hörmann 1995), or the articles of Gilks (1992) and Gilks and Wild (1992, 1993), who developed the ARS algorithm, which stands for adaptive rejection sampling. The original article of Gilks and Wild (1992) required also the derivative of f at various places (in black-box style), but Gilks (1992) did not need that. See also an exercise in Devroye (1986, Chapter 8.2) about adaptive rejection sampling. In any case, we are not aware of any uniform performance bounds for any of these methods.

The article is organized as follows. We begin in Secs. 2 and 3 with foundational inequalities for log-concave densities. In Sec. 4, we deal with densities for which the normalization constant is known or computable up to a certain explicit margin, and the mode is known. This algorithm is used in Secs. 5 through 9 to develop or outline uniformly fast algorithms for all gamma and beta densities, the Pearson IV density, Losev's density, and Chernoff's density. In Sec. 11, we deal with rather common situations in which a mode is not explicitly known, such as is the case for polynomially tilted secant and cosecant distributions, and Sitenko's distribution. In Sec. 12, the mode is known, but the normalization constant is not known. One can still develop a uniformly fast generator when the variance is known. In Sec. 13, we deal with an unknown mode and unknown normalization constant, provided that both mean and variance are known.

2. Some useful properties of univariate log-concave densities

A density is log-concave if $\log(f)$ is concave. Log-concave densities are unimodal, but may have an interval of modes. Examples of log-concave densities include the normal distribution, the gamma (a) distribution for $a \geq 1$, the exponential distribution, the uniform distribution on $[0, 1]$, the logistic distribution, Gumbel's extreme value distribution, the Laplace distribution, the hyperbolic secant distribution, the chi-square distribution with $n \geq 2$ degrees of freedom, the beta (a, b) distribution for $a, b \geq 1$, and the Weibull (a) distribution with $a \geq 1$.

Saumard and Wellner (2014) survey the main properties of log-concave distributions. Of particular utility to us is the distance between μ and m, starting with the Johnson and Rogers inequality (1951)

$$|m-\mu| \le \sqrt{3}\sigma,$$

which is valid for all unimodal distributions, which implies that it is valid for log-concave distributions. See also Dharmadhikari and Joag-Dev (1988) and Bottomley (2004). Bobkov and Ledoux (2019, proposition B2) give the following inequalities, valid for all log-concave densities with mode m, mean μ and variance σ^2:

$$\frac{1}{e\sqrt{3}} \le \sigma f(\mu) \le 1,$$
$$\frac{1}{\sqrt{12}} \le \sigma f(m) \le 1.$$

The left-hand side of the second inequality is valid for all densities, not just the log-concave densities (see Statulevicius (1965) and Hensley (1980)). The right-hand-side of the second inequality is due to Fradelizi, Guédon, and Pajor (2014); see also Bobkov and Chistyakov (2015). Other inequalities of this type include

$$f(x) \ge 2f(m)\min(F(x), 1-F(x)),$$

where F is the distribution function for f (Bobkov and Ledoux 2009, 2019). The above inequalities imply for example that $f(\mu) \le f(m) \le e\sqrt{3}f(\mu)$.

3. When the density can be computed exactly and its mode is known

When f can be computed in a black box format, and we know the position of a mode m, we are back in the situation dealt with in Devroye (1984, 1986). For completeness, we briefly recall the main inequality:

Theorem 1. *If f is log-concave with a mode at m, then*

$$f(x) \le M\ \min\left(1, e^{1-|x-m|M}\right), \tag{1}$$

where $M = f(m)$. The area under the upper bound (1) is 4. There is an infinite sequence of better bounds, the first one being

$$f(x) \le M\min\left(1, e^{-|x-m|M+|x-m|Me^{1-|x-m|M}}\right). \tag{2}$$

Proof. Without loss of generality, assume that $m=0$ and $M=1$. Then we only need to show that for all $x>1$, $f(x) \le e^{1-x}$. For such an x, the value of $f(x)$ is maximized if $f(y)=0$ for $y \notin [0,x]$ and $f(y)=e^{1-ay}$, $y \in [0,x]$. Since we must have a density, $\int_0^x e^{1-ay} = (1-e^{-ax})/a = 1$, so

$$f(x) \le e^{-ax}$$

where $1-a=e^{-ax}$ (note that $a=a(x)$ is a function of x, and is related to the Lambert W function (see, e.g. Lehtonen and Rees 2016), a multivalued function that represents the branches of the converse relation of the function $f(w)=we^w$, where w is any complex number. If $b(x)$ is any function with $1-b(x) \ge e^{-b(x)x}$, then $b(x) \le a(x)$, and therefore,

$$f(x) \le e^{-b(x)x}.$$

We first try $b(x) = 1 - 1/x$, recalling that $x > 1$. Then

$$\sup_{x>0} xe^{-x} = \frac{1}{e}$$

leads to the sought inequality,

$$1 - b(x) = \frac{1}{x} \geq e^{1-x} = e^{-b(x)x}.$$

But using this as a starting point, we can generate an infinite sequence of better lower bounds $b(x) = b_0(x) < b_1(x) < \cdots$, just by setting

$$b_{k+1}(x) = 1 - e^{-b_k(x)x}, k \geq 0.$$

For example, with

$$b_1(x) = 1 - e^{1-x},$$

we obtain the bound
$f(x) \leq e^{-x+xe^{1-x}}, x \geq 1.$ ■

Remark: MONOTONE LOG-CONCAVE FUNCTIONS If f is log-concave with mode at m, 0 to the left of m and nonincreasing to the right of m, then the areas under the upper bounds in Theorem 1 get halved, to 2 for (1) and $e/(e-1)$ for (2).

The rejection algorithm implied by (1) can be implemented as follows:

Algorithm LC $- f - m$

```
let m be the location of a mode of log-concave density f
set M = f(m)
repeat let B be Bernoulli (1/2)
        let S be a random sign
        let U be uniform on [0, 1]
        if B = 1,  then  let V be uniform on [0, 1]
                         set X ← m + SV/M
                         Accept ← [UM ≤ f(X)]
                   else  let E be exponential
                         set X ← m + S(1 + E)/M
                         Accept ← [UMe^{-E} ≤ f(X)]
until Accept
return X
        (note that X has density f)
```

When f is monotone decreasing log-concave with support on $[m, \infty)$, one can just set $S = 1$ in the above algorithm. In that case, the expected number of iterations is 2.

Conversely, we may apply the rejection method by an inversion trick, using the fact that

$$f(x) \leq \min(1, e^{1-x}), x > 0,$$

if and only if

$$x \leq 1 + \log\left(\frac{1}{f(x)}\right)$$

or

$$f^{\text{inv}}(y) \leq 1 + \log\left(\frac{1}{y}\right),$$

where f^{inv} denotes the inverse of f. The rejection method is based on the fact that if (X, Y) is uniformly distributed on $\stackrel{\text{def}}{=}\{(x,y) : y \le f(x)\}$, then X has density f. But for monotonically decreasing f on $[0,\infty)$, we also have $A = \{(x,y) : x \le f^{\text{inv}}(y)\}$. As $1 + \log(1/x)$ is the density of $U_1 U_2^B$, where U_1, U_2 are i.i.d. uniform random variables, and B is Bernoulli $(1/2)$, the following rejection method works for all decreasing log-concave densities on $[0,\infty)$ with mode at 0 and $M = 1$:

Algorithm LC − f − m, restated

```
Note: This assumes f is a decreasing log-concave density on [0,∞)
with mode f(0) = M = 1 at 0

repeat let U_1 be uniform on [0, 1], and set Z ← U_1
          with probability 1/2, set Z ← ZU_2, where U_2 is uniform on [0, 1]
            (note that Z has density 1 + log(1/z) on [0, 1])
          set Y ← 1 + log(1/Z)
          let U_3 be uniform on [0, 1]
          let X ← U_3 Y
            (note: (X, Y) is uniform in A defined above)
until Z ≤ f(X)
return X
            (note that X has density f)
```

The expected number of iterations of this algorithm is equal to the area of A, which is precisely two. If f is log-concave and is supported on $\mathbb{R}$ with mode at 0, and $f(0) = 1$, then the above algorithm is easily modified by treating the positive and negative axes separately by replacing U_3 by a uniform random variable on $[-1, 1]$. In that case, the expected number of iterations is 4. If X has a decreasing density on $[m,\infty)$ and is log-concave on that interval, then use the former algorithm with $f(x)$ replaced by $Mf(|x-m|M)$, where $M = f(m)$. If f is log-concave and is supported on $\mathbb{R}$ with mode at m, then use the latter algorithm with $f(x)$ replaced by $Mf(|x-m|M)$, where $M = f(m)$.

4. When the density can be computed up to a normalization constant

The log-concave target density with mode at m can be written as

$$f(x) = f(m)h(x),$$

where $M = f(m)$ is defined to be the normalization constant. Assume that the function h is easy to compute but the normalization constant, which in many cases involves transcendental functions or cumbersome integrals, is not. It is understood that the user has access to the function h adjusted such that $h(m) = 1$. Assume that we know that

$$M_- \le M \le M_+.$$

Then from (1), we have

$$h(x) \le \min\left(1, e^{1-|x-m|M}\right) \le \min\left(1, e^{1-|x-m|M_-}\right).$$

This leads straightforwardly to the following rejection algorithm:

Algorithm LC − g − m: version I

```
Let m be the location of a mode of log-concave density f
Note: We have access to a function h such that f(x) = f(m)h(x)
and know a constant M_ such that M = f(m) ≥ M_
```

```
repeat   let B be a fair coin flip
         let S be a random sign
         let U be uniform on [0, 1]
  if B = 1, then let V be uniform on [0, 1]
              set X ← m + SV/M_
              Accept ← [U ≤ h(X)]
         else  let E be exponential
              set X ← m + S(1 + E)/M_
              Accept ← [Ue^{-E} ≤ h(X)]
until Accept
return X
      (note that X has density f)
```

The expected number of iterations is

$$4 \times \frac{M}{M_-},$$

and thus depends only upon the relative accuracy of the inequality $M \geq M_-$. Two examples, shown in the next two sections, illustrate this procedure.

5. Example 1: the log-gamma distribution

There is no shortage of algorithms for efficiently generating gamma random variates G_a with parameter $a > 0$. However, the cases $a < 1$ and $a \geq 1$ are usually dealt with separately because of the different physionomies of the density – for $a \geq 1$, the density is log-concave, while for $a < 1$, it is monotone with an infinite peak at zero. However, $\log(G_a)$ has a log-concave density for all $a > 0$:

$$f(x) = \frac{e^{ax-e^x}}{\Gamma(a)}, x \in \mathbb{R}.$$

Its mode is $m = \log(a)$. We define

$$M = f(m) = \frac{1}{\Gamma(a)} \left(\frac{a}{e}\right)^a.$$

Thus, $f(x) = Mh(x)$, with $h(x) = \exp(a(x-m) + a - e^x)$. Assume now that we do not wish to rely on a program that computes the gamma function. We can get a good and easy-to-compute lower bound for M *via* an upper bound on $\Gamma(a)$. Among the myriad of bounds on the gamma function, we could use one developed by Batir (2008):

$$\frac{e^{4/9}}{\sqrt{\pi}} \left(\frac{a}{e}\right)^a \frac{\sqrt{2\pi(a+1/2)}}{a} e^{-\frac{1}{6(a+3/8)}} \leq \Gamma(a) \leq \left(\frac{a}{e}\right)^a \frac{\sqrt{2\pi(a+1/2)}}{a} e^{-\frac{1}{6(a+3/8)}}.$$

Using $\sqrt{\pi}/e^{4/9} < 1.136462649$, this leads to the bounds

$$M_- = \frac{a}{\sqrt{2\pi(a+1/2)}} e^{\frac{1}{6(a+3/8)}},$$
$$M_+ = 1.136462649 \frac{a}{\sqrt{2\pi(a+1/2)}} e^{\frac{1}{6(a+3/8)}}$$

The expected number of iterations in the algorithm that avoids computing $\Gamma(a)$ is

$$\frac{4M}{M_-} \leq \frac{4M_+}{M_-} = 4 * 1.136462649 < 4.55$$

.

The algorithm for the log-gamma distribution is given below. In the acceptance condition, it is convenient to replace $[U < h(X)]$ (where U is uniform on [0, 1]) by $-E' < \log(h(X))$, where E' is exponential.

Universal log-gamma generator

set $m = \log(a)$ /a mode of f/
set $M_- = \frac{a}{\sqrt{2\pi(a+1/2)}} \; e^{\frac{1}{6(a+3/8)}}$
repeat let B be a fair coin flip
let S be a random sign
let E' be exponential
if $B = 1$, then let V be uniform on [0, 1]
set $X \leftarrow m + SV/M_-$
Accept $\leftarrow [-E' \le a(X-m) + a - e^X]$
else let E be exponential
set $X \leftarrow m + S(1+E)/M_-$
Accept $\leftarrow [-E - E' \le a(X-m) + a - e^X]$
until Accept
return X
(note that $X \stackrel{\mathcal{L}}{=} \log(G_a)$)
(note that $e^X \stackrel{\mathcal{L}}{=} G_a$)

If one has access to the gamma function, one can replace M_- in the above algorithm by M throughout, and reduce the expected number of iterations to 4. We make no claims regarding the actual efficiency of this method when implemented. Luengo (2022) offers a recent survey and comparative timing. The fact that the log-gamma is log-concave for all parameters was already noted in Devroye (2012) and Xi, Tan, and Liu (2013). The rejection method of Schmeiser and Lal (1980) remains a robust standard.

6. Example 2: the beta distribution

A beta (a, b) random variate $B_{a,b}$ with $a, b > 0$ can be generated quite efficiently in a variety of direct ways, and also as a ratio of independent gamma random variables, $G_a/(G_a + G_b)$. If Y is $B_{a,b}$, and X is related to Y by the logistic transform

$$Y = \frac{1}{1+e^X},$$
$$X = \log\left(\frac{1-Y}{Y}\right),$$

then X has the log-concave density

$$f(x) = \frac{e^{bx}}{B(a,b)(1+e^x)^{a+b}}, x \in \mathbb{R},$$

where $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function. The mode is

$$m = \log\left(\frac{b}{a}\right).$$

Once again, we have the possibility of getting a simple uniformly fast algorithm over the entire parameter range. Here, we note that

$$M = f(m) = \frac{a^a b^b}{(a+b)^{a+b}\ B(a,b)}.$$

Using Batir's inequalities given above, setting

$$\rho == \frac{ab}{a+b}\sqrt{\frac{a+b+1/2}{2\pi(a+1/2)(b+1/2)}}\ e^{\frac{1}{6(a+3/8)}+\frac{1}{6(b+3/8)}-\frac{1}{6(a+b+3/8)}},$$

we get

$$M_- = \frac{\rho}{1.136462649}, M_+ = 1.136462649^2\rho.$$

Finally, we have $f(x) = f(m)h(x)$, with

$$h(x) = e^{b(x-m)}\left(\frac{e^{-m}+1}{e^{-m}+e^{x-m}}\right)^{a+b}.$$

The algorithm below uses an expected number of iterations not exceeding

$$\frac{4M_+}{M_-} = 4\times 1.136462649^3 = 5.87....$$

Universal beta generator

set $m = \log(b/a)$ /a mode of f/
set $M_- = \frac{1}{1.136462649}\frac{ab}{a+b}\sqrt{\frac{a+b+1/2}{2\pi(a+1/2)(b+1/2)}}\ e^{\frac{1}{6(a+3/8)}+\frac{1}{6(b+3/8)}-\frac{1}{6(a+b+3/8)}}$
repeat let B be a fair coin flip
  let S be a random sign
  let E' be exponential
  if $B=1$, then let V be uniform on [0, 1]
    set $X \leftarrow m + SV/M_-$
    Accept $\leftarrow \left[-E' \le b(X-m) + (a+b)\log\left(\frac{a+b}{a+be^{X-m}}\right)\right]$
  else let E be exponential
    set $X \leftarrow m + S(1+E)/M_-$
    Accept $\leftarrow \left[-E-E' \le b(X-m) + (a+b)\log\left(\frac{a+b}{a+be^{X-m}}\right)\right]$
until Accept
return X
  (note that $X \stackrel{\mathcal{L}}{=} \log((1/B_{a,b})-1)$)
  (note that $1/(1+e^X) \stackrel{\mathcal{L}}{=} B_{a,b}$)

If one has access to the gamma function, one can replace M_- in the algorithm above by M throughout, and reduce the expected number of iterations to 4. The early beta random variate algorithms were surveyed by Devroye (1986). The most recent comparative survey and study is by Luengo and Gragera (2023), who recommend Derflinger, Hörmann, and Leydold's NINIGL method (2010), Schmeiser and Babu's B4PE or b2pe algorithms (1980, 1983), the method of Schmeiser and Shalaby (1980), the switching method by Atkinson (1979) and Atkinson and Whittaker (1976, 1979), and Cheng's BA algorithm (1978). We do not claim that the algorithm given above is practically competitive.

7. Example 3: Chernoff's density

Chernoff's density is that of the symmetric random variable

$$X = \sup\{t \in \mathbb{R} : W(t) - t^2 \text{ is maximal}\},$$

where $W(t)$ denotes the standard two-sided Brownian motion starting from zero. An important density in the study of Grenander's estimate of a monotone density, it was shown by Balabdaoui and Wellner (2012, 2014) to be log-concave, as it can be written as a product of two log-concave functions, each having a Laplace transform that is inversely proportional to the Airy function. We know that Chernoff's density is symmetric about zero, but the value of the density at any point can only be obtained by iterative computation. Halting that numerical computation at any point can give bounds. In particular, the value of the mode at zero can be bounded from above and below. When developing a sampling method for Chernoff's density, which will be done elsewhere, a good starting point is the algorithm LC $-g-m$, which can do with bounds on the value at a mode.

8. Example 4: the log-Pearson IV density

Undoubtedly, the most enigmatic member of Pearson's family of distributions (Pearson 1895) is the Pearson IV distribution, which is characterized by two shape parameters, $a > 1/2$ and $s \in \mathbb{R}$. Its density on the real line is given by

$$f(x) = \frac{\rho \ e^{s\arctan(x)}}{(1+x^2)^a},$$

where, by Legendre's duplication formula,

$$\rho \stackrel{\text{def}}{=} \frac{|\Gamma(a-is/2)|^2}{\Gamma(a)\Gamma(a-1/2)\Gamma(1/2)} = \frac{4^{a-1}|\Gamma(a-is/2)|^2}{\pi\Gamma(2a-1)},$$

and Γ is the complex gamma function. We write $P_{a,s}$ to denote a Pearson type IV random variable with the given parameters. As $P_{a,s} \stackrel{\mathcal{L}}{=} -P_{a,-s}$, we can assume without loss of generality that $s \geq 0$. As noted in Exercise 1 on page 308 in Devroye (1986), when $a \geq 1$, $\arctan(P_{a,s})$ has a log-concave density on $[-\pi/2, \pi/2]$ given by

$$f(y) = \begin{cases} \rho e^{sy}(\cos^2(y))^{a-1} & \text{if } |y| \leq \frac{\pi}{2}, \\ 0 & \text{else.} \end{cases}$$

The mode of h occurs at $m = \arctan(\beta)$, where we set $\beta = s/(2(a-1))$. Also,

$$f(m) = \rho e^{s\arctan(\beta)} \left(\frac{1}{1+\beta^2}\right)^{a-1} = \rho \left(\frac{e^{2\beta\arctan(\beta)}}{1+\beta^2}\right)^{a-1}.$$

If we have constant time access to the value of ρ– which requires the complex gamma function – then we may proceed by the standard log-concave method given by algorithm LC $-f-m$. However, tight upper and lower bounds exist for the norms of the complex gamma function, requiring only standard mathematical operations, such as the exponential, logarithm and arc tangent. See, e.g. Boyd's inequality (Boyd, 1994; see also (5.11.11) in Olver et al. 2023).

These bounds can be used in algorithm based on LC $-g-m$. See Devroye and Hill (2024) for more details.

9. Example 5: Losev's density

Losev (1989) introduced the density

$$f(x)=\frac{\rho}{e^{-ax}+e^{bx}},x\in\mathbb{R},$$

where $a,b>0$ and the normalization constant is

$$\rho=\frac{a+b}{\pi}\sin\left(\frac{\pi b}{a+b}\right).$$

As this density is log-concave and its mode is located at

$$m=\frac{1}{a+b}\log\left(\frac{a}{b}\right),$$

we can apply algorithm LC $-f-m$ for uniformly fast performance. Losev also considered densities proportional to $(f(x))^r$ for $r>0$. Here, the normalization constant becomes unwieldy, yet we know a mode, which still is m. As f^r too is log-concave, we are in a situation in which we can hope to compute upper or lower bounds on the normalization constant. This is precisely the case dealt with in the algorithm LC $-g-m$. To be precise, the value of the normalized function h with $h(m)=1$ for simulating from f^r is given by

$$h(x)=\left(\frac{e^{-am}+e^{bm}}{e^{-ax}+e^{bx}}\right)^r.$$

What matters for computing bounds on the normalization constant is the behavior of h near m. Since e^{-ax} is close to $e^{-am}(1-a(x-m)+a^2(x-m)^2/2)$ and e^{bx} is close to $e^{bm}(1+b(x-m)+b^2(x-m)^2/2)$,

$$\begin{aligned}h(x)&\approx\left(\frac{e^{-am}+e^{bm}}{e^{-am}+e^{bm}+(be^{bm}-ae^{-am})(x-m)+(x-m)^2(b^2e^{bm}+a^2e^{-am})/2}\right)^r\\&=\left(\frac{1}{1+\delta(x-m)^2}\right)^r\end{aligned}$$

where

$$\delta=\frac{b^{\frac{2a+b}{a+b}}a^{\frac{b}{a+b}}+a^{\frac{2b+a}{a+b}}b^{\frac{a}{a+b}}}{2\left((b/a)^{\frac{a}{a+b}}+(a/b)^{\frac{b}{a+b}}\right)}.$$

Since $1=M\int h$, we see that if $\int h\leq H$, then $M\geq 1/H$, and thus we can set $M_-=1/H$ in our algorithm. Setting $\delta(x-m)^2=1/r$ would yield a value $h(x)\approx 1/e$ for r large enough. Thus, by lower bounding the area under h by that of a rectangle, we obtain

$$\int h\geq\frac{1}{\sqrt{r\delta}}\times\max\left(h\left(m+1/\sqrt{r\delta}\right),h\left(m-1/\sqrt{r\delta}\right)\right).$$

The upper bound should be of the same order of magnitude. This suggests the choice

$$M_- = \frac{\sqrt{r\delta}}{\max\left(h\left(m+1/\sqrt{r\delta}\right), h\left(m-1/\sqrt{r\delta}\right)\right)}.$$

This is all that is needed to apply algorithm LC $-g-m$.

10. Example 6: symmetric log-concave densities

The mode of any symmetric log-concave density is zero. Yet, we often do not explicitly know the normalization constant, as is the case of a density proportional to

$$h(x) = \exp\left(-\sum_{i=0}^{K} a_{2i}x^{2i}\right),$$

where $a_0, ..., a_K > 0$. Noting that $h(0) = 0$, we can compute an appropriate value for M_-, mimicking the argument presented above for Losev's density so that algorithm LC $-g-m$ can be applied. More specifically, borrowing an example from Gradshteyn and Ryzhik (2015, formula 3.324), we can consider

$$h(x) = \exp\left(-a^2x^2 - 2x^4\right).$$

Its mode is at the origin, and the integral is

$$\frac{e^{\frac{1}{2a^2}}K_{1/4}(1/(2a^2))}{2a\sqrt{32}},$$

where K denotes the Bessel function of imaginary argument. Those wishing to avoid the computation of the latter function can still generate random variables from this log-concave density *via* algorithm LC $-g-m$ (version I) provided that simple inequalities for this function are known.

11. An unknown mode

Sometimes, we have a log-concave density f in a black box setting, but do not know the location of a mode m. This happens for example when the analytic form of f is unwieldy. As we will see in this section, it suffices to have a location parameter to anchor the density. For example, knowledge of the mean μ will do. We begin with a few examples, which illustrate the point that it is quite often the case that one has easier access to the mean than to a mode.

11.1. Example 7: Sums of log-concave random variables

As the convolution of two log concave functions is log concave, the sum $X+Y$ of two independent log-concave random variables is a log-concave random variable. The mean is the sum of the means, but often, a mode of $X+Y$ is not readily available. That would be the case if we add a gaussian random variable with a beta random variable, for example, when both beta parameters are at least one. If we have the sum of k independent log-concave random variables, then generating them independently and summing them would incur a time complexity at least equal to k. So, it is advantageous to generate that sum directly, but then a mode is unknown, while the mean and variance are. ■

11.2. Example 8: Polynomially tilted hyperbolic secant and cosecant distributions

In what follows, ζ denotes Riemann's zeta function. The following polynomially tilted hyperbolic secant distributions are all log-concave on $[0, \infty)$:

$$\rho_1(a)\frac{x^{a-1}}{\cosh(x)}, \text{ where } \rho_1(a) = \frac{2^{a-1}}{\Gamma(a)\sum_{k=0}^{\infty}(-1)^k\left(\frac{2}{2k+1}\right)^a}, a \geq 1,$$
$$\rho_2(a)\frac{x^{a-1}}{\sinh(x)}, \text{ where } \rho_2(a) = \frac{2^{a-1}}{(2^a-1)\Gamma(a)\zeta(a)}, a \geq 2,$$
$$\rho_3(a)\frac{x^{a-1}}{\cosh^2(x)}, \text{ where } \rho_3(a) = \frac{2^{a-2}}{(1-2^{2-a})\Gamma(a)\zeta(a-1)}, a > 1,$$
$$\rho_4(a)\frac{x^{a-1}}{\sinh^2(x)}, \text{ where } \rho_4(a) = \frac{2^{a-2}}{\Gamma(a)\zeta(a-1)}, a \geq 3,$$

In the third example, we have $\rho_3(a) = 1/\log(2)$ when $a = 2$. To verify that these are indeed proper distributions, see Gradshteyn and Ryzhik (2015, formulas 3.523 and 3.527). In the four examples, the mode can only be computed by binary search or another numerical method. On the other hand, the means are explicitly available: we have

$$\mu_i = \frac{\rho_i(a)}{\rho_i(a+1)}, 1 \leq i \leq 4.$$

In the same vein, we even have simple explicit expressions for the variances. ■

11.3. Example 9: Polynomially and exponentially tilted hyperbolic secant distributions

Consider the log-concave density

$$f(x) = \rho(a,b)x^{a-1}e^{-bx}\sinh(x), x \geq 0,$$

with parameters $b > a \geq 1$, and normalization constant (see Gradshteyn and Ryzhik 2105, 3.501)

$$\frac{1}{\rho(a,b)} = \frac{\Gamma(a)}{2}\left((b-1)^{-a} - (b+1)^{-a}\right).$$

Its mode can only be a computed by an iterative algorithm. Yet, we know the mean, $\rho(a+1,b)/\rho(a,b)$. ■

11.4. Example 10: Sitenko's and related distributions

The error function

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}\, dz$$

is of crucial importance in electromagnetics, and appears in many distributions in that literature. It is the distribution function of $|N|/\sqrt{2}$, where N is a standard normal random variable. The following are some densities on the positive halfline, drawn from Gradshteyn and Ryzhik (2015, 8.28):

$$f_1(x) = \rho_1(a)x^{2a-1}(1-\Phi(x)), a \geq 1/2,$$
$$f_2(x) = \rho_2(a)(1-\Phi(x))e^{-ax^2}, a > 0,$$
$$f_3(x) = \rho_3(a)\Phi(x)e^{-ax^2}, a > 0,$$
$$f_4(x) = \rho_4(a)x\Phi(x)e^{-ax^2}, a > 0.$$

Density f_4 is called Sitenko's function (1982). The log-concavity of all densities can be verified using Mills' ratio. The normalization constants are relatively simple functions:

$$\rho_1(a) = \frac{2\sqrt{\pi a}}{\Gamma(2a-1)},$$
$$\rho_2(a) = \frac{\sqrt{\pi a}}{\arctan\left(\sqrt{a}\right)},$$
$$\rho_3(a) = \frac{\sqrt{\pi a}}{\pi/2 - \arctan\left(\sqrt{a}\right)},$$
$$\rho_4(a) = 2a\sqrt{a+1}.$$

Unfortunately, the modes are not explicitly computable. Yet, the means are readily available. To wit, the mean of f_3 is $\rho_3(a)/\rho_4(a)$, and that of f_1 is $\rho_1(a+1/2)/\rho_1(a)$. Again, we are in the situation dealt with in this section. To apply the rejection methods of this article, one would need to integrate the alternating series method to make the correct acceptance conditions (see Devroye 1981, 1986). For this, it suffices to have convergent series for $\Phi(x)$ or for Mills' ratio $(1-\Phi(x))e^{x^2}$. Explicit algorithms will be dealt with elsewhere. ∎

11.5. The algorithms

Theorem 3. For any log-concave density with mean μ and variance σ^2, for which we know functions M_+ and M_- of μ and/or σ^2 such that

$$M_- \le M = f(m) \le M_+,$$

we have (Figure 1)

$$f(x) \le \begin{cases} M_+ & \text{if } |x-\mu| \le \dfrac{1+\sqrt{3}}{M_+}, \\ \dfrac{1+\sqrt{3}}{|x-\mu|} & \text{if } \dfrac{1+\sqrt{3}}{M_+} \le |x-\mu| \le \dfrac{1+\sqrt{3}}{M_-}, \\ M_- \exp\left(1+\sqrt{3}-|x-\mu|M_-\right) & \text{if } |x-\mu| \ge \dfrac{1+\sqrt{3}}{M_-}. \end{cases}$$

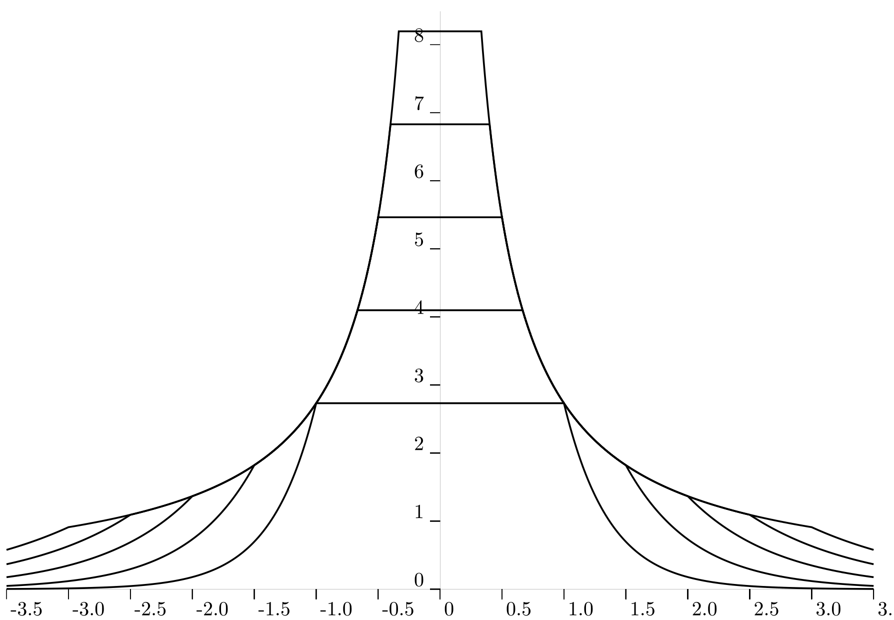

Figure 1. Five bounding envelopes are shown, for the values $M = 1+\sqrt{3}$, $\mu = 0$, $M_+ = \alpha M$, $M_- = M/\alpha$, and $\alpha \in \{1, 1.5, 2, 2.5, 3\}$.

The area under the bounding curve is

$$4 + 2\sqrt{3} + 2\left(1 + \sqrt{3}\right) \log\left(\frac{M_+}{M_-}\right).$$

Proof. Using inequality (1), we have

$$\begin{aligned}
f(x) &\le M\ \min\left(1, e^{1-|x-m|M}\right) \\
&\le M\ \min\left(1, e^{1+M\sigma\sqrt{3}-|x-\mu|M}\right) \quad \left(\text{since } |m-\mu| \le \sigma\sqrt{3}\right) \\
&\le M\ \min\left(1, e^{1+\sqrt{3}-|x-\mu|M}\right) \quad (\text{since } M\sigma \le 1) \\
&\le \max_{M_- \le t \le M_+} t\ \min\left(1, e^{1+\sqrt{3}-|x-\mu|t}\right) \\
&= \begin{cases} M_+ & \text{if } |x-\mu| \le \dfrac{1+\sqrt{3}}{M_+}, \\ \dfrac{1+\sqrt{3}}{|x-\mu|} & \text{if } \dfrac{1+\sqrt{3}}{M_+} \le |x-\mu| \le \dfrac{1+\sqrt{3}}{M_-}, \\ M_- \exp\left(1+\sqrt{3}-|x-\mu|M_-\right) & \text{if } |x-\mu| \ge \dfrac{1+\sqrt{3}}{M_-}. \end{cases}
\end{aligned}$$

The area under the bounding function is

$$2\left(1+\sqrt{3}\right) + 2\left(1+\sqrt{3}\right)\log\left(\frac{M_+}{M_-}\right) + 2.$$

■

Examples for the choices of M_- and M_+ include the following:

i. When only μ is known, we can use the choices as suggested by these inequalities:

$$M_- \stackrel{\text{def}}{=} f(\mu) \le M = f(m) \le f(\mu)e\sqrt{3} \stackrel{\text{def}}{=} M_+.$$

As $M_+/M_- = e\sqrt{3}$, we obtain an area under the bounding curve equal to

$$6 + 4\sqrt{3} + \left(1+\sqrt{3}\right)\log(3) = 15.92\ldots.$$

ii. When both μ and σ^2 are known, we could opt for these choices:

$$M_- \stackrel{\text{def}}{=} \frac{1}{\sigma\sqrt{12}} \le M = f(m) \le \frac{1}{\sigma} \stackrel{\text{def}}{=} M_+.$$

As $M_+/M_- = \sqrt{12}$, we obtain an area under the bounding curve equal to

$$4 + 2\sqrt{3} + \left(1+\sqrt{3}\right)\log(12) = 14.25\ldots.$$

iii. When both μ and σ^2 are known, and one does not mind computing $f(\mu)$, we could combine the inequalities given above as follows:

$$M_- = \max\left(f(\mu), \frac{1}{\sigma\sqrt{12}}\right),$$
$$M_+ = \min\left(f(\mu)e\sqrt{3}, \frac{1}{\sigma}\right).$$

This yields an area bound that is better than both (i) and (ii):

$$4+2\sqrt{3}+2\left(1+\sqrt{3}\right)\log\left(\sqrt{12}\,\frac{\min(1,f(\mu)\sigma e\sqrt{3})}{\max(1,f(\mu)\sigma\sqrt{12})}\right).$$

Depending upon the value of $f(\mu)\sigma$, a quantity sandwiched between $1/\left(e\sqrt{3}\right)$ and 1, the upper bound can be as good as $4+2\sqrt{3}=7.46...$, and as bad as the bound given in (ii), $4+2\sqrt{3}+\left(1+\sqrt{3}\right)\log(12)=14.25....$

Lemma 1. A random variate with density proportional to $1/x$ on *[a, b]* with $0<a<b$ can be obtained as $a^U b^{1-U}$ where U is uniform on *[0, 1]*.

The generic rejection algorithm based on the bound of Theorem 3 is as follows:

Algorithms LC $-f-\mu$ and LC $-f-\mu-\sigma$, version I

```
let μ and σ² be the mean and variance of a log-concave density f
let M₊ and M₋ be functions of μ and σ
  as in Theorem 3 and the examples given above
set p1 = 1 + √3, p2 = (1 + √3) log (M₊/M₋), p3 = 1, q = p1 + p2 + p3
repeat let S be a random sign
        let U and V be uniform on [0, 1]
        if Vq ≤ p1 then let W be uniform [0, 1]
                        set X ← μ + SW(1 + √3)/M₊
                        Accept ← [UM₊ ≤ f(X)]
        else if Vq ≤ p1 + p2   let W be uniform on [0, 1]
                              set Y ← (1 + √3)/(M₊^W M₋^(1−W))
                              set X ← μ + SY
                              Accept ← [U(1 + √3)/Y ≤ f(X)]
        else let E be exponential
              set X ← μ + S(1 + √3 + E)/M₋
              Accept ← [UM₋e^(−E) ≤ f(X)]
until Accept
return X
        (note that X has density f)
```

One can slightly improve matters with a more careful analysis, at the expense of an algorithm that takes fewer iterations on average, but requires a few more computations in each iteration. This is based on the following Theorem.

Theorem 4. For any log-concave density with mean μ and variance $\sigma^2>0$, we have

$$f(x)\le\begin{cases}\dfrac{1}{\sigma} & \text{if } |x-\mu|\le\left(1+\sqrt{3}\right)\sigma,\\[2mm] \dfrac{1}{|x-\mu|-\sigma\sqrt{3}} & \text{if } \left(\sqrt{3}+\sqrt{12}\right)\sigma\ge|x-\mu|\ge\left(1+\sqrt{3}\right)\sigma,\\[2mm] \dfrac{1}{\sigma\sqrt{12}}e^{\frac{3}{2}-\frac{|x-\mu|}{\sigma\sqrt{12}}} & \text{if } |x-\mu|\ge\left(\sqrt{3}+\sqrt{12}\right)\sigma.\end{cases}$$

The area under the bounding curve is $2\left(2+\sqrt{3}\right)+\log(12)=9.94900....$

Proof. We begin with the inequality

$$f(x)\le M\,\min\left(1,e^{1-|x-m|M}\right),x\in\mathbb{R}.$$

By the Johnson and Rogers inequality (1951), $|x-m|\ge|x-\mu|-|\mu-m|\ge|x-\mu|-\sigma\sqrt{3}$. Thus,

$$
\begin{aligned}
f(x) &\le M\ \min\left(1, e^{1+\sqrt{3}\sigma M-|x-\mu|M}\right) \\
&\le \max_{\frac{1}{\sqrt{12}\sigma}=M_-\le t\le M_+=\frac{1}{\sigma}} t\ \min\left(1, e^{1+\sqrt{3}\sigma t-|x-\mu|t}\right) \\
&= \begin{cases} M_+ & \text{if } |x-\mu| \le \sqrt{3}\sigma + \frac{1}{M_+}, \\ \frac{1}{|x-\mu|-\sigma\sqrt{3}} & \text{if } \sqrt{3}\sigma + \frac{1}{M_-} \ge |x-\mu| \ge \sqrt{3}\sigma + \frac{1}{M_+}, \\ M_- e^{1+\sqrt{3}\sigma M_- - |x-\mu|M_-} & \text{if } |x-\mu| \ge \sqrt{3}\sigma + \frac{1}{M_-}. \end{cases} \\
&= \begin{cases} \frac{1}{\sigma} & \text{if } |x-\mu| \le \left(1+\sqrt{3}\right)\sigma, \\ \frac{1}{|x-\mu|-\sigma\sqrt{3}} & \text{if } \left(\sqrt{3}+\sqrt{12}\right)\sigma \ge |x-\mu| \ge \left(1+\sqrt{3}\right)\sigma, \\ \frac{1}{\sigma\sqrt{12}} e^{\frac{3}{2}-\frac{|x-\mu|}{\sigma\sqrt{12}}} & \text{if } |x-\mu| \ge \left(\sqrt{3}+\sqrt{12}\right)\sigma. \end{cases}
\end{aligned}
$$

The area under the bounding curve is

$$
2\left(1+\sqrt{3}\right) + 2\log\left(\sqrt{12}\right) + \frac{\sqrt{12}}{\sqrt{3}} = 2\left(2+\sqrt{3}\right) + \log(12) = 9.94900\ldots.
$$

■

The rejection method based on the bound of Theorem 4 requires the weights (areas) of each of the three parts of the bounding curve, namely, $p_1 = 1+\sqrt{3}$, $p_2 = \log\left(\sqrt{12}\right)$, and $p_3 = 1$.

Algorithm LC $-f-\mu-\sigma$, version II

```
let μ and σ² be the mean and variance of a log-concave density f
set p₁ = 1 + √3, p₂ = log(√12), p₃ = 1, q = p₁ + p₂ + p₃
repeat let S be a random sign
        let U and V be uniform on [0, 1]
        if Vq ≤ p₁ then let W be uniform [0, 1]
                        set X ← μ + SW(1 + √3)σ
                        Accept ← [U/σ ≤ f(X)]
        else if Vq ≤ p₁ + p₂ let W be uniform [0, 1]
                            set Y ← (√12)^W
                            set X ← μ + Sσ(√3 + Y)
                            Accept ← [U/(σY) ≤ f(X)]
        else let E be exponential
              set X ← μ + S(√3 + √12(1 + E))σ
              Accept ← [Ue^(−E)/(σ√12) ≤ f(X)]
until Accept
return X
        (note that X has density f)
```

The penalty for not knowing a mode m is at worst less than 150%. In any case, the expected complexity of the algorithm is uniformly bounded over all choices of μ and σ^2, so the algorithm given above can be used "off the shelf". One can imagine that there are situations in which one knows about the log-concavity of f and is given μ but not σ^2. Thanks to the inequality in Theorem 5, a uniformly fast sampler is still available.

Theorem 5. For any log-concave density with mean μ, we have

$$f(x) \leq \begin{cases} f(\mu)e\sqrt{3} & \text{if } |x-\mu| \leq \frac{1+1/\left(e\sqrt{3}\right)}{f(\mu)}, \\ \frac{1}{|x-\mu|-1/f(\mu)} & \text{if } \frac{2}{f(\mu)} \geq |x-\mu| \geq \frac{1+1/\left(e\sqrt{3}\right)}{f(\mu)}, \\ f(\mu)\exp\left(2-\frac{|x-\mu|}{f(\mu)}\right) & \text{if } |x-\mu| \geq \frac{2}{f(\mu)}. \end{cases}$$

The area under the bounding curve is $6+2e\sqrt{3}+\log(3) = 16.51\ldots$.

Proof. We begin with the inequality

$$f(x) \leq M \min\left(1, e^{1-|x-m|M}\right), x \in \mathsf{R}.$$

By a simple argument and the unimodality of f, $|\mu - m|f(\mu) \leq 1$. Thus, using the inequality $f(\mu) \leq M \leq f(\mu)e\sqrt{3}$,

$$\begin{aligned} f(x) &\leq M \min\left(1, e^{1+\frac{M}{f(\mu)}-|x-\mu|M}\right) \\ &\leq \max_{f(\mu)=M_- \leq t \leq M_+ = f(\mu)e\sqrt{3}} t \min\left(1, e^{1+\frac{t}{f(\mu)}-|x-\mu|t}\right) \\ &= \begin{cases} M_+ & \text{if } |x-\mu| \leq \frac{1}{f(\mu)}+\frac{1}{M_+}, \\ \frac{1}{|x-\mu|-1/f(\mu)} & \text{if } \frac{1}{f(\mu)}+\frac{1}{M_+} \leq |x-\mu| \leq \frac{1}{f(\mu)}+\frac{1}{M_-}, \\ M_- e^{1+\frac{M_-}{f(\mu)}-|x-\mu|M_-} & \text{if } |x-\mu| \geq \frac{1}{f(\mu)}+\frac{1}{M_-} \end{cases} \\ &= \begin{cases} f(\mu)e\sqrt{3} & \text{if } |x-\mu| \leq \frac{\left(1+\frac{1}{e\sqrt{3}}\right)}{f(\mu)}, \\ \frac{1}{|x-\mu|-\frac{1}{f(\mu)}} & \text{if } \frac{\left(1+\frac{1}{e\sqrt{3}}\right)}{f(\mu)} \leq |x-\mu| \leq \frac{2}{f(\mu)}, \\ f(\mu)e^{2-\frac{|x-\mu|}{f(\mu)}} & \text{if } |x-\mu| \geq \frac{2}{f(\mu)}. \end{cases} \end{aligned}$$

The area under the bounding function is

$$2\left(f(\mu)e\sqrt{3}\frac{\left(1+\frac{1}{e\sqrt{3}}\right)}{f(\mu)}+\log\left(e\sqrt{3}\right)+\frac{f(\mu)}{f(\mu)}\right) = 6+2e\sqrt{3}+\log(3) = 16.51\ldots.$$

Algorithm LC $-f-\mu$, version II

let μ be the mean of a log-concave density f
set $p_1 = 1+e\sqrt{3}$, $p_2 = 1+\log\left(\sqrt{3}\right)$, $p_3 = 1$, $q = p_1+p_2+p_3$
let $M_+ = f(\mu)e\sqrt{3}$ and $M_- = f(\mu)$
repeat let S be a random sign
    let U and V be uniform on [0, 1]

```
        if Vq ≤ p₁    then   let W be uniform on [0, 1]
                             set X ← μ + S(1 + 1/(e√3))W/M₋
                             Accept ← [UM₊ ≤ f(X)]
        else if Vq ≤ p₁ + p₂   let W be uniform on [0, 1]
                               set Y ← 1/(M₊^W M₋^(1−W))
                               set X ← μ + S(1/M₋ + Y)
                               Accept ← [U/Y ≤ f(X)]
        else let E be exponential
             set X ← μ + S(2 + E)/M₋
             Accept ← [UM₋e^(−E) ≤ f(X)]
until Accept
return X
        (note that X has density f)
```

12. The mode is known but not the normalization constant

Assume that we can compute h, a function proportional to the target density f. Then $h(x) = f(x)/M$, where $M = f(m)$ and m is a mode of f. Thus, we have $h(m) = 1$ and $h(x) \leq \exp(1 - |x - m|M)$. As we need an upper bound that does not depend upon the computation of f at any point, it is necessary to obtain a lower bound for M to proceed, such as

$$M \geq \frac{1}{\sigma\sqrt{12}}.$$

This yields the bound

$$h(x) \leq \min\Big(1, \exp\left(1 - (|x - m|/(\sigma\sqrt{12}))\right)\Big),$$

which leads directly to a rejection algorithm with bounding area constant

$$\frac{4\sigma\sqrt{12}}{\int h} = 8\sqrt{3}f(m)\sigma \leq 8\sqrt{3} = 13.85.....$$

As $f(m)\sigma \geq 1/\sqrt{12}$, the upper bound above can be as good as 4.

Algorithm LC $- g - m - \sigma$: version II

```
let m be the location of a mode of log-concave density f
let h be a function proportional to f scaled so that h(m) = 1
let σ² be the variance of f
repeat let B be a fair coin flip
        let S be a random sign
        let U be uniform on [0, 1]
        if B = 1, then let V be uniform on [0, 1]
                       set X ← m + SVσ√12
                       Accept ← [U ≤ h(X)]
                  else let E be exponential
                       set X ← m + S(1 + E)σ√12
                       Accept ← [Ue^(−E) ≤ h(X)]
until Accept
return X
      (note that X has density f)
```

13. Neither a mode nor the normalization constant is known

In some applications, we do not know the location of a mode, and do not have access to the values $f(x)$ but only to those of a function h proportional to f. Assume however that we know the mean μ and variance σ^2. We calibrate h such that $h(\mu) = 1$. Combining the bound of the previous section with the Johnson-Rogers inequality, and $h(m) \le h(\mu) \times e\sqrt{3} = e\sqrt{3}$, we obtain

$$\begin{aligned} h(x) = \frac{f(x)}{f(\mu)} &\le h(m)\min\left(1, \exp\left(1 - \frac{|x-m|}{\sigma\sqrt{12}}\right)\right) \\ &\le h(m)\min\left(1, \exp\left(1 + \frac{|m-\mu|}{\sigma\sqrt{12}} - \frac{|x-\mu|}{\sigma\sqrt{12}}\right)\right) \\ &\le h(m)\min\left(1, \exp\left(\frac{3}{2} - \frac{|x-\mu|}{\sigma\sqrt{12}}\right)\right) \\ &\le e\sqrt{3}\ \min\left(1, \exp\left(\frac{3}{2} - \frac{|x-\mu|}{\sigma\sqrt{12}}\right)\right). \end{aligned}$$

The ratio of the areas under the bounding curve and $\int h = 1/f(\mu)$ is

$$\frac{2e\sqrt{3}}{1/f(m)} \times \left(\frac{3\sigma\sqrt{12}}{2} + \sigma\sqrt{12}\right) = 30ef(m)\sigma \le 30e = 81.54...,$$

as $f(m)\sigma \le 1$.

Algorithm LC $-\ g - \mu - \sigma$

```
let h be a function proportional to f calibrated so that h(μ) = 1
let μ and σ² be the mean and variance of f
repeat let B be Bernoulli (3/5)
       let S be a random sign
       let U be uniform on [0, 1]
       if B = 1, then let V be uniform on [0, 1]
                      set X ← μ + SVσ√27
                      Accept ← [Ue√3 ≤ h(X)]
                 else let E be exponential
                      set X ← μ + S(1 + E)σ√12
                      Accept ← [U√3e^(1−E) ≤ h(X)]
until Accept
return X
       (note that X has density f)
```

14. Conclusion

We developed "off-the-shelf" algorithms that are uniformly efficient across the entire family of univariate log-concave densities. These generators are versatile and operate under a variety of conditions. For instance, if the density f is provided in a black-box format, knowing just a mode or the mean is sufficient. When f is available in analytic form, even more options become accessible, and the normalization constant is not required. For example, knowledge of the mean and variance alone suffices. Furthermore, by extending the methods presented in this article, generators can be developed even when the mode, mean, or variance are only known within certain bounds.

Extending these results to log-concave densities in $\mathbb{R}^d$ introduces new challenges. Avoiding an exponential explosion in computational complexity with respect to dimension appears nearly inevitable. Of particular interest are uniform densities on compact convex sets in R^d, which present both theoretical and practical opportunities for further exploration.

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