On exact simulation algorithms for some distributions related to Brownian motion and Brownian meanders

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ABSTRACT. We survey and develop exact random variate generators for several distributions related to Brownian motion, Brownian bridge, Brownian excursion, Brownian meander, and related restricted Brownian motion processes. Various parameters such as maxima and first passage times are dealt with at length. We are particularly interested in simulating process variables in expected time uniformly bounded over all parameters.

KEYWORDS AND PHRASES. Random variate generation. Brownian motion. Brownian meander. Rejection method. Simulation. Monte Carlo method. Expected time analysis. Probability inequalities.

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Introduction

The purpose of this note is to propose and survey efficient algorithms for the exact generation of various functionals of Brownian motion $\{B(t), 0 \le t \le 1\}$. Many applications require the simulation of these processes, often under some restrictions. For example, financial stochastic modeling (Duffie and Glynn, 1995, Calvin, 2001, McLeish, 2002) and the simulation of solutions of stochastic differential equations (Kloeden and Platen, 1992, Beskos and Roberts, 2005) require fast and efficient methods for generating Brownian motion restricted in various ways. Exact generation of these processes is impossible as it would require an infinite effort. But it is possible to exactly sample the process at a finite number of points that are either fixed beforehand or chosen "on the fly", in an adaptive manner. Exact simulation of various quantities related to the processes, like maxima, first passage times, occupation times, areas, and integrals of functionals, is also feasible. Simulation of the process itself can be achieved by three general strategies.

(i) Generate the values of $B(t)$ at $0 = t_0 < t_1 < \cdots < t_n = 1$, where the t_i's are given beforehand. This is a <u>global</u> attack of the problem.

(ii) Simulation by subdivision. In the popular binary division (or "bridge sampling") method (see, e.g., Fox, 1999), one starts with $B(0)$ and $B(1)$, then generates $B(1/2)$, then $B(1/4)$ and $B(3/4)$, always refining the intervals dyadically. This can be continued until the user is satisfied with the accuracy. One can imagine other situations in which intervals are selected for sampling based on some criteria, and the sample locations may not always be deterministic. We call these methods <u>local</u>. The fundamental problem here is to generate $B(\lambda t + (1-\lambda)s)$ for some $\lambda \in (0,1)$, given the values $B(t)$ and $B(s)$.

(iii) Generate the values of $B(t)$ sequentially, or by extrapolation. That is, given $B(t)$, generate $B(t+s)$, and continue forward in this manner. We call this a <u>linear</u> method, or simply, an <u>extrapolation</u> method.

We briefly review the rather well-known theory for all strategies. Related simulation problems will also be discussed. For example, in case (ii), given an interval with certain restrictions at the endpoints, exact simulation of the minimum, maximum, and locations of minima and maxima in the interval becomes interesting. Among the many possible functionals, maxima and minima stand out, as they provide a rectangular cover of the sample path $B(t)$, which may of interest in some applications. Brownian motion may be restricted in various ways, e.g., by being nonnegative (Brownian meander), by staying within an interval (Brownian motion on an interval), or by attaining a fixed value at $t = 1$ (Brownian bridge). This leads to additional simulation challenges that will be discussed in this paper.

We keep three basic principles in mind, just as we did in our book on random variate generation (Devroye, 1986). First of all, we are only concerned with exact simulation methods, and to achieve this, we assume that real numbers can be stored on a computer, and that standard algebraic operations, and standard functions such as the trigonometric, exponential and logarithmic functions, are exact. Secondly, we assume that we have a source capable of producing an i.i.d. sequence of uniform $[0,1]$ random variables $U_1, U_2, U_3, \ldots$. Thirdly, we assume that all standard operations, function evaluations, and accesses to the uniform random variate generator take one unit of time. Computer scientists refer to this as the RAM model of computation. Under the latter hypothesis, we wish to achieve uniformly bounded expected complexity (time) for each of the distributions that we will be presented with. The uniformity is with respect to the parameters of the distribution. Users will appreciate not having to worry about bad input parameters. Developing a uniformly fast algorithm is often a challenging and fun exercise. Furthermore, this aspect has often been neglected in the literature, so we hope that this will make many applications more efficient.

The paper is divided into logical sections:

We blend a quick survey of known results with several new algorithms that we feel are important in the exact simulation of Brownian motion, and for which we are not aware of uniformly efficient exact methods. The new algorithms apply, for example, to the joint location and value of the maximum of a Brownian bridge, the value of a Brownian meander on a given interval when only the values at its endpoints are given, and the the maximum of a Brownian meander with given endpoint.

This paper is a first in a series of papers dealing with the simulation of Brownian processes, focusing mainly on the process itself and simple parameters such as the maximum and location of the maximum in such processes. Further work is needed for the efficient and exact simulation of passage times, occupation times, areas (like the area of the Brownian excursion, which has the Airy distribution, for which no exact simulation algorithm has been published to date), the maximum of Bessel bridges and Bessel processes of all dimensions.

Notation

We adopt Pitman's notation (see, e.g., Pitman, 1999) and write

$B(t)$	Brownian motion, $B(0) = 0$,
$B_r(t)$	Brownian bridge: same as B conditional on $B(1) = r$,
$B^{\rm br}(t)$	standard Brownian bridge: same as B_r with $r = 0$,
$B^{\rm me}(t)$	Brownian meander: same as B conditional on $B(t) \geq 0$ on $[0, 1]$,
$B_r^{\rm me}(t)$	restricted Brownian meander: same as $B^{\rm me}$ conditional on $B^{\rm me}(1) = r$,
$B^{\rm ex}(t)$	Brownian excursion: same as $B_r^{\rm me}$ with $r = 0$.

Conditioning on zero probability events can be rigorously justified either by weak limits of some lattice walks or as weak limits of processes conditioned on ϵ-probability events and letting $\epsilon \downarrow 0$ (see, e.g., Durrett, Iglehart and Miller (1997), and consult Bertoin and Pitman (1994) or Borodin and Salminen (1996) for further references). The absolute values of the former three processes, also called reflected Brownian motion and reflected Brownian bridge will only be briefly mentioned.

The maxima of these processes on $[0, 1]$ are denoted, respectively, by

$$M, M_r, M^{\rm br}, M^{\rm me}, M_r^{\rm me}, M^{\rm ex}.$$

In what follows, we reserve the notation $N, N', N'', N_1, N_2, \ldots$ for i.i.d. standard normal random variables, $E, E', E'', E_1, E_2, \ldots$ for i.i.d. exponential random variables, $U, U', U'', U_1, U_2, \ldots$ for i.i.d. uniform $[0, 1]$ random variables, and G_a for a gamma random variable of shape parameter $a > 0$. All random variables appearing together in an expression are independent. Thus, $U - U = 0$ but $U - U'$ has a triangular density. The symbol $\stackrel{\mathcal{L}}{=}$ denotes equality in distribution. We use ϕ for the normal density, and Φ for its distribution function. Convergence in distribution is denoted by $\stackrel{\mathcal{L}}{\to}$. The notation $\equiv$ means equality in distribution as a process. Also, we use "$X \in dy$" for "$X \in [y, y + dy]$".

Brownian motion: Survey of global and local strategies

We recall that $B(1) \stackrel{\mathcal{L}}{=} N$ and that for $0 \leq t \leq 1$, $\{B(ts), 0 \leq t \leq 1\} \stackrel{\mathcal{L}}{=} \{\sqrt{s} B(t), 0 \leq t \leq 1\}$. Furthermore, there are many constructions that relate the sample paths of the processes. Most useful is the definition, which states that for any $t_0 < t_1 < \cdots, < t_n$, we have that

$$(B(t_1) - B(t_0), \cdots, B(t_n) - B(t_{n-1})) \stackrel{\mathcal{L}}{=} \left(\sqrt{t_1 - t_0}\, N_1, \ldots, \sqrt{t_n - t_{n-1}}\, N_n\right).$$

The simplest representation of Brownian bridges is the drift decomposition of B_r: assuming a bridge on $[0, 1]$ with endpoint r, we have

$$B_r(t) \equiv B(t) + t(r - B(1)), 0 \leq t \leq 1.$$

Thus, given $B(t_i)$ at points $t_0 = 0 < t_1 < \cdots < t_n = 1$, we immediately have $B_r(t_i)$ by the last formula.

As $B(1)$ is a sum of two independent components, $B(t)$ and $B(1) - B(t) \stackrel{\mathcal{L}}{=} B(1-t)$, so that for a fixed t,

$$B_r(t) \stackrel{\mathcal{L}}{=} tr + \sqrt{t}\,(1-t)N_1 + t\sqrt{1-t}\,N_2 \stackrel{\mathcal{L}}{=} tr + \sqrt{t(1-t)}\,N.$$

This permits one to set up a simple local strategy. Given shifted Brownian motion (i.e., Brownian motion translated by a value a) with values $B(0) = a, B(1) = b$, then interval splitting can be achieved by the recipe

$$B(t) = a + t(b-a) + \sqrt{t(1-t)}\,N.$$

Introducing scaling, we have, with $B(0) = a, B(T) = b$,

$$B(t) = a + \frac{t}{T}(b-a) + \sqrt{\frac{t}{T}\left(1 - \frac{t}{T}\right)}\,N\sqrt{T}, 0 \le t \le T.$$

All further splitting can be achieved with fresh independent normal random variates. Extrapolation beyond t for Brownian motion is trivial, as $B(t+s) \stackrel{\mathcal{L}}{=} B(t) + N\sqrt{s}$, $s > 0$.

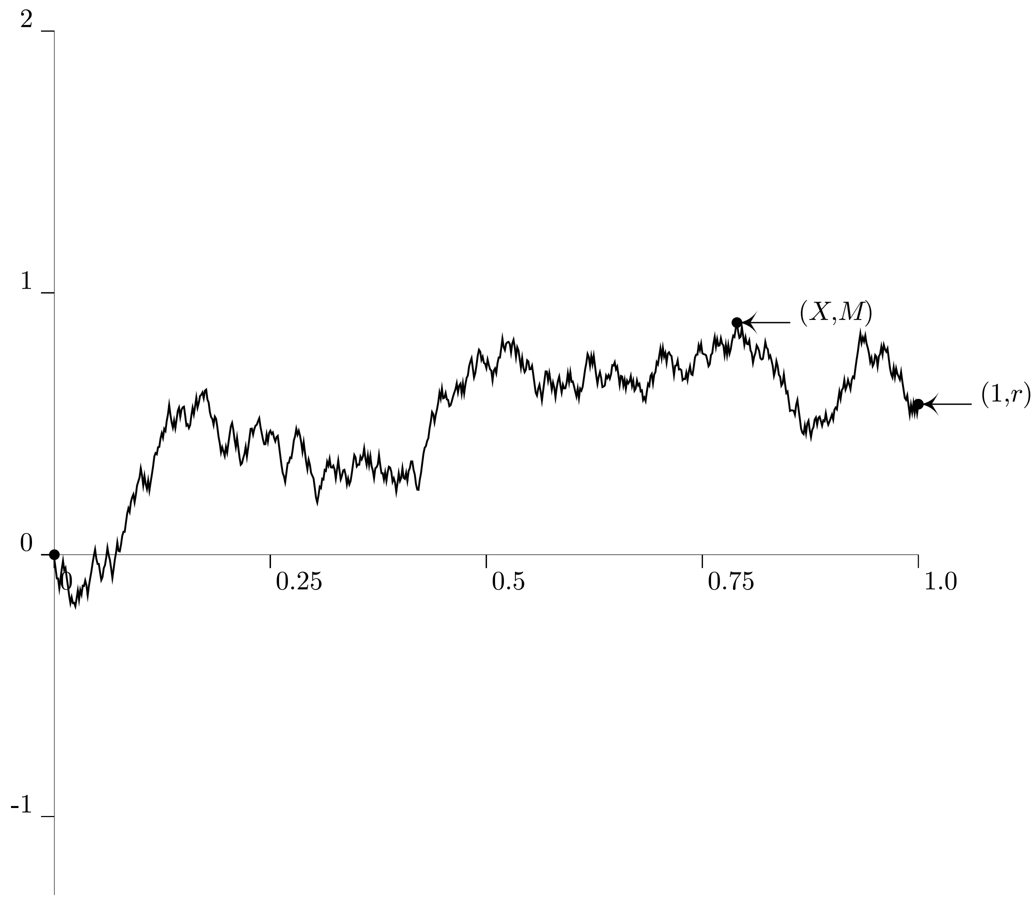

Figure 1. Simulation of Brownian motion.

In 1999, Jim Pitman published an important paper on the joint law of the various Brownian motion processes sampled at the order statistics of a uniform $[0,1]$ cloud of points. These yield various distributional identities but also fast methods of simulation. For the sake of completeness, we briefly recall his results here. The sampling period is $[0,1]$. the order statistics of n i.i.d. uniform $[0,1]$ random variables are denoted by

$$0 = U_{(0)} < U_{(1)} < \cdots < U_{(n)} < U_{(n+1)} = 1.$$

It is well-known that this sample can be obtained from a sequence of i.i.d. exponential random variables $E_1, E_2, \ldots$ in the following manner, denoting $S_i = E_1 + \cdots + E_i$:

$$U_{(i)} = \frac{S_i}{S_{n+1}}, 1 \le i \le n+1.$$

See, e.g. Shorack and Wellner (1986). Denote by $X(t)$ any of the processes defined at the outset of this paper, and let it be independent of the uniform sample. Let T_i be a time in $[U_{(i-1)}, U_{(i)}]$ when X attains its infimum on that interval. Consider then the $2n+2$-dimensional random vector

$$X_n \stackrel{\text{def}}{=} \left(X(T_1), X(U_{(1)}), X(T_2), X(U_{(2)}), \ldots, X(T_{n+1}), X(U_{(n+1)}) \right).$$

Obtain an independent uniform sample

$$0 = V_{(0)} < V_{(1)} < \cdots < V_{(n)} < V_{(n+1)} = 1,$$

which is based on an independent collection of exponentials with partial sums S'_i, $1 \le i \le n+1$, so $V_{(i)} = S'_i / S'_{n+1}$.

Proposition (Pitman, 1999). *If $X \equiv B$ and $n \ge 0$, then*

$$X_n \stackrel{\mathcal{L}}{=} \sqrt{2G_{n+3/2}} \left(\frac{S_{i-1} - S'_i}{S_{n+1} + S'_{n+1}}, \frac{S_i - S'_i}{S_{n+1} + S'_{n+1}}; 1 \le i \le n+1 \right).$$

If $X \equiv B^{\rm me}$ and $n \ge 0$, then

$$X_n \stackrel{\mathcal{L}}{=} \sqrt{2G_{n+1}} \left(\frac{S_{i-1} - S'_i}{S_{n+1} + S'_{n+1}}, \frac{S_i - S'_i}{S_{n+1} + S'_{n+1}}; 1 \le i \le n+1 \Big| \cap_{i=1}^n [S_i > S'_i] \right).$$

If $X \equiv B^{\rm br}$ and $n \ge 0$, then

$$X_n \stackrel{\mathcal{L}}{=} \sqrt{\frac{G_{n+1}}{2}} \left(U_{(i-1)} - V_{(i)}, U_{(i)} - V_{(i)}; 1 \le i \le n+1 \right).$$

If $X \equiv B^{\rm ex}$ and $n \ge 0$, then

$$X_n \stackrel{\mathcal{L}}{=} \sqrt{\frac{G_{n+1}}{2}} \left(U_{(i-1)} - V_{(i-1)}, U_{(i)} - V_{(i-1)}; 1 \le i \le n+1 \Big| \cap_{i=1}^n [U_{(i)} > V_{(i)}] \right).$$

If $X \equiv B_r$ and $n \ge 0$, then

$$X_n \stackrel{\mathcal{L}}{=} \frac{\sqrt{r^2 + 2G_{n+1}} - |r|}{2} \left(U_{(i-1)} - V_{(i)}, U_{(i)} - V_{(i)}; 1 \le i \le n+1 \right) + r \left(U_{(i-1)}, U_{(i)}; 1 \le i \le n+1 \right).$$

The random vectors thus described, with one exception, are distributed as a square root of a gamma random variable multiplied with a random vector that is uniformly distributed on some polytope of $\mathbb{R}^{2n+2}$. Global sampling for all these processes in time $O(n)$ is immediate, provided that one can generate a gamma random variates G_a in time $O(a)$. Since we need only integer values of a or integer values plus $1/2$, one can achive this by using $G_n \stackrel{\mathcal{L}}{=} E_1 + \cdots + E_n$ and $G_{n+1/2} \stackrel{\mathcal{L}}{=} E_1 + \cdots + E_n + N^2/2$.

However, there are also more sophisticated methods that take expected time $O(1)$ (see, e.g., Devroye, 1986).

There are numerous identities that follow from Pitman's proposition. For example,

$$\left(\min_{0\le t\le U} B^{\text{br}}(t), B^{\text{br}}(U), \min_{U\le t\le 1} B^{\text{br}}(t)\right) \stackrel{\mathcal{L}}{=} \sqrt{\frac{G_2}{2}}\left(-U', U-U', U-1\right).$$

This implies that $|B^{\text{br}}(U)| \stackrel{\mathcal{L}}{=} \sqrt{G_2/2}\,|U-U'| \stackrel{\mathcal{L}}{=} U\sqrt{E/2}$.

In the last statement of Pitman's result, we replaced a random variable $L_{n,r}$ (with parameter $r>0$) in section 8 of Pitman by the equivalent random variable $\sqrt{r^2+2G_{n+1}}-r$. It is easy to verify that it has the density

$$\frac{y^n(y+r)(y+2r)^n}{n!2^n} \times \exp\left(-\frac{y^2}{2}-ry\right), y>0.$$

Brownian motion: Extremes and locations of extremes

The marginal distributions of the maximum M and its location X for B on $[0,1]$ are well-known. We mention them for completeness (see, e.g., Karatzas and Shreve, 1998): X is arc sine distributed, and

$$M \stackrel{\mathcal{L}}{=} |N|.$$

The arc-sine, or beta $(1/2,1/2)$ distribution, corresponds to random variables that can be represented equivalently in all these forms, where C is standard Cauchy:

$$\frac{G_{1/2}}{G_{1/2}+G'_{1/2}} \stackrel{\mathcal{L}}{=} \frac{N^2}{N^2+N'^2} \stackrel{\mathcal{L}}{=} \frac{1}{1+C^2} \stackrel{\mathcal{L}}{=} \sin^2(2\pi U) \stackrel{\mathcal{L}}{=} \sin^2(\pi U) \stackrel{\mathcal{L}}{=} \sin^2(\pi U/2)$$
$$\stackrel{\mathcal{L}}{=} \frac{1+\cos(2\pi U)}{2} \stackrel{\mathcal{L}}{=} \frac{1+\cos(\pi U)}{2}.$$

In simulation, M is rarely needed on its own. It is usually required jointly with other values of the process. The distribution function of M_r (see Borodin and Salminen (2002, p. 63)) is

$$F(x) = 1-\exp\left(\frac{1}{2}\left(r^2-(2x-r)^2\right)\right), x \ge \max(r,0).$$

By the inversion method, this shows that

$$M_r \stackrel{\mathcal{L}}{=} \frac{1}{2}\left(r+\sqrt{r^2+2E}\right). \tag{1}$$

This was used by McLeish (2002) in simulations. Therefore, replacing r by N, we have the following joint law:

$$(M, B(1)) \stackrel{\mathcal{L}}{=} \left(\frac{1}{2}\left(N+\sqrt{N^2+2E}\right), N\right).$$

Putting $r=0$ in (1), we observe that $M^{\text{br}} \stackrel{\mathcal{L}}{=} \sqrt{E/2}$, a result due to Lévy (1939, (20)). It is also noteworthy that

$$M \stackrel{\mathcal{L}}{=} |N| \stackrel{\mathcal{L}}{=} M - B(1).$$

The rightmost result is simply due to Lévy's observation (1948) that $|B(t)|$ is equivalent as a process to $M(t) - B(t)$ where $M(t)$ is the maximum of B over $[0, t]$.

Pitman's Proposition together with the observation that $2G_{3/2} \stackrel{\mathcal{L}}{=} N^2 + 2E''$, show that

$$
\begin{aligned}
(M, B(1)) &\stackrel{\mathcal{L}}{=} \sqrt{2G_{3/2}} \times \left(\frac{E}{E+E'}, \frac{E-E'}{E+E'} \right) \\
&\stackrel{\mathcal{L}}{=} \sqrt{N^2 + 2E''} \times \left(\frac{E}{E+E'}, \frac{E-E'}{E+E'} \right) \\
&\stackrel{\mathcal{L}}{=} \left(U\sqrt{N^2+2E}, (2U-1)\sqrt{N^2+2E} \right).
\end{aligned}
$$

Furthermore, Pitman's results allow us to rediscover Lévy's result $M^{\text{br}} \stackrel{\mathcal{L}}{=} \sqrt{E/2}$. Using $E/(E+E') \stackrel{\mathcal{L}}{=} U$, we also have

$$
M \stackrel{\mathcal{L}}{=} U\sqrt{2G_{3/2}} \stackrel{\mathcal{L}}{=} U\sqrt{N^2+2E} \stackrel{\mathcal{L}}{=} \frac{1}{2}\left(N + \sqrt{N^2+2E} \right).
$$

For $x > 0$, we define the first passage time (also called hitting time)

$$
T_x = \min\{t : B(t) = x\}.
$$

For $t > 0$,

$$
\begin{aligned}
\mathbb{P}\{T_x > t\} &= \mathbb{P}\left\{ \max_{0 \le s \le t} B(s) < x \right\} \\
&= \mathbb{P}\left\{ \max_{0 \le s \le 1} B(s) < x/\sqrt{t} \right\} \\
&= \mathbb{P}\left\{ \frac{1}{2}\left(N + \sqrt{N^2+2E} \right) < x/\sqrt{t} \right\} \\
&= \mathbb{P}\left\{ \left(\frac{2x}{N + \sqrt{N^2+2E}} \right)^2 > t \right\},
\end{aligned}
$$

and therefore,

$$
T_x \stackrel{\mathcal{L}}{=} \left(\frac{x}{M} \right)^2.
$$

Simulating hitting times and maxima are in fact equivalent computational questions. The same argument can be used for Brownian meanders: the hitting time of $x > 0$ for a Brownian meander is distributed as

$$
\left(\frac{x}{M^{\text{me}}} \right)^2.
$$

Consider now the joint density of the triple $(X, M, B(1))$. Using (x, m, y) as the running coordinates for $(X, M, B(1))$, Shepp (1979) [see also Karatzas and Shreve (1998, p. 100)] showed that this density is

$$
\frac{m(m-y)}{\pi x^{3/2}(1-x)^{3/2}} \times \exp\left(-\frac{m^2}{2x} - \frac{(m-y)^2}{2(1-x)} \right), m \ge y \in \mathbb{R}, x \in (0,1).
$$

This suggests a simple method for their joint generation:

$$
(X, M, B(1)) \stackrel{\mathcal{L}}{=} \left(X \stackrel{\text{def}}{=} \frac{1+\cos(2\pi U)}{2}, \sqrt{2XE}, \sqrt{2XE} - \sqrt{2(1-X)E'} \right).
$$

This is easily seen by first noting that if $(X, M) = (x, m)$, then $B(1) \stackrel{\mathcal{L}}{=} m - \sqrt{2(1-x)E'}$. Then, given $X = x$, $M \stackrel{\mathcal{L}}{=} \sqrt{2xE}$.

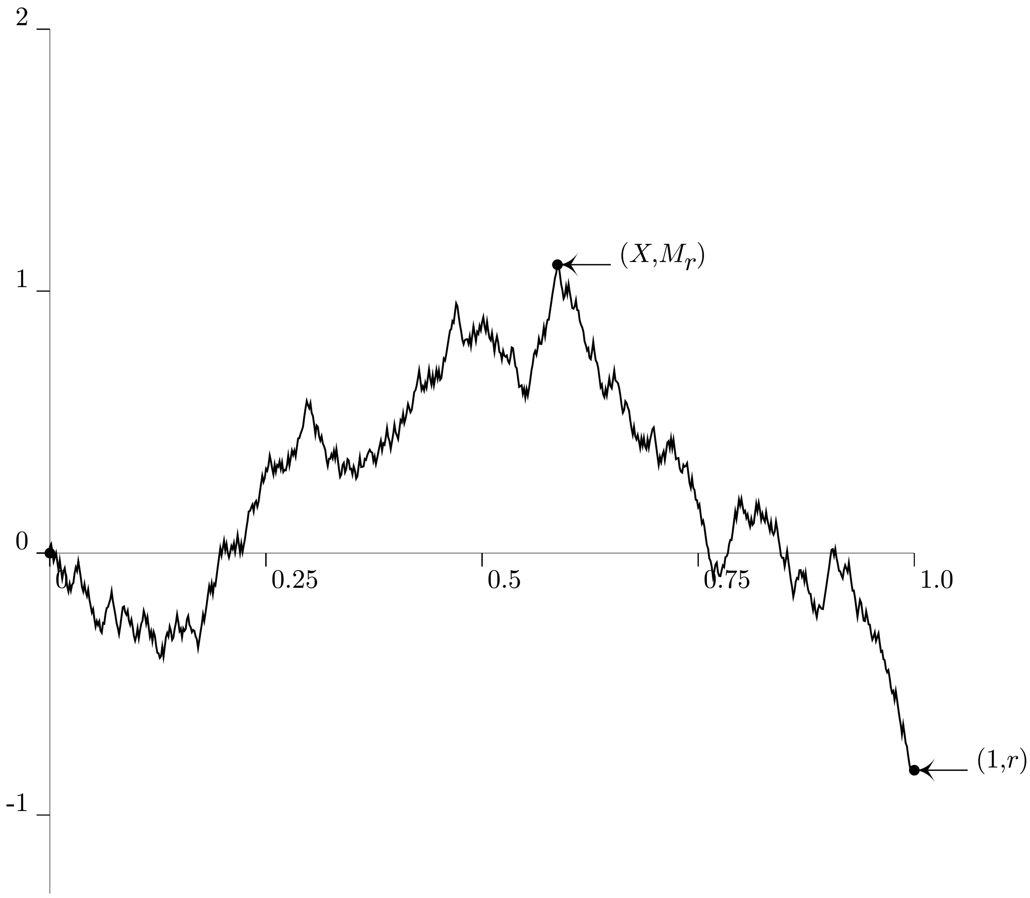

Figure 2. A simulation of a Brownian bridge from 0 to $B(1) = r$. X is the location of the maximum M_r.

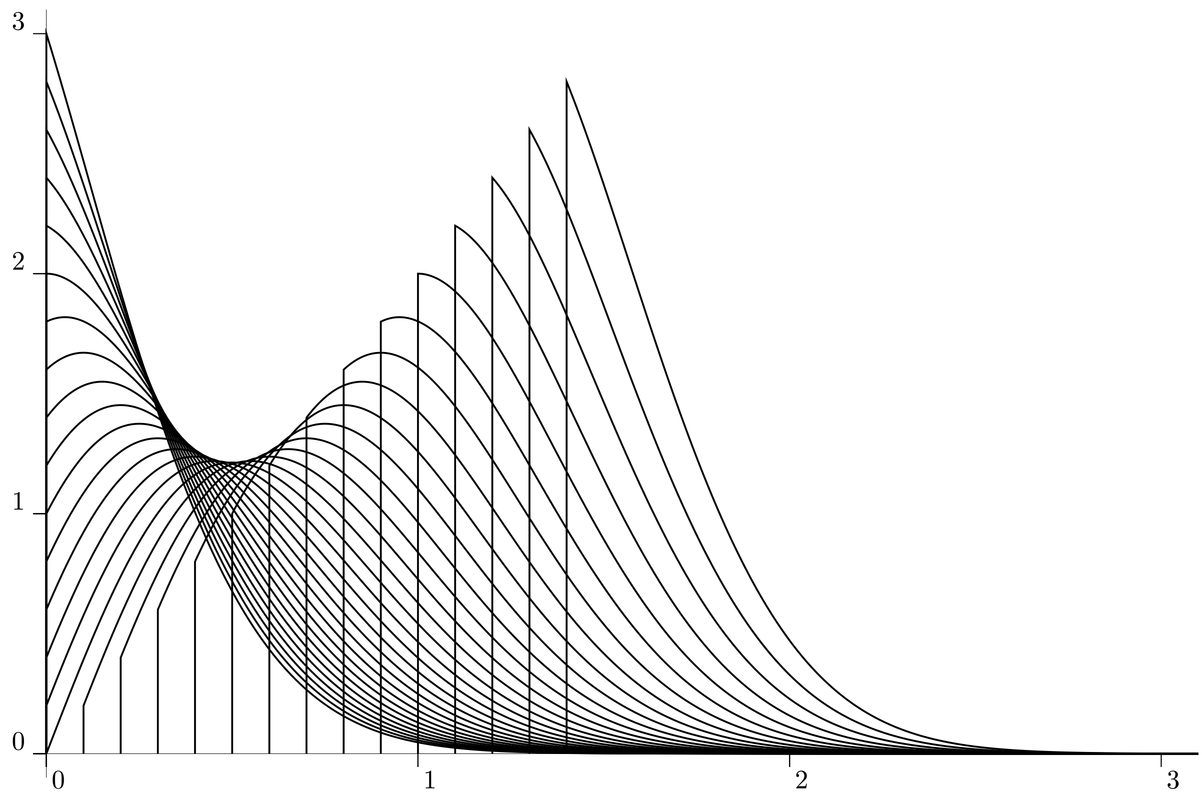

Figure 3. The density of M_r for various values of r from -1.5 to 1.4 in increments of 0.1. For all values of r, 0 excepted, there is a discontinuity at $\max(0, r)$. At $r = 0$, we recover the scaled Rayleigh density $4x\exp(-2x^2), x > 0$.

Finally, we consider the joint law of (X, M_r) for B_r. This is a bit more cumbersome, especially if we want to simulate it with expected complexity uniformly bounded over all r. The joint density can be written as

$$\frac{m(m-r)\sqrt{2\pi}\, e^{r^2/2}}{\pi x^{3/2}(1-x)^{3/2}} \times \exp\left(-\frac{m^2}{2x} - \frac{(m-r)^2}{2(1-x)}\right), 0 \le x \le 1, m \ge \max(r, 0).$$

The standard Brownian bridge: $r = 0$. The special case of the standard Brownian bridge ($r = 0$) has a simple solution. Indeed, the joint density reduces to

$$\frac{2m^2}{\sqrt{2\pi}x^{3/2}(1-x)^{3/2}} \times \exp\left(-\frac{m^2}{2x(1-x)}\right), 0 \le x \le 1, m \ge 0.$$

Integrating with respect to dm shows that X is uniform on $[0,1]$. And given X, we see that $M^{\text{br}} \stackrel{\mathcal{L}}{=} \sqrt{2X(1-X)G_{3/2}}$. Thus,

$$\left(X, M^{\text{br}}\right) \stackrel{\mathcal{L}}{=} \left(U, \sqrt{2U(1-U)G_{3/2}}\right).$$

Using Lévy's result about $M^{\rm br}$, this implies that

$$\sqrt{2U(1-U)G_{3/2}} \stackrel{\mathcal{L}}{=} \sqrt{U(1-U)(N^2+2E)} \stackrel{\mathcal{L}}{=} \sqrt{\frac{E}{2}}.\square$$

The remainder of this section deals with the more complicated case $r \neq 0$. We will simulate in two steps by the conditional method. First, the maximum M_r is generated as in (1): $(1/2)(r+\sqrt{r^2+2E})$. Call this value m for convenience. Then the random variable X is generated, which has density proportional to

$$\frac{\exp\left(-\frac{m^2}{2x} - \frac{(m-r)^2}{2(1-x)}\right)}{x^{3/2}(1-x)^{3/2}}, 0 < x < 1. \tag{2}$$

For this, we propose a rejection algorithm with rejection constant (the expected number of iterations before halting, or, equivalently, one over the acceptance probability) $R(m,r)$ depending upon m and r, uniformly bounded in the following sense:

$$\sup_r \mathbb{E}\{R(M_r, r)\} < \infty. \tag{3}$$

Note that $\sup_{r, m \geq \max(r,0)} R(m,r) = \infty$, but this is of secondary importance. In fact, by insisting only on (3), we can design a rather simple algorithm. Since we need to refer to it, and because it is fashionable to do so, we will give this algorithm a name, MAXLOCATION.

ALGORITHM "MAXLOCATION"

Case I ($m \geq \sqrt{2}$)

  Repeat Generate U, N. Set $Y \leftarrow 1 + \frac{(m-r)^2}{N^2}$

    Until $U \exp(-m^2/2) \leq Y \exp(-Y m^2/2)$

  Return $X \leftarrow 1/Y$

Case II ($m - r \geq \sqrt{2}$)

  Repeat Generate U, N. Set $Y \leftarrow 1 + \frac{m^2}{N^2}$

    Until $U \exp(-(m-r)^2/2) \leq Y \exp(-Y(m-r)^2/2)$

  Return $X \leftarrow 1 - 1/Y$

Case III ($m - r \leq \sqrt{2}$, $m \leq \sqrt{2}$)

  Repeat Generate U, N. Set $X \leftarrow$ beta $(1/2, 1/2)$

    Until $U \frac{4}{\sqrt{X(1-X)} e^2 m^2 (m-r)^2} \leq \frac{\exp(-m^2/2X - (m-r)^2/2(1-X))}{(X(1-X))^{3/2}}$

  Return X

No attempt was made to optimize the algorithm with respect to its design parameters like the cut-off points. Our choices facilitate easy design and analysis. Note also that the three cases in MAXLOCATION overlap. In overlapping regions, any choice will do. Gou (2009) has another algorithm for this, but it is not uniformly fast. However, for certain values of the paparmeters, it may beat MAXLOCATION in given implementations.

THEOREM 1. *Algorithm* MAXLOCATION *generates a random variable X with density proportional to (2). Furthermore, if m is replaced by $M_r = (1/2)(r + \sqrt{r^2 + 2E})$, then (X, M_r) is distributed as the joint location and value of the maximum of a Brownian bridge B_r. Finally, the complexity is uniformly bounded over all values of r in the sense of (3).*

PROOF. The first two cases are symmetric—indeed, X for given input values m, r is distributed as $1 - X'$, where X' has input parameters $m - r$ and $-r$. This follows from considering the Brownian motion backwards. Case I: Let X have density proportional to (2), and let $Y = 1/X$. Then Y has density proportional to

$$y \exp\left(-\frac{m^2 y}{2}\right) \times (y-1)^{-\frac{3}{2}} \exp\left(-\frac{(m-r)^2 y}{2(y-1)}\right), y > 1.$$

If $m \geq \sqrt{2}$, then the leftmost of the two factors is not larger than $\exp(-m^2/2)$, while the rightmost factor is proportional to the density of $1 + (m-r)^2/N^2$, as is readily verified. This confirms the validity of the rejection method for cases I and II. Case III: note that (2) is bounded by

$$\frac{4}{\sqrt{x(1-x)}e^2 m^2 (m-r)^2},$$

which is proportional to the beta $(1/2, 1/2)$ density. To see this, observe that $(m^2/(2x)) \exp(-(m^2/(2x))) \leq 1/e$, and $((m-r)^2/(2(1-x))) \exp(-((m-r)^2/(2(1-x)))) \leq 1/e$.

Finally, we verify (3) when the supremum is taken over te parameter ranges that correspond to the three cases. It is helpful to note that m is now random and equal to $(1/2)(r + \sqrt{r^2 + 2E})$. Thus, $m(m-r) = E/2$, a property that will be very useful. The acceptance rate in case I (using the notation of the algorithm) is

$$\begin{aligned}
\mathbb{P}\left\{U \exp(-m^2/2) \leq Y \exp(-Ym^2/2)\right\} &= \mathbb{E}\left\{Y \exp((1-Y)m^2/2)\right\} \\
&= \mathbb{E}\left\{Y \exp(-m^2(m-r)^2/2N^2)\right\} \\
&\geq \mathbb{E}\left\{\exp(-m^2(m-r)^2/2N^2)\right\} \\
&= \mathbb{E}\left\{\exp(-E^2/8N^2)\right\} \\
&\stackrel{\text{def}}{=} \delta > 0.
\end{aligned}$$

The acceptance rate for case II is dealt with in precisely the same manner—it is also at least δ. Finally, in case III, the acceptance rate is

$$\begin{aligned}
&\mathbb{P}\left\{U \frac{4}{\sqrt{X(1-X)}e^2 m^2 (m-r)^2} \leq \frac{\exp(-m^2/2X - (m-r)^2/2(1-X))}{(X(1-X))^{3/2}}\right\} \\
&= \mathbb{E}\left\{\frac{e^2 m^2 (m-r)^2 \exp(-m^2/2X - (m-r)^2/2(1-X))}{4X(1-X)}\right\} \\
&\geq \mathbb{E}\left\{\frac{e^2 m^2 (m-r)^2 \exp(-1/X - 1/(1-X))}{4X(1-X)}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left\{ \frac{e^2 E^2 \exp(-1/X(1-X))}{16X(1-X)} \right\} \\
&= \mathbb{E}\left\{ \frac{e^2 \exp(-1/X(1-X))}{8X(1-X)} \right\} \\
&\geq \mathbb{E}\left\{ \frac{e^2 \exp(-16/3)}{8(3/16)} \mathbb{1}_{[X\in[1/4,3/4]]} \right\} \\
&= \frac{1}{3e^{10/3}}.
\end{aligned}
$$

Therefore,

$$\mathbb{E}\{R(M_r, r)\} \leq \max(1/\delta, 3e^{10/3}). \square$$

The joint maximum and minimum of Brownian bridge and Brownian motion. The joint maximum and minimum of B_r can be done in two steps. First, we generate $M_r = (1/2)(r + \sqrt{r^2 + 2E})$ and then apply maxlocation to generate the location X of the maximum. Using a decomposition of Williams (1974) and Denisov (1984), we note that the process cut at X consists of two Brownian meanders, back to back. More specifically,

$$M_r - B_r(X + t), 0 \leq t \leq 1 - X,$$

is a Brownian meander with endpoint $B^{\text{me}}(1 - X) = M_r - r$. The maximum Z_1 of this process is distributed as

$$\sqrt{1-X} \times M_s^{\text{me}} \text{ with } s = \frac{M_r - r}{\sqrt{1-X}}.$$

The value M_s^{me} is generated by our algorithm maxmeander, which will be developed further on in the paper. Similarly, the process

$$M_r - B_r(X - t), 0 \leq t \leq X,$$

is a Brownian meander with endpoint $B^{\text{me}}(X) = M_r$. The maximum Z_2 of this process is distributed as

$$\sqrt{X} \times M_s^{\text{me}} \text{ with } s = \frac{M_r}{\sqrt{X}}.$$

The value M_s^{me} is again generated by our algorithm maxmeander. Putting things together, and using the Markovian nature of Brownian motion, we see that the minimum of B_r on $[0, 1]$ is equal to

$$M_r - \max(Z_1, Z_2).$$

The joint maximum and minimum for B is dealt with as above, for B_r, when we start with $r = N$. $\square$

Brownian meander: Global methods

Simple computations involving the reflection principle show that the density of $B(1)$ for Brownian motion started at $a > 0$ and restricted to remain positive on $[0, 1]$ is

$$f(x) \stackrel{\text{def}}{=} \frac{\exp\left(-\frac{(x-a)^2}{2}\right) - \exp\left(-\frac{(x+a)^2}{2}\right)}{\sqrt{2\pi}\mathbb{P}\{|N| \le a\}}, x > 0.$$

The limit of this as $a \downarrow 0$ is the Rayleigh density $x \exp(-x^2/2)$, i.e., the density of $\sqrt{2E}$. An easy scaling argument then shows that

$$B^{\text{me}}(t) \stackrel{\mathcal{L}}{=} \sqrt{2tE}.$$

This permits simulation at a single point, but cannot be used for a sequence of points.

A useful property (see Williams, 1970, or Imhof, 1984) of the Brownian meander permits care-free simulation: a restricted Brownian meander B_r^{me} can be represented as a sum of three independent standard Brownian bridges:

$$B_r^{\text{me}}(t) \equiv \sqrt{(rt + B^{\text{br}}(t))^2 + (B^{\text{br}'}(t))^2 + (B^{\text{br}''}(t))^2}.$$

This is called the three-dimensional Bessel bridge from 0 to r. We obtain B^{me} as B_r^{me} with $r = \sqrt{2E}$. B^{me} can also be obtained from the sample path of B directly: let $\tau = \sup\{t \in [0, 1] : B(t) = 0\}$. Then

$$B^{\text{me}}(t) \equiv \frac{|B(\tau + t(1-\tau))|}{\sqrt{1-\tau}}, 0 \le t \le 1.$$

This is not very useful for simulating B^{me} though.

For the standard Brownian bridge, $B^{\text{br}}(t) \equiv B(t) - tB(1)$. Maintaining three independent copies of such bridges gives a simple global algorithm for simulating B_r^{me} at values $0 = t_0 < t_1 < \cdots < t_n = 1$, based on the values $B^{\text{br}}(t_i)$.

There is also a way of simulating B_r^{me} inwards, starting at $t = 1$, and then obtaining the values at points $1 = t_0 > t_1 > t_2 > \cdots > 0$. Using $B^{\text{br}}(t) \stackrel{\mathcal{L}}{=} \sqrt{t(1-t)}\, N$, we have

$$\begin{aligned} B_r^{\text{me}}(t) &\stackrel{\mathcal{L}}{=} \sqrt{(rt + \sqrt{t(1-t)}\, N)^2 + t(1-t)(N_2^2 + N_3^2)} \\ &\stackrel{\mathcal{L}}{=} \sqrt{(rt + \sqrt{t(1-t)}\, N)^2 + 2Et(1-t)} \\ &\stackrel{\text{def}}{=} Z(t, r). \end{aligned}$$

So, we have $B(t_0) = B(1) = r$. Then

$$B(t_{n+1}) \stackrel{\mathcal{L}}{=} Z\left(\frac{t_{n+1}}{t_n}, B(t_n)\right), n \ge 0,$$

where the different realizations of $Z(\cdot, \cdot)$ can be generated independently, so that $B(t_n), n \ge 0$ forms a Markov chain imploding towards zero.

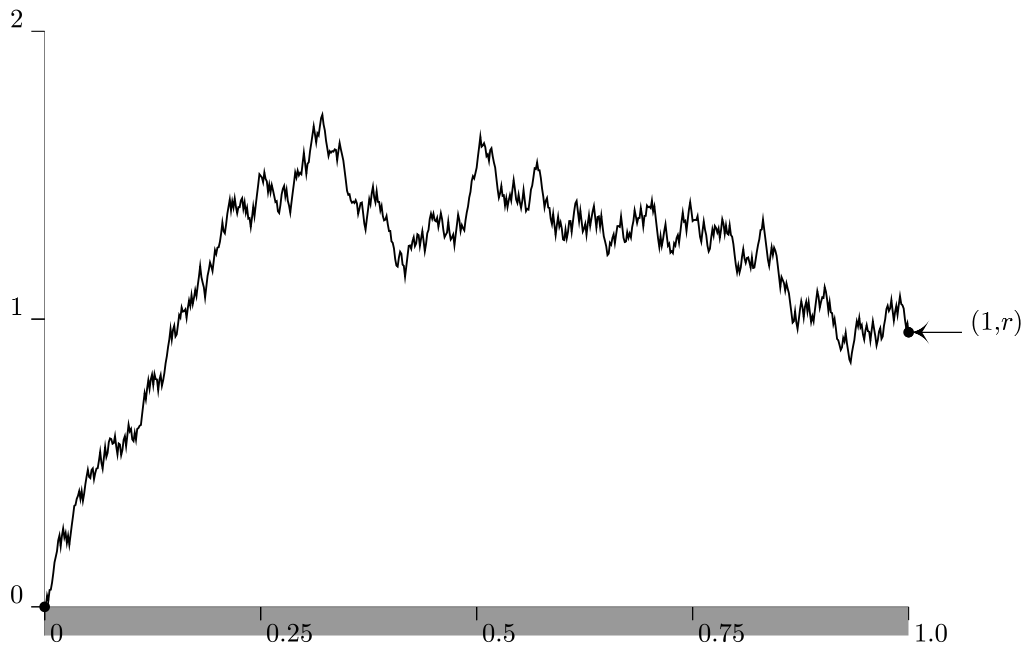

Figure 4. Simulation of Brownian meander with end value r.

For B^{ex} , a simple construction by circular rotation of a standard Brownian bridge B^{br} is possible (Vervaat, 1979; Biane, 1986). As noted above, for B^{br} on $[0,1]$, the minimum is located at $X \stackrel{\mathcal{L}}{=} U$. Construct now the new process

$$Y(t) = \begin{cases} B^{\text{br}}(X+t) - B^{\text{br}}(X) & \text{if } 0 \le t \le 1-X \ ; \\ B^{\text{br}}(X+t-1) - B^{\text{br}}(X) & \text{if } 1-X \le t \le 1 \ . \end{cases}$$

Then $Y \equiv B^{\text{ex}}$ on $[0,1]$. Furthermore, the process B^{ex} just constructed is independent of X. This construction permits the easy simulation of B^{br} given B^{ex}, by cutting and pasting starting at a randomly generated uniform $[0,1]$ position U. But vice versa, the benefits for simulating B^{ex} given B^{br} are not so clear.

Brownian meander: Local methods

The local simulation problem for Brownian meanders can be summarized as follows: given $a, b \ge 0$, and $B^{\text{me}}(0) = a$, $B^{\text{me}}(1) = b$, generate the value of $B^{\text{me}}(t)$ for given $t \in (0,1)$ in expected time bounded uniformly over a, b, t. Armed with such a tool, we can continue subdividing intervals at unit expected cost per subdivision. We may need to rescale things. Let us denote by $B^{\text{me}}(t; a, b, s)$ the value $B^{\text{me}}(t)$ when $0 \le t \le s$, given that $B^{\text{me}}(0) = a$, $B^{\text{me}}(s) = b$. Then

$$B^{\text{me}}(t; a, b, s) \stackrel{\mathcal{L}}{=} \sqrt{s} B^{\text{me}} \left(\frac{t}{s}; \frac{a}{\sqrt{s}}, \frac{b}{\sqrt{s}}, 1 \right) .$$

Random variate generation can be tackled by a variant of the global method if one is willing to store and carry through the values of all three Brownian motions in the three-dimensional Bessel bridge approach. However, if this is not done, and the boundaries of an interval are fixed, then one must revert to a truly local method. This section discusses the simulation of $B^{\mathrm{me}}(t;a,b,1)$. In the remainder of this section, we will write $B^{\mathrm{me}}(t)$ instead of $B^{\mathrm{me}}(t;a,b,1)$.

The (known) density of $B^{\mathrm{me}}(t)$ can be derived quite easily. We repeat the easy computations because some intermediate results will be used later. Let us start from the well-known representation for Brownian motion $X(t)$ restricted to $X(0)=a, X(1)=b$ with $0 \le a < b$:

$$X(t) \stackrel{\mathcal{L}}{=} a + B(t) + t(b-a-B(1)), 0 \le t \le 1.$$

Writing $B(1) = B(t) + B'(1-t)$ (B' being independent of B), and replacing $B(t) = \sqrt{t}\,N$, $B'(1-t) = \sqrt{1-t}\,N'$, we have

$$\begin{aligned} X(t) &\stackrel{\mathcal{L}}{=} a + t(b-a) + \sqrt{t}\,(1-t)N - \sqrt{1-t}\,tN' \\ &\stackrel{\mathcal{L}}{=} a + t(b-a) + \sqrt{t(1-t)}\,N. \end{aligned}$$

For the Brownian bridge B_r on $[0,1]$, we know that

$$M_r \stackrel{\mathcal{L}}{=} \frac{1}{2}\left(r + \sqrt{r^2+2E}\right),$$

and thus, the minimum is distributed as

$$\frac{1}{2}\left(r - \sqrt{r^2+2E}\right).$$

Since $X(t)$ is just $a + B_r(t)$ with $r = b-a$,

$$\begin{aligned} &\mathbb{P}\left\{\min_{0\le t\le 1} X(t) \ge 0\right\} \\ &= \mathbb{P}\left\{a + \frac{1}{2}\left(b-a-\sqrt{(b-a)^2+2E}\right) \ge 0\right\} \\ &= \mathbb{P}\left\{\sqrt{(b-a)^2+2E} \le a+b\right\} \\ &= \mathbb{P}\left\{(b-a)^2+2E \le (a+b)^2\right\} \\ &= \mathbb{P}\left\{E \le 2ab\right\} \\ &= 1-\exp(-2ab). \end{aligned}$$

For $x > 0$, assuming $ab > 0$,

$$\begin{aligned} &\mathbb{P}\left\{B^{\mathrm{me}}(t) \in dx\right\} \\ &= \mathbb{P}\left\{X(t)\in dx \,\middle|\, \min_{0\le s\le 1} X(s) \ge 0\right\} \\ &= \frac{\mathbb{P}\left\{X(t)\in dx, \min_{0\le s\le 1} X(s)\ge 0\right\}}{\mathbb{P}\left\{\min_{0\le s\le 1} X(s)\ge 0\right\}} \\ &= \frac{\mathbb{P}\left\{X(t)\in dx\right\}\mathbb{P}\left\{\min_{0\le s\le t} Y(s)\ge 0\right\}\mathbb{P}\left\{\min_{t\le s\le 1} Z(s)\ge 0\right\}}{\mathbb{P}\left\{\min_{0\le s\le 1} X(s)\ge 0\right\}} \end{aligned}$$

where $Y(s)$ is Brownian motion on $[0,t]$ with endpoint values a,x, and $Z(s)$ is Brownian motion on $[t,1]$ with endpoint values x,b. The decomposition into a product in the numerator follows from the Markovian nature of Brownian motion. Using scaling, we see that

$$\mathbb{P}\left\{\min_{0\le s\le t} Y(s)\ge 0\right\} = 1-\exp(-2(a/\sqrt{t})(x/\sqrt{t})) = 1-\exp(-2ax/t),$$

and similarly,

$$\mathbb{P}\left\{\min_{t\le s\le 1} Z(s)\ge 0\right\} = 1-\exp(-2bx/(1-t)).$$

Therefore, putting $\mu = a+t(b-a)$,

$$\begin{aligned}&\mathbb{P}\left\{B^{\mathrm{me}}(t)\in dx\right\}\\ &\quad = \mathbb{P}\left\{a+t(b-a)+\sqrt{t(1-t)}N\in dx\right\}\times\frac{(1-\exp(-2ax/t))\,(1-\exp(-2bx/(1-t)))}{1-\exp(-2ab)}.\end{aligned}$$

The density of $B^{\mathrm{me}}(t)$ is

$$f(x) = g(x)\times h(x),$$

where, for $x>0$,

$$g(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi t(1-t)}}\exp\left(-\frac{(x-\mu)^2}{2t(1-t)}\right),$$

$$h(x) \stackrel{\text{def}}{=} \begin{cases} \frac{(1-\exp(-2ax/t))(1-\exp(-2bx/(1-t)))}{1-\exp(-2ab)} & \text{if } ab>0,\\ \frac{x}{bt}\,(1-\exp(-2bx/(1-t))) & \text{if } a=0,b>0,\\ \frac{x}{a(1-t)}\,(1-\exp(-2ax/t)) & \text{if } a>0,b=0,\\ \frac{2x^2}{t(1-t)} & \text{if } a=b=0,\end{cases}$$

When $a=0$ or $b=0$ or both, the density was obtained by a continuity argument. The case $a=0,b>0$ corresponds to Brownian meander started at the origin and ending at b, and the case $a=b=0$ is just Brownian excursion. In the latter case, the density is

$$\frac{2x^2}{\sqrt{2\pi(t(1-t))^3}}\exp\left(-\frac{x^2}{2t(1-t)}\right), x>0,$$

which is the density of

$$\sqrt{2t(1-t)G_{3/2}} \stackrel{\mathcal{L}}{=} \sqrt{t(1-t)(N^2+2E)}\,.$$

More interestingly, we already noted the 3d representation of Brownian meanders, which gives for $a=0$ the recipe

$$B^{\mathrm{me}}(t) \stackrel{\mathcal{L}}{=} \sqrt{\left(bt+\sqrt{t(1-t)}N\right)^2+2Et(1-t)},$$

and, by symmetry, for $b=0$,

$$B^{\mathrm{me}}(t) \stackrel{\mathcal{L}}{=} \sqrt{\left(a(1-t)+\sqrt{t(1-t)}N\right)^2+2Et(1-t)}.$$

We rediscover the special case $a=b=0$. We do not know a simple generalization of these sampling formulae for $ab>0$. In the remainder of this section, we therefore develop a uniformly fast rejection method for f.

If $ab \geq 1/2$, we have $1 - \exp(-2ab) \geq 1 - 1/e$, and thus

$$h(x) \leq \frac{e}{e-1}.$$

Since g is a truncated normal density, the rejection method is particularly simple:

```
Repeat Generate U,N.  Set X ← μ + √(t(1−t)) N
       Until X ≥ 0 and Ue/(e−1) ≤ h(X)
Return X
```

The expected number of iterations is the integral under the dominating curve, which is $e/(e-1)$.

Consider now the second case, $ab \leq 1/2$. Using the general inequality $1-e^{-u} \leq u$, and $1-e^{-u} \geq u(1-1/e)$ for $u \leq 1$, we have

$$h(x) \leq \frac{e}{e-1} \times \frac{x}{bt} \left(1 - \exp(-2bx/(1-t))\right),$$

where on the right hand side, we recognize the formula for h when $a = 0, b > 0$ discussed above. Thus, using the sampling formula for that case , we obtain a simple rejection algorithm with expected number of iterations again equal to $e/(e-1)$.

```
Repeat Generate U,N,E.  Set X ← √((bt + √(t(1−t)) N)^2 + 2Et(1−t))
       Until (e/(e−1)) (UX/(bt)) ≤ (1−exp(−2aX/t)) / (1−exp(−2ab))
Return X
```

Brownian meander: Extrapolation

Given $B^{\text{me}}(t) = a$, we are asked to simulate $B^{\text{me}}(t+s)$. We recall first that Brownian meanders are translation invariant, i.e., $B^{\text{me}}(t; a, b, t'), 0 \leq t \leq t'$ is equivalent to $B^{\text{me}}(t+s; a, b, t'+s), 0 \leq t \leq t'$ for all $s > 0$. Also, it is quickly verified that $B^{\text{me}}(t; a, b, t'), 0 \leq t \leq t'$ is equivalent to Brownian motion on $[0, t']$ starting from a and ending at b, conditional on staying positive (if a or b are zero, then limits must be taken). Finally, scaling is taken care of by noting that given $B^{\text{me}}(t) = a$, $B^{\text{me}}(t+s)$ is distributed as $\sqrt{s}B^{\text{me}}(t+1)$ started at $a/\sqrt{s}$. These remarks show that we need only be concerned with the simulation of Brownian motion $B(1)$ on $[0,1]$, given $B(0) = a > 0$ and conditional on $\min_{0 \leq t \leq 1} B(t) > 0$. The case $a = 0$ reduces to standard Brownian meander $B^{\text{me}}(1)$, which we know is distributed as $\sqrt{2E}$.

As we remarked earlier, simple computations involving the reflection principle show that the density of $B(1)$ under the above restrictions is

$$f(x) \stackrel{\text{def}}{=} \frac{\exp\left(-\frac{(x-a)^2}{2}\right) - \exp\left(-\frac{(x+a)^2}{2}\right)}{\sqrt{2\pi}\mathbb{P}\{|N| \leq a\}}, x > 0.$$

The limit of this as $a \downarrow 0$ is $x \exp(-x^2/2)$, the density of $\sqrt{2E}$. The distribution function is given by

$$F(x) = 1 - \frac{\Phi(x+a) - \Phi(x-a)}{\Phi(a) - \Phi(-a)},$$

where we recall that Φ is the distribution function of N. This is not immediately helpful for random variate generation. We propose instead the following simple rejection algorithm, which has uniformly bounded expected time.

(Case $a \geq 1$)
Repeat Generate U uniform $[0,1]$, N standard normal
    $X \leftarrow a + N$
Until $X > 0$ and $U \geq \exp(-2aX)$
Return X
(Case $0 < a \leq 1$)
Repeat Generate U uniform $[0,1]$, E exponential
    $X \leftarrow \sqrt{2E/(1-a^2/3)}$
Until $2aUX \exp(a^2X^2/6) \leq \exp(aX) + \exp(-aX)$
Return X

For $a \geq 1$, we apply rejection with as dominating curve the first term in the expression of f, which is nothing but the normal density with mean a. The acceptance condition is simply $U \geq \exp(-2aX)$, which we leave as a simple exercise. The probability of rejection is

$$\begin{aligned} \mathbb{P}\{[a+N<0] \cup [U \leq \exp(-2a(a+N))]\} &\leq \mathbb{P}\{N > a\} + \mathbb{E}\{\exp(-2a(a+N))\} \\ &= 1 - \Phi(a) + \exp(-4a^2) \\ &\leq 1 - \Phi(1) + e^{-4} \\ &< 0.18. \end{aligned}$$

This method applies for all a, but as $a \downarrow 0$, we note with disappointment that the rejection probability approaches 1. For $0 < a \leq 1$, we rewrite the numerator in f as

$$\exp\left(-\frac{x^2}{2} - \frac{a^2}{2}\right) \times \left(e^{ax} + e^{-ax}\right),$$

and bound

$$e^{ax} + e^{-ax} \leq 2axe^{(ax)^2/6},$$

which is easily verified by comparing Taylor series on both sides. This explains the rejection condition. Furthermore, the dominating curve is proportional to

$$x \exp\left(-\frac{x^2(1-a^2/3)}{2}\right),$$

which in turn is proportional to the density of $\sqrt{2E/(1-a^2/3)}$. The probability of acceptance is one over the integral of the dominating curve, which is

$$\frac{2ae^{-a^2/2}}{\sqrt{2\pi}\mathbb{P}\{|N| \leq a\}(1-a^2/3)} \leq \frac{1}{1-a^2/3} \leq \frac{3}{2}.$$

Thus, the rejection probability is less than 1/3. Therefore, the expected time taken by the algorithm above is uniformy bounded over all choices of a.

Brownian meander: Extremes

The maxima related to B^{me} are slightly more complicated to describe:

$$\mathbb{P}\{M^{\mathrm{me}} \le x\} = 1 + 2\sum_{k=1}^{\infty}(-1)^k \exp\left(-k^2x^2/2\right), x > 0$$

(Chung, 1975, 1976; and Durrett and Iglehart (1977)). This is also known as the (scaled) Kolmogorov-Smirnov limit distribution. For this distribution, fast exact algorithms exist (Devroye, 1981)—more about this in the last section. Furthermore,

$$\mathbb{P}\{M^{\mathrm{ex}} \le x\} = 1 + 2\sum_{k=1}^{\infty}(1-4k^2x^2) \exp\left(-2k^2x^2\right), x > 0$$

(Chung, 1975, 1976; and Durrett and Iglehart (1977)). This is also called the theta distribution. For this too, we have fast exact methods (Devroye, 1997).

The remainder of this section deals with M_r^{me}. Once we can simulate this, we also have

$$(M^{\mathrm{me}}, B^{\mathrm{me}}(1)) \equiv (M_r^{\mathrm{me}}, r) \text{ with } r = \sqrt{2E},$$
$$(M^{\mathrm{ex}}, B^{\mathrm{ex}}(1)) \equiv (M_r^{\mathrm{me}}, r) \text{ with } r = 0.$$

The starting point is the following joint law,

$$\mathbb{P}\{M^{\mathrm{me}} \le x, B^{\mathrm{me}}(1) \le y\} = \sum_{k=-\infty}^{\infty}\left[\exp\left(-(2kx)^2/2\right) - \exp\left(-(2kx+y)^2/2\right)\right], x \ge y \ge 0,$$

as obtained by Durrett and Iglehart (1977) and Chung (1976). Straightforward calculations then show

$$\mathbb{P}\{M^{\mathrm{me}} \le x, B^{\mathrm{me}}(1) \in dy\} = \sum_{k=-\infty}^{\infty}(2kx+y)\exp\left(-(2kx+y)^2/2\right)\, dy, x \ge y \ge 0.$$

Because $B^{\mathrm{me}}(1)$ has density $y\exp(-y^2/2)$, we see that the distribution function of M_r^{me} is

$$\sum_{k=-\infty}^{\infty}\frac{2kx+r}{r}\exp\left(r^2/2-(2kx+r)^2/2\right), x \ge r > 0.$$

Its density is

$$f(x) \stackrel{\mathrm{def}}{=} r^{-1}e^{r^2/2}\sum_{k=-\infty}^{\infty}2k\left(1-(2kx+r)^2\right)\exp\left(-(2kx+r)^2/2\right), x \ge r > 0. \quad (4)$$

It helps to rewrite the density (4) of M_r^{me} by grouping the terms:

$$f(x) = \sum_{k=1}^{\infty}f_k(x) \stackrel{\mathrm{def}}{=} \sum_{k=1}^{\infty}2ke^{-2k^2x^2}g(r,k,x),$$

with

$$g(r,k,x) = \frac{1}{r} \times \left(\left(1-(r+2kx)^2\right) e^{-2kxr} - \left(1-(r-2kx)^2\right) e^{2kxr} \right)$$
$$= \frac{1}{r} \times \left((r^2+4k^2x^2-1)\sinh(2kxr) - 4kxr\cosh(2kxr) \right).$$

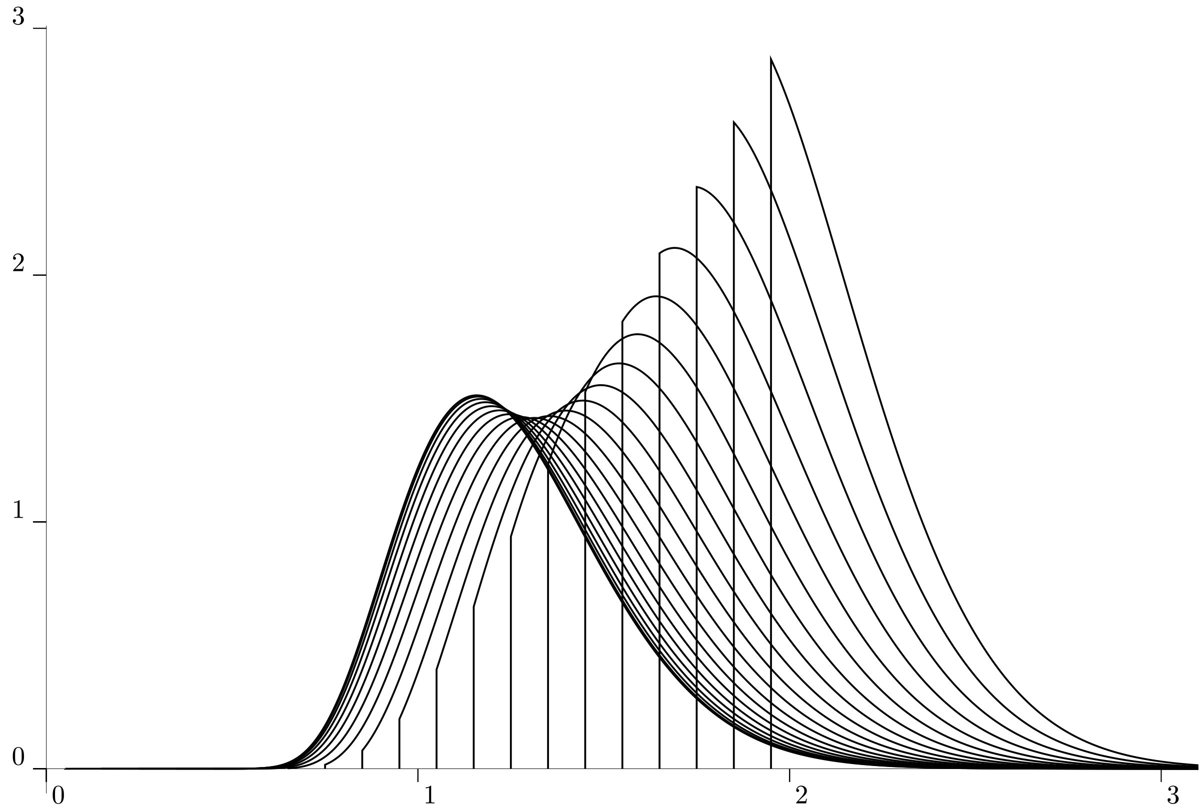

Figure 5. The density of M_r^{me} for r varying from 0.05 to 1.95 in steps of 0.1. Note the accumulation at $r=0$, which corresponds to M^{ex} (the theta distribution).

The Jacobi theta function

$$\theta(x) = \sum_{n=-\infty}^{\infty} \exp\left(-n^2\pi x\right), x > 0,$$

has the remarkable property that $\sqrt{x}\theta(x) = \theta(1/x)$, which follows from the Poisson summation formula, and more particularly from Jacobi's theta function identity

$$\frac{1}{\sqrt{\pi x}} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{(n+y)^2}{x}\right) = \sum_{n=-\infty}^{\infty} \cos(2\pi n y) \exp\left(-n^2\pi^2 x\right), y \in \mathbb{R}, x > 0.$$

Taking derivatives with respect to y then shows the identity

$$\frac{1}{\sqrt{\pi x^3}} \sum_{n=-\infty}^{\infty} (n+y) \exp\left(-\frac{(n+y)^2}{x}\right) = \sum_{n=-\infty}^{\infty} \pi n \sin(2\pi n y) \exp\left(-n^2\pi^2 x\right), y \in \mathbb{R}, x > 0.$$

A term by term comparison yields the alternative representation

$$\begin{aligned}
&\mathbb{P}\{M^{\mathrm{me}} \le x, B^{\mathrm{me}}(1) \in dy\} \\
&= 2x \sum_{k=-\infty}^{\infty} \left(k + \frac{y}{2x}\right) \exp\left(-\frac{\left(k+\frac{y}{2x}\right)^2}{1/(2x^2)}\right) \, dy \\
&= 2x\sqrt{\frac{\pi}{8x^6}} \sum_{n=-\infty}^{\infty} \pi n \sin\left(2\pi n \frac{y}{2x}\right) \exp\left(-\frac{n^2\pi^2}{2x^2}\right) \, dy \\
&= \sqrt{\frac{\pi}{2x^4}} \sum_{n=-\infty}^{\infty} \pi n \sin\left(2\pi n \frac{y}{2x}\right) \exp\left(-\frac{n^2\pi^2}{2x^2}\right) \, dy, x \ge y \ge 0.
\end{aligned}$$

The distribution function of M_r^{me} can also be written as

$$F(x) = \sum_{n=1}^{\infty} F_n(x) \sin\left(\frac{\pi n r}{x}\right), x \ge r \ge 0. \tag{5}$$

where

$$F_n(x) = \sqrt{2\pi} x^{-2} r^{-1} e^{r^2/2} \pi n \exp\left(-\frac{n^2\pi^2}{2x^2}\right).$$

This yields the density

$$\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} F_n'(x) \sin(\pi n r/x) - \sum_{n=1}^{\infty} F_n(x)(\pi n r/x^2) \cos(\pi n r/x) \\
&= \sum_{n=1}^{\infty} F_n(x) \left(\left(\frac{n^2\pi^2 - 2x^2}{x^3}\right) \sin(\pi n r/x) - \left(\frac{\pi n r}{x^2}\right) \cos(\pi n r/x)\right) \qquad (6) \\
&\stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \psi_n(x).
\end{aligned}$$

Armed with the dual representations (4) and (5), we develop an algorithm called MAXMEANDER. It is based upon rejection combined with the series method developed by the author in 1981 (see also Devroye, 1986). The challenge here is to have an expected time uniformly bounded over all choices of r. For rejection, one should make use of the properties of the family when r approaches its extremes. as $r \downarrow 0$, the figure above suggests that $M_r \xrightarrow{\mathcal{L}} M_0$, and that bounds on the density for M_0 should help for small r. As $r \to \infty$, the distribution "escapes to infinity". In fact, $2r(M_r - r) \xrightarrow{\mathcal{L}} E$, a fact that follows from our bounds below. Thus, we should look for exponential tail bounds that hug the density tightly near $x = r$. We have two regimes, $r \ge 3/2$, and $r \le 3/2$.

REGIME I: $r \ge 3/2$.

LEMMA 1. *Assume $r \ge 3/2$. For every $K \ge 1$, $x \ge r \ge 3/2$,*

$$-\frac{1}{1-\zeta} \times \frac{4K(1+4Kxr)}{r} \times e^{-2K^2x^2+2Kxr} \le \sum_{k=K}^{\infty} f_k(x) \le \frac{1}{1-\xi} \times 2K(r + 4K^2x^2/r) \times e^{-2K^2x^2+2Kxr}$$

where $\xi = 6.8\, e^{-9}$ *and* $\zeta = 2.2\, e^{-9}$*. Next, when* $r \le 3/2$*,* $x \ge 3/2$*, we have*

$$-\frac{8K^2 x e^{2Kxr-2K^2x^2}}{1-\tau} \le \sum_{k=K}^{\infty} f_k(x) \le \frac{164}{9(1-\nu)}\, K^4 x^3 e^{2Kxr-2K^2x^2},$$

where $\nu = 16e^{-9}$ *and* $\tau = 4e^{-9}$*.*

This leads to an algorithm for $r \ge 3/2$. Note that for this, we need a bound on f. By Lemma 1,

$$f(x) \le s(x) \stackrel{\text{def}}{=} \frac{2r^2 + 8x^2}{(1-\xi)r} \times e^{-2x^2+2xr}$$

and the upper bound is easily checked to be log-concave for $x \ge r \ge 3/2$. Thus,

$$s(x) \le s(r)\exp((\log s)'(r)(x-r)), x \ge r,$$

and since $(\log s)'(r) = (8 - 10r^2)/(5r)$, we have

$$f(x) \le g(x) \stackrel{\text{def}}{=} \frac{10r}{(1-\xi)} \times \exp\left(-\frac{10r^2-8}{5r}(x-r)\right), x \ge r.$$

We have

$$\begin{aligned}\int_r^{\infty} g(x)\, dx &= \frac{10r \times 5r}{(1-\xi)(10r^2-8)} \\ &= \frac{5}{(1-\xi)\,(1-8/(10r^2))} \\ &\le \frac{5}{(1-\xi)\,(1-32/90)} \\ &< 7.77.\end{aligned}$$

This suggests that using g as a dominating curve for rejection yields an algorithm that is uniformly fast when $r \ge 3/2$. Also, the function g is proportional to the density of $r + cE$ with $c = 5r/(10r^2 - 8)$.

```
Algorithm MAXMEANDER (for r ≥ 3/2)
Repeat Generate X = r + cE where c = 5r/(10r^2 − 8)
        Generate V uniformly on [0,1], and set Y ← Vg(X) = 10rVe^{−E}/(1−ξ)
        k ← 2, S ← f_1(X)
        Decision ← ‘‘Undecided’’
        Repeat If Y ≤ S − 1/(1−ζ) × 4k(1+4kXr)/r × e^{−2k^2X^2+2kXr} then Decision ← ‘‘Accept’’
               If Y ≥ S + 1/(1−ξ) × 2k(r + 4k^2X^2/r) × e^{−2k^2X^2+2kXr} then Decision ← ‘‘Reject’’
               S ← S + f_k(X)
               k ← k + 1
        Until Decision ≠ ‘‘Undecided’’
Until Decision = ‘‘Accept’’
Return X
```

REGIME II: $r \le 3/2$. The next Lemma provides approximation inequalities for small values of x, thanks to the Jacobi-transformed representation (5).

LEMMA 2. *For every* $K \ge 1$, $x \le 3/2$, $r \le 3/2$,

$$\left| \sum_{k=K}^{\infty} \psi_k(x) \right| \le \frac{1}{1-\mu} F_K(x) \times \frac{K^3 \pi^3 r}{x^4},$$

where $\mu = 16 \exp\left(-\frac{2\pi^2}{3}\right) = 0.0222\ldots$.

Consider first $x \le 3/2$. Rejection can be based on the inequality

$$\begin{aligned} f(x) &\le \frac{1}{1-\mu} F_1(x) \left(\frac{\pi^3 r}{x^4} \right) \\ &= \frac{\sqrt{2\pi} e^{r^2/2} \pi^4}{(1-\mu) x^6} \times \exp\left(-\frac{\pi^2}{2x^2} \right) \\ &\le g(x) \stackrel{\text{def}}{=} \frac{\sqrt{2\pi} e^{9/8} \pi^4}{(1-\mu) x^6} \times \exp\left(-\frac{\pi^2}{2x^2} \right). \end{aligned} \tag{7}$$

It is remarkable, but not surprising, that the dominating function does not depend upon r. It is uniformly valid over the range. We will apply it over $\mathbb{R}^+$. The random variable $\pi/\sqrt{2G_{5/2}} = \pi/\sqrt{N^2 + 2E_1 + 2E_2}$ has density

$$\frac{\sqrt{2\pi}\pi^4}{3x^6} \exp\left(-\frac{\pi^2}{2x^2} \right), x > 0,$$

which is $g(x)/p$ with $p = 3e^{9/8}/(1-\mu)$. Thus, $\int_0^\infty g(x)\,dx = p$, and rejection is universally efficient.

Consider next $x \ge 3/2$, a situation covered by the ineqalities of Lemma 1. Here we first need an upper bound to be able to apply rejection. Once again, an exponential bound is most appropriate. To see this, not that

$$f(x) \le \frac{164}{9(1-\nu)} x^3 e^{2xr - 2x^2}, x \ge 3/2.$$

The upper bound is log-concave in x, and we can apply the exponential tail technique for log-concave densities, which yields the further bound

$$f(x) \le g^*(x) \stackrel{\text{def}}{=} q \times (4-2r) e^{-(4-2r)(x-3/2)}, x \ge 3/2, \tag{8}$$

where

$$q \stackrel{\text{def}}{=} \int_{3/2}^{\infty} g^*(x)\,dx = \frac{123 \times e^{3r-9/2}}{2(1-\nu)(4-2r)}.$$

The function g^* is proportional to the density of $3/2 + E/(4-2r)$. We are thus set up to apply rejection with a choice of dominating curves, one having weight p for $x \le 3/2$, and one of weight q for $x \ge 3/2$. The algorithm, which has an expected time uniformly bounded over the range $r \le 3/2$ (since $p+q$ is uniformly bounded) can be summarized as follows:

```
Algorithm MAXMEANDER (for r ≤ 3/2)
Set p = 3e^{9/8}/(1−μ), q = 123×e^{3r−9/2}/(2(1−ν)(4−2r)).
Repeat Generate U,V uniformly on [0,1]
        If U ≤ p/(p+q) then X ← π/√(N²+2E₁+2E₂)
                    Y ← Vg(X) [g is as in (7)]
                    k ← 2, S ← ψ₁(X)
                    Decision ← ``Undecided''
                    Repeat If X ≥ 3/2 then Decision ← ``Reject''
                            Set T ← (1/(1−μ)) F_k(X) × k³π³r/X⁴
                            If Y ≤ S − T then Decision ← ``Accept''
                            If Y ≥ S + T then Decision ← ``Reject''
                            S ← S + ψ_k(X)
                            k ← k + 1
                    Until Decision ≠ ``Undecided''
                else X ← 3/2 + E/(4−2r)
                    Y ← Vg*(X) [g* is as in (8)]
                    k ← 2, S ← f₁(X)
                    Decision ← ``Undecided''
                    Repeat If Y ≤ S − 8k²Xe^{2kXr−2k²X²}/(1−τ) then Decision ← ``Accept''
                            If Y ≥ S + (164/(9(1−ν))) k⁴X³e^{2kXr−2k²X²} then Decision ← ``Reject''
                            S ← S + f_k(X)
                            k ← k + 1
                    Until Decision ≠ ``Undecided''
Until Decision = ``Accept''
Return X
```

Extensions. Using the ideas of this section, it is possible to develop a uniformly fast generator for M^{me} when both endpoints are fixed and nonzero: $B^{\mathrm{me}}(0) = a$ and $B^{\mathrm{me}}(1) = b$. Majumdar, Randon-Furling, Kearney and Yor (2008) describe the distributions of the locations of maxima in several constrained Brownian motions, including Brownian meanders. It is also possible to develop uniformly fast exact simulation algorithms for them. □

Reflected Brownian motion

Reflected Brownian motion with one reflecting barrier at 0 is just $|B(t)|$. If one is merely interested in simulating this process at given time epochs, it suffices to simulate B and take absolute values. Extrapolation beyond t when the value $|B(t)| = x$ is easy too, as

$$|B(t+s)| \stackrel{\mathcal{L}}{=} |B(t) + \sqrt{s}N| \stackrel{\mathcal{L}}{=} |r + \sqrt{s}N|,$$

where we used the reflection principle to note that conditioning on $|B(t)| = x$ is equivalent to conditioning on $B(t) = x$. Interpolation between $|B(t)| = x$ and $|B(t')| = r$ for $t < s < t'$ follows easily after one notes that $|B(s)|$ has the same distribution when only conditioned on $B(t) = x$ and $B(t') = r$, and so this case reduces to interpolation for B, which we dealt with earlier. The simulation of

$$\max_{t<s<t'} |B(s)|$$

conditional on the values a, b at both endpoints poses no problems either. Consider a Brownian bridge taking the values a and b at t and t' respectively. Simulate the joint values of maximum and minimum as explained earlier (this requires two accesses to a generator for the maximum of a conditional Brownian meander). Finally, return the maxim of the absolute values of the maximum and minimum.

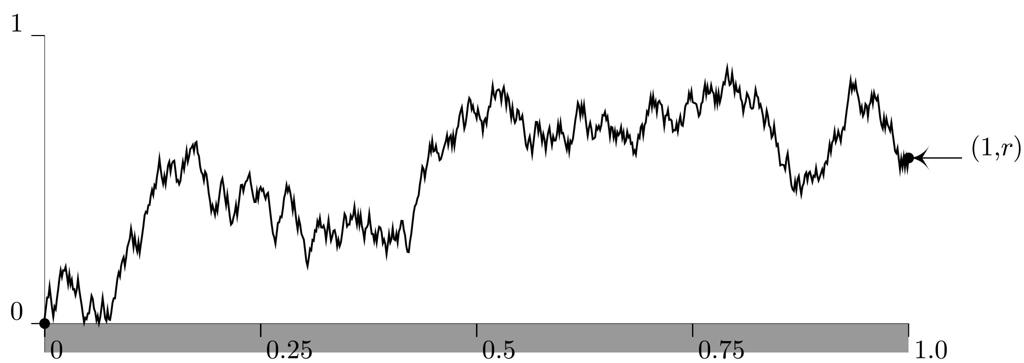

Figure 6. Simulation of reflected Brownian motion.

With two reflecting barriers, at 0 and $a > 0$, and calling the process $R(t)$, simple considerations show that we have the following folding formula,

$$R(t) \stackrel{\mathcal{L}}{=} \begin{cases} B(t) - \lfloor B(t)/a \rfloor a & \text{if } \lfloor B(t)/a \rfloor \text{ is even,} \\ a - (B(t) - \lfloor B(t)/a \rfloor a) & \text{if } \lfloor B(t)/a \rfloor \text{ is odd.} \end{cases}$$

So, for global simulation, it again suffices to simulate B. It is more interesting to study extrapolation and interpolation. Extrapolation from $R(0) = x \in (0, a)$ is trivial by the Markov property of Brownian motion: $R(t)$ can be obtained from $B(t)$ by the folding formula if we start with $B(0) = x$. Interpolation between $R(0) = x \in (0, a)$ and $R(1) = y \in (0, a)$ can also be done by the folding formula. First generate $B(t)$ as in a Brownian bridge between $B(0) = x$ and $B(1) = y$, and then compute $R(t)$ from $B(t)$ by the folding formula.

Brownian motion on an interval

By Brownian motion on an interval $[0, a]$, we mean B conditional on B staying in $[0, a]$. Define the event

$$A(t, t') = \left[0 \leq \inf_{t<s<t'} B(s) \leq \sup_{t<s<t'} B(s) \leq a \right] .$$

The values at $0 < t_1 < \cdots < t_n$ can be obtained by consecutive extrapolations, as

$$(B(0) = x, B(t_1), B(t_2), \ldots, B(t_n)) \text{ given } A(0, t_n)$$

is distributed as the random vector in which $B(t_i)$ given $B(t_{i-1}) = r$ is generated independently of the past up to t_{i-1}, after noting that $B(t_i)$ is just like $B(t_i - t_{i-1})$ starting from $B(0) = x$ and conditional on $A(0, t_i - t_{i-1})$. This follows from the Markovian nature of Brownian motion. The only snake in the grass is when $x \in \{0, a\}$, because then the events $A(0, s)$ for $s > 0$ have zero probability, and one must proceed by taking limits of approximating events (barriers at $-\epsilon$ and $a + \epsilon$, and $\epsilon \downarrow 0$) to obtain a meaningful law. More about this later.

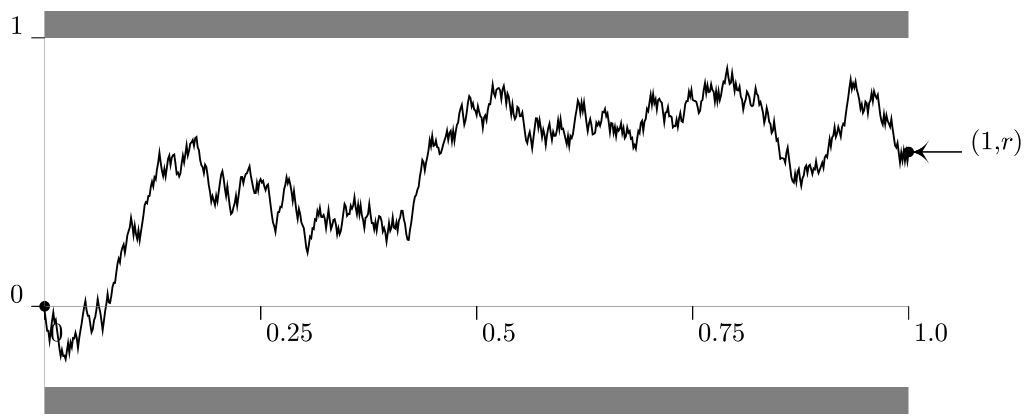

Figure 7. Simulation of Brownian motion restricted to the interval $[-0.3, 1]$ and started at 0.

Simulation can thus be driven be a concatenation of extrapolation steps. In general, we need to simulate $B(t)$ starting at $B(0) = x \in (0, a)$, conditional on $A(0, t)$. Let $T_0 = \inf\{s : B(s) = 0\}$, and $T_a = \inf\{s : B(s) = a\}$ be the first passage times of the barriers. Karatzas and Shreve (1998, pp. 97–100) provide the following formula:

$$\begin{aligned} &\mathbb{P}\{B(t) \in dy, \min(T_0, T_a) > t\} \\ &= \sum_{n=-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y + 2na - x)^2}{2t} \right) - \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y + 2na + x)^2}{2t} \right) \right) dy, 0 < y < a. \end{aligned}$$

The density of $B(t)$ given $\min(T_0, T_a) > t$ is proportional to the expression above, and is thus a weighted sum of pieces of the normal density. The n-th positive part corresponds to a normal density centered at $x - 2na$ with variance t, and restricted to $[0, a]$.

Let us denote bt $B(t,x,a)$ the process (for $t \ge 0$) with parameters $0 < x < a$. Note that

$$B(t,x,a) \stackrel{\mathcal{L}}{=} \sqrt{t} \times B(1, x/\sqrt{t}, a/\sqrt{t}),$$

so we can without loss of generality assume that $t = 1$. This leaves a two-parameter family.

While it is easy to give simple algorithms for generating $B(1,x,a)$, it is harder to find one whose expected time is uniformly bounded over all values of the parameters. This is due to the fact that there are indeed three regimes. When a and x are both very large (relative to $t = 1$), the boundaries do not matter, and we are close to ordinary Brownian motion ($B(1,x,a)$ is close to normal). When x is near zero, and a is large, only the zero boundary matters, and in fact, the behavior is like that of a Brownian meander ($B(1,x,a)$ is roughly like $\sqrt{2E}$). Finally, when a is very small, we are restricted to a thin sausage, and the law of $B(1,x,a)$ is basically independent of the starting point x—in fact, the density of $B(1,x,a)$ is close to the positive period of the sine function.

This section first develops the theory for $0 < x \le a/2$. Indeed, since $B(1,x,a) \stackrel{\mathcal{L}}{=} a - B(1, a-x, a)$, by symmetry, we do not need to consider $a/2 \le x \le 1$. Then by taking limits, we can also handle $B(1,0,a)$.

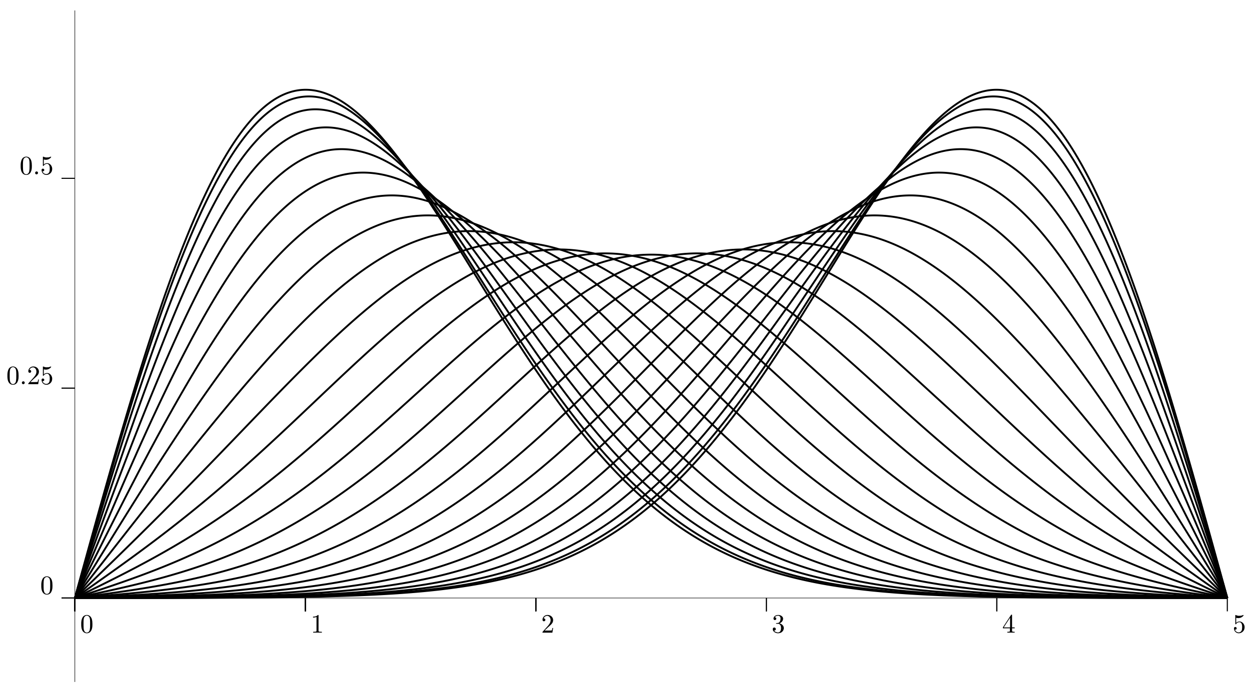

Figure 8. Densities of $B(1,x,a)$ for $a = 5$ held fixed, and x varied from 0.1 to 4.9 in steps of 0.2.

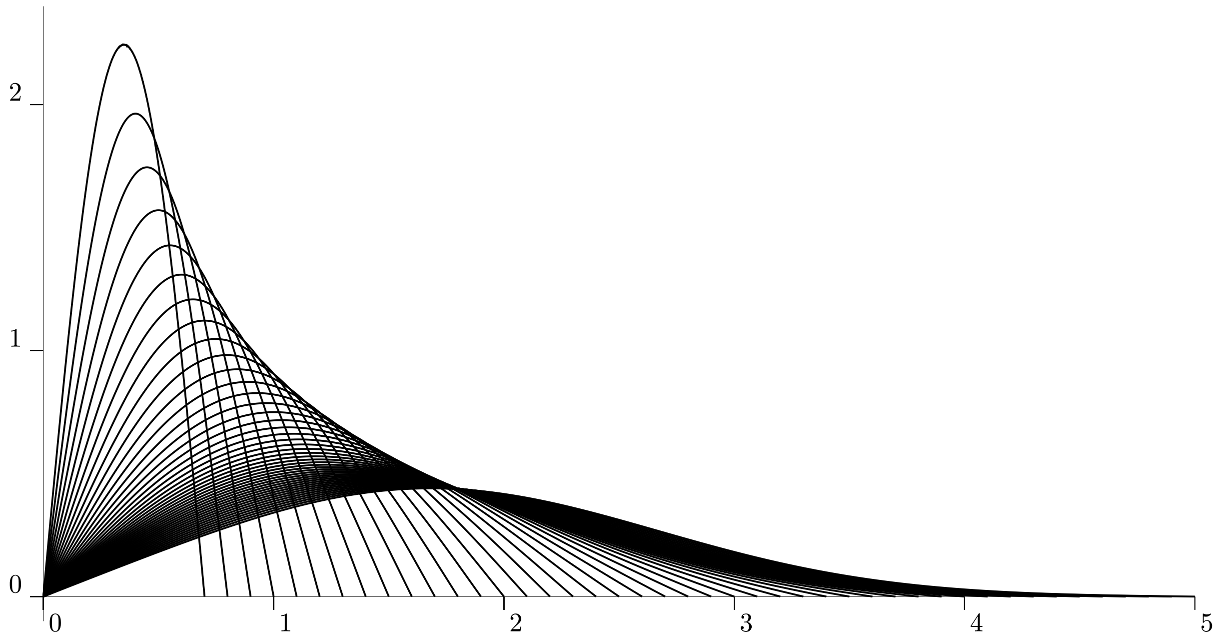

Figure 9. Densities of $B(1,x,a)$ when a varies from 0.7 to 5 in steps of 0.1. In all cases, $x = a/3$. For small values of a, the density is a one-period sinusoid.

The density of $Y \stackrel{\text{def}}{=} B(1,x,a)$ is proportional to

$$
\begin{aligned}
f(y) &= \sum_{n=-\infty}^{\infty} f_n(y) \\
&\stackrel{\text{def}}{=} \sum_{n=-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y+2na-x)^2}{2} \right) - \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y+2na+x)^2}{2} \right) \right) dy, 0 < y < a.
\end{aligned}
$$

It is clear that $f_0(y) \geq 0$, that $f_1(y) > f_2(y) > \cdots > 0$ and that $f_{-1}(y) < f_{-2}(y) < \cdots < 0$. Note also that these ordered sequences can be interleaved when $y \in [0,a]$, $0 \leq x \leq a/2$, and $a \geq 2$:

$$|f_{-1}(y)| \geq f_1(y) \geq |f_{-2}(y)| \geq f_2(y) \geq \cdots.$$

Proof. We will use the monotonicity of the standard normal density ϕ, and its convexity outside $[-1,1]$. Note that $f_n(y) = \phi(y+2na-x) - \phi(y+2na+x)$, and thus that for $n > 0$, $|f_{-n}(y)| \geq f_n(y)$ if

$$\phi(y+2na+x) + \phi(y-2na+x) \geq \phi(y+2na-x) + \phi(y-2na-x).$$

This is equivalent to asking

$$\phi(2na+y+x) + \phi(2na-y-x) \geq \phi(2na+y-x) + \phi(2na-y+x),$$

which in turn follows from the convexity when $2na - y - x \geq 1$. This is satisfied for all $y \in [0,a]$, $0 \leq x \leq a/2$ and $n \geq 1$ if $a \geq 2$. Next, we must show that $f_n(y) \geq |f_{-(n+1)}(y)|$ for all $n \geq 1$. This follows from

$$\phi(y+2na-x) - \phi(y+2na+x) \geq \phi(y-2na-2a+x) - \phi(y-2na-2a-x),$$

or, equivalently,

$$\phi(2na + y - x) + \phi(2na + 2a - y + x) \geq \phi(2na + y + x) + \phi(2na + 2a - y - x),$$

which again follows by convexity and since $2na + y - x \geq 3a/2 \geq 3$ for $a \geq 2$. □

The remainder of this section has three parts. First, we develop a uniformly fast generator for f_0, valid for all values of a and all $0 < x \leq a/2$. Then we develop a uniformly fast generator for f when $a \geq 2$. Finally, the case $a < 2$ is dealt with using a different uniformly fast generator.

UNIFORMLY FAST SIMULATION OF f_0.

Recall

$$\sqrt{2\pi} f_0(y) = \exp\left(-\frac{(y-x)^2}{2}\right) - \exp\left(-\frac{(y+x)^2}{2}\right), 0 < y < a.$$

This is not a density, as a normalization factor is missing. By expanding the exponents, and using $\exp(-z) \geq 1 - z, z > 0$, it is easy to see that

$$\sqrt{2\pi} f_0(y) \leq 2xy \exp\left(-\frac{(y-x)^2}{2}\right), 0 < y < a.$$

To get a uniformly fast method in both parameters, a and $0 \leq x \leq a/2$, we consider three regions: region I ($a \leq 1$), region II ($x \geq 1$), and region III ($xle1$, $x \leq a/2$, $a \geq 1$), which is the trickiest.

In region I, we use

$$\sqrt{2\pi} f_0(y) \leq 2xy, 0 < y < a.$$

A random variate with density proportional to the upper bound can be drawn as $a\sqrt{U}$, and rejection is simple:

```
(Generator for f_0.  Efficient for a ≤ 1 (region I).)
Repeat Generate U,V uniformly on [0,1]
        X ← a√U
Until V × 2xX ≤ exp(-(X-x)^2/2) - exp(-(X+x)^2/2)
Return X
```

Consider region II, $a/2 \geq x \geq 1$. Here we use the trivial (and most obvious) bound

$$f_0(y) \leq \phi(y - x), y \in \mathbb{R}.$$

The rejection algorithm, which uses the fact that

$$\exp\left(\frac{(y-x)^2}{2} - \frac{(y+x)^2}{2}\right) = \exp(-2yx),$$

can be written as follows:

```
(Generator for f_0.  Efficient in region II: a/2 ≥ x ≥ 1.)
Repeat Generate V uniformly on [0,1]
        X ← x + N
Until V ≥ exp(−2Xx) and X ∈ [0,a]
Return X
```

Since $V \stackrel{\mathcal{L}}{=} \exp(-E)$, we can describe the acceptance condition equivalently as

$$E \leq 2Xx = 2x(x+N).$$

So, the algorithm keeps generating independent pairs (E, N) until for the first time $E/(2x) \leq x+N \leq a$, and then returns $X = x+N$.

In region III, we use

$$\sqrt{2\pi} f_0(y) \leq g(y) \stackrel{\text{def}}{=} 2x(y-x)_+ \exp\left(-\frac{(y-x)^2}{2}\right) + 2x^2 \exp\left(-\frac{(y-x)^2}{2}\right), y \in \mathbb{R}.$$

The leftmost of these functions is the density of $x + \sqrt{2E}$ times a factor x. The rightmost is the density of $x+N$ times a factor $2x^2\sqrt{2\pi}$. The factors determine the relative weights of these densities in the mixture g. Thus, the rejection method can be used for f_0:

```
(Generator for f_0.  Efficient in region III (xle1, x ≤ a/2, a ≥ 1))
Repeat Generate U,V uniformly on [0,1]
        If U(2x + x^2 √(8π)) ≤ 2x
           then X ← x + √(2E)
           else X ← x + N
        Y ← V g(X)
Until Y ≤ exp(−(X−x)^2/2) − exp(−(X+x)^2/2) and X ∈ [0,a]
Return X
```

LEMMA 3. *Each of the three algorithms takes expected time uniformly bounded over all parameters in its region.* [Proof in the Appendix.]

GENERATOR FOR f WHEN $a \geq 2$.

The converging alternating series expression for f implies that for any $N \geq 0$,

$$\sum_{|i| \leq N} f_i(y) + f_{-(N+1)} \leq f(y) \leq \sum_{|i| \leq N} f_i(y) \leq f_0(y).$$

Both upper and lower bound converge monotonically to f at all $y \in [0, a]$. Therefore, the following alternating series method is valid.

```
(It is assumed that a ≥ 2, x ≤ a/2.)
Repeat Generate X with density proportional to f_0 on [0,a]
       Generate V uniformly on [0,1], and set Y ← V f_0(X)
       n ← 1, S ← f_0(X)
       Decision ← ``Undecided''
       Repeat S ← S + f_{-n}(X)
              If Y ≤ S then Decision ← ``Accept''
              S ← S + f_n(X)
              If Y ≥ S then Decision ← ``Reject''
              n ← n + 1
       Until Decision ≠ ``Undecided''
Until Decision = ``Accept''
Return X
```

LEMMA 4. *The algorithm above takes expected time uniformly bounded over* $a \geq 2$, $x \leq a/2$. [Proof in the Appendix.]

Generator for f when $a \le 2$. For $0 < x \le a/2$, $a \le 2$. we propose a uniform bound on $[0,1]$. In this range, the target time (1) is far, relative to the width of the interval, and one expects that the density of $B(1,x,a)$ is probably best bounded by a uniform law. By Jacobi's identity,

$$\sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y+2na-x)^2}{2}\right) = \frac{1}{2a} \sum_{n=-\infty}^{\infty} \cos\left(\frac{\pi n(y-x)}{a}\right) \exp\left(-\frac{n^2\pi^2}{2a^2}\right).$$

A similar identity holds with x replaced by $-x$. Thus, the density of $X = B(1,x,a)$ can be written as $1/\mathbb{P}\{\min(T_0,T_a) > 1\}$ times $f(y)$ where

$$\begin{aligned} f(y) &\stackrel{\text{def}}{=} \frac{1}{2a} \sum_{n=-\infty}^{\infty} \left(\cos\left(\frac{\pi n(y-x)}{a}\right) - \cos\left(\frac{\pi n(y+x)}{a}\right)\right) \exp\left(-\frac{n^2\pi^2}{2a^2}\right) \\ &= \frac{1}{a} \sum_{n=-\infty}^{\infty} \sin\left(\frac{\pi n y}{a}\right) \sin\left(\frac{\pi n x}{a}\right) \exp\left(-\frac{n^2\pi^2}{2a^2}\right) \\ &= \frac{2}{a} \sum_{n=1}^{\infty} \sin\left(\frac{\pi n y}{a}\right) \sin\left(\frac{\pi n x}{a}\right) \exp\left(-\frac{n^2\pi^2}{2a^2}\right) \\ &\stackrel{\text{def}}{=} \sum_{n=1}^{\infty} f_n(y). \end{aligned}$$

First we provide an upper bound which will be appropriate for the two-parameter range. Using $|\sin\theta| \le |\theta|$, we see that

$$f(y) \le \sum_{n=1}^{\infty} \frac{2n^2\pi^2 xy}{a^3} \exp\left(-\frac{n^2\pi^2}{2a^2}\right).$$

The ratio of the $(n+1)$-st term to the n-th term in this sum is

$$\left(1+\frac{1}{n}\right)^2 \exp\left(-\frac{(2n+1)\pi^2}{2a^2}\right) \le 4e^{-3\pi^2/8} \stackrel{\text{def}}{=} \rho < 1.$$

Thus, we have a further bound

$$f(y) \le g(y) \stackrel{\text{def}}{=} \frac{2\pi^2 xy}{a^3(1-\rho)} \exp\left(-\frac{\pi^2}{2a^2}\right), 0 \le y \le a.$$

Function g is proportional to the density of $a\sqrt{U}$. Also note that the tail sums in the definition of f are easily bounded as well;

$$\begin{aligned} \left| \frac{2}{a} \sum_{n=N}^{\infty} \sin\left(\frac{\pi n y}{a}\right) \sin\left(\frac{\pi n x}{a}\right) \exp\left(-\frac{n^2\pi^2}{2a^2}\right) \right| &\le \sum_{n=N}^{\infty} \frac{2n^2\pi^2 xy}{a^3} \exp\left(-\frac{n^2\pi^2}{2a^2}\right) \\ &\le h_N(y) \\ &\stackrel{\text{def}}{=} \frac{2N^2\pi^2 xy}{a^3(1-\rho)} \exp\left(-\frac{N^2\pi^2}{2a^2}\right). \end{aligned}$$

Since for every $N \ge 1$,

$$\left| f(y) - \sum_{n=1}^{N} f_n(y) \right| \le h_{N+1}(y),$$

we can use the following series method.

```
(It is assumed that a ≤ 2, x ≤ a/2.)
Repeat Generate X ← a√U
       Generate V uniformly on [0,1], and set Y ← V g(X)
       n ← 1, S ← 0
       Decision ← ``Undecided''
       Repeat S ← S + f_n(X)
              If Y ≤ S − h_{n+1}(X) then Decision ← ``Accept''
              If Y ≥ S + h_{n+1}(X) then Decision ← ``Reject''
              n ← n + 1
       Until Decision ≠ ``Undecided''
Until Decision = ``Accept''
Return X
```

LEMMA 5. *The algorithm above takes expected time uniformly bounded over its region, $a \le 2$, $x \le a/2$.* [Proof in the Appendix.]

This concludes the development of uniformly fast extrapolation algorithm for $B(1, x, a)$, $0 < x < a$.

SPECIAL CASE: STARTING AT $x = 0$. We will briefly comment on $Y_0 \stackrel{\text{def}}{=} B(1, 0, a)$. By taking limits, we see that the density of Y_0 is proportional to

$$\sum_{n=-\infty}^{\infty} (y + 2na) \exp\left(-\frac{(y + 2na)^2}{2}\right), 0 \le y \le a.$$

SPECIAL CASE: BROWNIAN MEANDER. A second special case occurs when $a = \infty$, which corresponds to Brownian meander started at x, $B(1, x, \infty)$. The density is easily seen to be

$$\frac{\exp\left(-\frac{(y-x)^2}{2}\right) - \exp\left(-\frac{(y+x)^2}{2}\right)}{\sqrt{2\pi}\mathbb{P}\{|N| \le x\}},$$

when $x > 0$, and by taking limits, the density of $B(1, 0, \infty)$ is

$$y e^{-y^2/2}, y > 0,$$

which is the well-known density of $B^{\mathrm{me}}(1) \stackrel{\mathcal{L}}{=} \sqrt{2E}$. The former density was called f_0 earlier on in this section, where two different algorithms were given, depending upon whether x is less than or more than one.

Interpolation for Brownian motion on an interval. Interpolation is quadratically more complicated, and will not be dealt with here for fear of making this tedious paper even more unreadable. The fundamental problem, after having rescaled things, is to generate $B(t, x, a)$ for fixed $0 < t < 1$ given $B(1, x, a) = z$. We could denote such a four-parameter random variable by $B(1, t, x, z, a)$.

Let $f(y; x, a)$ denote the density of $B(1, x, a)$ at the point y. By the Markov property, and by flipping the time scale of Brownian motion in the second part, it is easy to see that the density of $B(1, t, x, z, a)$ is proportional to the product of the densities of $B(t, x, a)$ and $B(1 - t, z, a)$, i.e., to

$$f(y/\sqrt{t}, x/\sqrt{t}, a/\sqrt{t}) \times f(y/\sqrt{1-t}, x/\sqrt{1-t}, a/\sqrt{1-t}), 0 < y < a.$$

We have seen that there were three rather different regimes to deal with each one of these f's. Depending upon the situation, that leads to at least nine different possible combinations, not taking into account that $x \le a/2$ and $y \ge a/2$ can be of different polarity. In fact, eliminating all obvious symmetries, there are twelve different combined regimes. A thorough treatment of this case deserves a separate study. It should be noted though that most of this has already been done. We can use as dominating curves the products of the simple dominating curves used by us, which were gaussian, or Maxwell, or constant). Furthermore, the converging series approximations can now be used in product form without further technical hurdles.

Interpolation for boundary-restricted Brownian meander. One case is of special interest, namely when $a = \infty$, the Brownian meander, restricted on both sides.

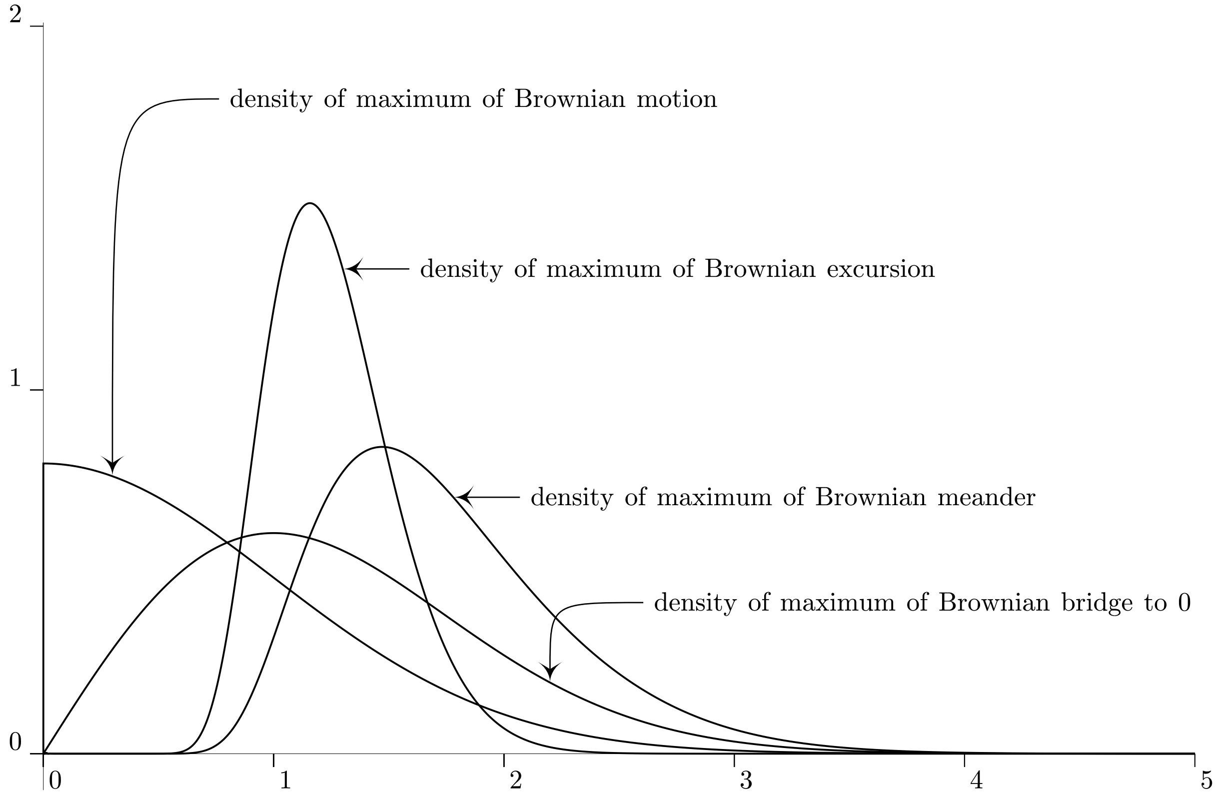

Figure 10. The four main densities dealt with in this paper. M has the half-normal density. $M^{\rm ex}$ has the theta distribution. $M^{\rm me}$ is distributed as $2K$, where K has the Kolmogorov-Smirnov law. Finally, $M^{\rm br} \stackrel{\mathcal{L}}{=} \sqrt{E/2}$ has the Rayleigh density.

The Kolmogorov-Smirnov statistic has the limit distribution function

$$F(x) = \sum_{n=-\infty}^{\infty} (-1)^n e^{-2n^2x^2}, x > 0$$

(Kolmogorov, 1933). We call this the Kolmogorov-Smirnov distribution and denote its random variable by K. It is known that

$$2K \stackrel{\mathcal{L}}{=} M^{\rm me}.$$

Exact random variate generation for the Kolmogorov-Smirnov law was first proposed by Devroye (1981), who used the so-called alternating series method, which is an extension of von Neumann's rejection method. This method is useful whenever densities can be written as infinite sums,

$$f(x) = \sum_{n=0}^{\infty} (-1)^n a_n(x),$$

where $a_n(x) \geq 0$ and for fixed x, $a_n(x)$ is eventually decreasing in n. Jacobi functions are prime examples of such functions. In the present paper, we proposed an algorithm for M_r^{me} that is uniformly fast over all r, and is thus more general. Replacing r by $\sqrt{E/2}$ yields a method for simulating $2K$.

We say that T is theta distributed if it has distribution function

$$G(x) = \sum_{n=-\infty}^{\infty} \left(1 - 2n^2x^2\right) e^{-n^2x^2}, x > 0.$$

We warn that some authors use a different scaling: we call a random variable with distribution function G a theta random variable, and denote it by T. It appears as the limit law of the height of random conditional Galton-Watson trees (see, e.g., Rényi and Szekeres, 1967, de Bruijn, Knuth and Rice, 1972, Chung, 1975, Kennedy, 1975, Meir and Moon, 1978, and Flajolet and Odlyzko, 1982). Furthermore,

$$\frac{T}{\sqrt{2}} \stackrel{\mathcal{L}}{=} M^{\mathrm{ex}}$$

(see, e.g., Pitman and Yor, 2001). Devroye (1997) published an exact algorithm for T that uses the principle of a converging series representation for the density. The algorithm presented in this paper for M_r^{me} with $r = 0$ can also be used.

Both T and K are thus directly related to the maxima dealt with in this paper. But they are connected in a number of other ways that are of independent interest. To describe the relationships, we introduce the random variables J and J^* where the symbol J refers to Jacobi. The density of J is

$$f(x) = \frac{d}{dx} \sum_{n=-\infty}^{\infty} (-1)^n \exp\left(-\frac{n^2\pi^2x}{2}\right) = \sum_{n=1}^{\infty} (-1)^{n+1} n^2\pi^2 \exp\left(-\frac{n^2\pi^2x}{2}\right).$$

The density of J^* is

$$f^*(x) = \pi \sum_{n=0}^{\infty} (-1)^n \left(n + \frac{1}{2}\right) \exp\left(-\frac{(n+1/2)^2\pi^2x}{2}\right).$$

We note that all moments are finite, and are expressible in terms of the Riemann zeta function. The properties of these laws are carefully laid out by Biane, Pitman and Yor (2001). Their Laplace transforms are given by

$$\mathbb{E}\left\{e^{-\lambda J}\right\} = \frac{\sqrt{2\lambda}}{\sinh\left(\sqrt{2\lambda}\right)} \quad , \quad \mathbb{E}\left\{e^{-\lambda J^*}\right\} = \frac{1}{\cosh\left(\sqrt{2\lambda}\right)}.$$

Using Euler's formulae

$$\sinh z = z \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{n^2\pi^2}\right) \quad , \quad \cosh z = \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{(n-1/2)^2\pi^2}\right),$$

it is easy to see that J and J^* are indeed positive random variables, and that they have the following representation in terms of i.i.d. standard exponential random variables $E_1, E_2, \ldots$:

$$J \stackrel{\mathcal{L}}{=} \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{E_n}{n^2} \quad , \quad J^* \stackrel{\mathcal{L}}{=} \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{E_n}{(n-1/2)^2}.$$

It is known that J^* is the first passage time of Brownian motion started at the origin for absolute value 1, and J is similarly defined for the Bessel process of dimension 3 (which is the square root of the sum of the squares of three independent Brownian motions). See, e.g., Yor (1992, 1997). An exact algorithm for J^* is given by Devroye (2009).

Watson (1961) first observed that $K \stackrel{\mathcal{L}}{=} (\pi/2)\sqrt{J}$, and so we have

$$M^{\mathrm{me}} \stackrel{\mathcal{L}}{=} \pi\sqrt{J} \stackrel{\mathcal{L}}{=} 2K.$$

In addition, M^{me} is distributed as twice the maximum absolute value of a Brownian bridge on $[0,1]$ (Durrett, Iglehart and Miller, 1977; Kennedy, 1976; Biane and Yor, 1987; Borodin and Salminen, 2002).

Let us write $K(1), K(2), \ldots$ for a sequence of i.i.d. copies of a Kolmogorov-Smirnov random variable K. As noted by Biane, Pitman and Yor (2001), the distribution function of the sum $J(1)+J(2)$ of two independent copies of J is given by

$$\sum_{n=-\infty}^{\infty} \left(1 - n^2\pi^2 x\right) e^{-n^2\pi^2 x/2}, x > 0.$$

Thus, we have the distributional identity

$$\frac{\pi^2}{2}(J(1)+J(2)) \stackrel{\mathcal{L}}{=} T^2.$$

Using $J \stackrel{\mathcal{L}}{=} (4/\pi^2)K^2$, we deduce

$$T \stackrel{\mathcal{L}}{=} \sqrt{2(K(1)^2 + K(2)^2)}.$$

This provides a route to the simulation of T via a generator for K.

It is also noteworthy that

$$J \stackrel{\mathcal{L}}{=} \frac{J(1)+J(2)}{(1+U)^2}$$

where U is uniform $[0,1]$ and independent of the $J(i)$'s (Biane, Pitman and Yor (2001, section 3.3)). Thus we have the further identities

$$J \stackrel{\mathcal{L}}{=} \frac{2T^2}{\pi^2(1+U)^2} \stackrel{\mathcal{L}}{=} \frac{4K^2}{\pi^2}.$$

Finally,

$$K \stackrel{\mathcal{L}}{=} \frac{T}{(1+U)\sqrt{2}}.$$

Further properties of K and of maxima of Bessel bridges are given by Pitman and Yor (1999).

Appendix

Proof of Lemma 1. We deal with $r \geq 3/2$ first/. Clearly, using $x \geq r$,

$$g(r,k,x) \leq (r + 4k^2x^2/r)\exp(2kxr).$$

Define

$$h(r,k,x) = 2k(r + 4k^2x^2/r) \times e^{-2k^2x^2+2kxr}.$$

Also,

$$g(r,k,x) \geq -(2/r)\,(1+4kxr)\exp(2kxr).$$

Define

$$h^*(r,k,x) = \frac{4k(1+4kxr)}{r} \times e^{-2k^2x^2+2kxr}.$$

We have

$$-h^*(r,k,x) \leq f_k(x) \leq h(r,k,x).$$

For $k \geq K \geq 1$,

$$\begin{aligned}
\frac{h(r,k+1,x)}{h(r,k,x)} &= (1+1/k)\ \frac{1+4(k+1)^2(x/r)^2}{1+4k^2(x/r)^2} \times e^{-(4k+2)x^2+2xr} \\
&\leq 2 \times \frac{1+16}{1+4} \times e^{-6x^2+2xr} \\
&\leq \frac{34}{5} \times e^{-4x^2} \leq \frac{34}{5} \times e^{-4r^2} \leq \frac{34e^{-9}}{5} \stackrel{\text{def}}{=} \xi.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{k=K}^{\infty} f_k(x) &\leq \sum_{k=K}^{\infty} h(r,k,x) \\
&\leq \frac{1}{1-\xi} \times h(r,K,x) \\
&= \frac{1}{1-\xi} \times 2K(r+4K^2x^2/r) \times e^{-2K^2x^2+2Kxr}.
\end{aligned}$$

Reasoning in a similar way,

$$\begin{aligned}
\frac{h^*(r,k+1,x)}{h^*(r,k,x)} &= \left(1+\frac{1}{k}\right) \times \left(1+\frac{4xr}{1+4kxr}\right) \times e^{-(4k+2)x^2+2xr} \\
&\leq 2\left(\frac{2+4r^2}{1+4r^2}\right) \times e^{-6x^2+2xr} \\
&\leq 2 \times \frac{11}{10}\, e^{-4x^2} \\
&\leq 2.2\, e^{-4r^2} \leq 2.2\, e^{-9} \stackrel{\text{def}}{=} \zeta.
\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=K}^{\infty} f_k(x) &\geq -\sum_{k=K}^{\infty} h^*(r,k,x)\\ &\geq -\frac{1}{1-\zeta} \times h^*(r,K,x)\\ &\geq -\frac{1}{1-\zeta} \times \frac{4K(1+4Kxr)}{r} \times e^{-2K^2x^2+2Kxr}.\end{aligned}$$

Consider next the case $r \leq 3/2$ but $x \geq 3/2$. Observe that in this range, $r^2 + 4k^2x^2 - 1 \in [8, 5/4 + 4k^2x^2] \subseteq [8, (41/9)k^2x^2]$. Also, for $\theta \geq 0$, $\sinh\theta \in [\theta, \theta e^{\theta}]$. Thus,

$$-r_k(x) \overset{\text{def}}{=} -8k^2xe^{2kxr-2k^2x^2} \leq f_k(x) \leq \frac{4\times 41}{9} k^4x^3e^{2kxr-2k^2x^2} \overset{\text{def}}{=} R_k(x).$$

For $k \geq 1$,

$$\frac{R_{k+1}(x)}{R_k(x)} = (1+1/k)^4 e^{2xr-2(2k+1)x^2} \leq 16e^{2xr-6x^2} \leq 16e^{3r-27/2} \leq 16e^{-9} \overset{\text{def}}{=} \nu.$$

Thus, $\sum_{k\geq K} f_k(x) \leq R_K(x)/(1-\nu)$ for all $K \geq 1$. Similarly,

$$\frac{r_{k+1}(x)}{r_k(x)} = (1+1/k)^2 e^{2xr-2(2k+1)x^2} \leq 4e^{-9} \overset{\text{def}}{=} \tau.$$

Thus, $\sum_{k\geq K} f_k(x) \geq -r_K(x)/(1-\tau)$ for all $K \geq 1$. $\square$

Proof of Lemma 2.

For the first part, assume $x \leq 3/2$, and let ψ_k and F_k be as defined in (6). For $x \leq \pi/\sqrt{2}$ (which is $\geq \Delta$), using $|\sin x| \leq |x|$, and, for $\alpha, \beta, \theta \geq 0$, $|\alpha\sin\theta - \beta\cos\theta| \leq \alpha\theta + \beta$,

$$\begin{aligned}\psi_k(x) &\leq F_k(x)\left(\frac{k^3\pi^3r}{x^4} - \frac{\pi kr}{x^2}\right)\\ &\leq F_k(x)\left(\frac{k^3\pi^3r}{x^4}\right)\\ &\overset{\text{def}}{=} H_k(x),\end{aligned}$$

and $\psi_k(x) \geq -H_k(x)$. For $k \geq 1$,

$$\frac{H_{k+1}(x)}{H_k(x)} = (1+1/k)^4 \exp\left(-\frac{(2k+1)\pi^2}{2x^2}\right) \leq 16\exp\left(-\frac{3\pi^2}{2x^2}\right) \leq 16\exp\left(-\frac{2\pi^2}{3}\right) \overset{\text{def}}{=} \mu.$$

We conclude that

$$\sum_{k=K}^{\infty} \psi_k(x) \leq \sum_{k=K}^{\infty} H_k(x) \leq \frac{1}{1-\mu} H_K(x).$$

Similarly, on the bottom side,

$$\sum_{k=K}^{\infty} \psi_k(x) \geq -\sum_{k=K}^{\infty} H_k(x) \geq -\frac{1}{1-\mu} H_K(x). \square$$

Proof of Lemma 3.

We will repeatedly use the fact that in a rejection method, the expected number of iterations is not more than the integral of the dominating curve divided by

$$\int_0^a \sqrt{2\pi} f_0(y)\, dy = \sqrt{2\pi}\left(\mathbb{P}\{N \in [-x,x]\} - \mathbb{P}\{N \in [a-x,a+x]\}\right) \geq \sqrt{2\pi}\left(2x\phi(x) + x(\phi(0)-\phi(x)) - 2x\phi(a-x)\right),$$

where the last inequality is only true for $x \leq 1$, as it uses the concavity of ϕ on $[-1,1]$.

For region I, the area under the dominating curve is xa^2. The expected number of iterations is not more than

$$\begin{aligned}
&\frac{xa^2}{\sqrt{2\pi}\left(2x\phi(x) + x(\phi(0)-\phi(x)) - 2x\phi(a-x)\right)} \\
&= \frac{a^2}{\sqrt{2\pi}\left(2\phi(x) + (\phi(0)-\phi(x)) - 2\phi(a-x)\right)} \\
&\leq \frac{a^2}{\sqrt{2\pi}\left(\phi(0) + \phi(x) - 2\phi(a-x)\right)} \\
&= \frac{a^2}{(1+\exp(-x^2/2) - 2\exp(-(a-x)^2/2))} \\
&\leq \frac{a^2}{(1-\exp(-(a-x)^2/2))} \\
&\leq \frac{a^2}{\frac{(a-x)^2}{2} e^{-1/2}} \\
&\quad \text{(by using Taylor's series with remainder)} \\
&\leq \frac{a^2}{\frac{a^2}{8\sqrt{e}}} = 8\sqrt{e}.
\end{aligned}$$

Region II: Using the convexity in the tails of ϕ, the expected number of iterations is

$$\begin{aligned}
&\frac{\mathbb{P}\{N \in [-x,x]\}}{\mathbb{P}\{N \in [-x,x]\} - \mathbb{P}\{N \in [a-x,a+x]\}} \\
&\leq \frac{\mathbb{P}\{N \in [-x,x]\}}{\mathbb{P}\{N \in [-x,x]\} - \mathbb{P}\{N \geq a-x\}} \\
&\leq \frac{\mathbb{P}\{N \in [-x,x]\}}{\mathbb{P}\{N \in [-x,x]\} - \mathbb{P}\{N \geq x\}} \\
&\leq \frac{\mathbb{P}\{|N| \leq 1\}}{\mathbb{P}\{|N| \leq 1\} - \mathbb{P}\{N \geq 1\}} \\
&= \frac{2\Phi(1)-1}{3\Phi(1)-2}
\end{aligned}$$

Again, the bound is uniformly valid over the region.

For region III, note that

$$\int_{\mathbb{R}} g(y)\, dy = 2x + 2x^2\sqrt{2\pi}.$$

The expected number of iterations is not more than

$$\frac{2x + 2x^2\sqrt{2\pi}}{\sqrt{2\pi}\,(2x\phi(x) + x(\phi(0) - \phi(x)) - 2x\phi(a-x))}$$
$$\le \frac{2 + 2x\sqrt{2\pi}}{\sqrt{2\pi}\,(\phi(0) + \phi(x) - 2\phi(a-x))}$$
$$\le \frac{2 + 2x\sqrt{2\pi}}{\sqrt{2\pi}\,(\phi(0) - \phi(a-x))}$$
$$\le \frac{2 + 2x\sqrt{2\pi}}{\sqrt{2\pi}\,(\phi(0) - \phi(a/2))}$$
$$\le \frac{2 + 2\sqrt{2\pi}}{\sqrt{2\pi}\,(\phi(0) - \phi(1/2))}$$
$$= \frac{2 + 2\sqrt{2\pi}}{1 - e^{-1/8}}. \square$$

Proof of Lemma 4.

Taking the normalization constant into account, The expected number of iterations of this simple algorithm is

$$\frac{\int_0^a f_0(y)\,dy}{\int_0^a f(y)\,dy} \le \frac{\int_0^a f_0(y)\,dy}{\int_0^a (f_0(y) + f_{-1}(y))\,dy}.$$

It is easy to check that

$$\int_0^a f_0(y)\,dy = \mathbb{P}\{N \in [-x, a-x]\} - \mathbb{P}\{N \in [x, a+x]\} = \mathbb{P}\{N \in [-x, x]\} - \mathbb{P}\{N \in [a-x, a+x]\}.$$

Similarly,

$$\int_0^a f_{-1}(y)\,dy = \mathbb{P}\{N+2a \in [-x, a-x]\} - \mathbb{P}\{N+2a \in [x, a+x]\} = \mathbb{P}\{N \in [-2a-x, -2a+x]\} - \mathbb{P}\{N \in [-a-x, -a+x]\}.$$

By monotonicity over the parameter range of interest, $\mathbb{P}\{N \in [a-x, a+x]\} \le x\phi(a) + x\phi(a-x)$. Thus,

$$\frac{\int_0^a f_0(y)\,dy}{\int_0^a (f_0(y) + f_{-1}(y))\,dy}$$
$$= \frac{\mathbb{P}\{N \in [-x, x]\}}{\mathbb{P}\{N \in [-x, x]\} + \mathbb{P}\{N \in [2a-x, 2a+x]\} - 2\mathbb{P}\{N \in [a-x, a+x]\}}$$
$$\le \frac{\mathbb{P}\{N \in [-x, x]\}}{\mathbb{P}\{N \in [-x, x]\} - 2x\phi(a-x) - 2x\phi(a)}$$
$$\le \frac{\mathbb{P}\{N \in [-x, x]\}}{\mathbb{P}\{N \in [-x, x]\} - 2x\phi(a/2) - 2x\phi(a)}$$
$$\le \frac{\mathbb{P}\{N \in [-a/2, a/2]\}}{\mathbb{P}\{N \in [-a/2, a/2]\} - a\phi(a/2) - a\phi(a)}$$
$$\le \frac{\mathbb{P}\{|N| \le 1\}}{\mathbb{P}\{|N| \le 1\} - 2\phi(1) - 2\phi(2)} \quad \stackrel{\text{def}}{=} \rho^*.$$

This uniform bound ρ^* is valid for the entire parameter region $a \ge 2$, $x \le a/2$. $\square$

PROOF OF LEMMA 5. The expected number of iterations is given by

$$\frac{\int_0^a g(y)\,dy}{\int_0^a f(y)\,dy}.$$

Verify that

$$\int_0^a g(y)\,dy = \frac{\pi^2 x}{a(1-\rho)} \exp\left(-\frac{\pi^2}{2a^2}\right).$$

Also, using $\sin\theta \ge (2/\pi)\theta$, $0 \le \theta \le \pi/2$, and the fact that $\int_0^a f_n(y)\,dy = 0$ for even values of n,

$$\begin{aligned}
\int_0^a f(y)\,dy &\ge \int_0^a f_1(y)\,dy - \int_0^a h_3(y)\,dy \\
&= \int_0^a f_1(y)\,dy - \int_0^a h_3(y)\,dy \\
&= \frac{4}{\pi}\sin\left(\frac{\pi x}{a}\right)\exp\left(-\frac{\pi^2}{2a^2}\right) - \frac{9\pi^2 x}{a(1-\rho)}\exp\left(-\frac{9\pi^2}{2a^2}\right) \\
&\ge \frac{8x}{\pi a}\exp\left(-\frac{\pi^2}{2a^2}\right) - \frac{9\pi^2 x}{a(1-\rho)}\exp\left(-\frac{9\pi^2}{2a^2}\right).
\end{aligned}$$

The ratio we seek is thus at most

$$\begin{aligned}
\frac{\pi^2}{\frac{8(1-\rho)}{\pi} - 9\pi^2\exp\left(-\frac{8\pi^2}{2a^2}\right)} &= \frac{\pi^3}{8(1-\rho) - 9\pi^3\exp\left(-\frac{4\pi^2}{a^2}\right)} \\
&\le \frac{\pi^3}{8(1-\rho) - 9\pi^3\exp\left(-\pi^2\right)} \\
&\overset{\text{def}}{=} \chi < \infty. \ \square
\end{aligned}$$

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