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# *On random variate generation for the generalized hyperbolic secant distributions*

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We give random variate generators for the generalized hyperbolic secant distribution and related families such as Morris's skewed generalized hyperbolic secant family and a family introduced by Laha and Lukacs. The rejection method generators are uniformly fast over the parameter space and are based upon a complex function representation of the distributions due to Harkness and Harkness

*Keywords:* Random variate generation, generalized hyperbolic secant distribution, gamma function, natural exponential family, probability inequalities

## 1. Introduction

Natural exponential families of distributions have probability mass functions of the form $[\exp(\theta x)]\mu(dx)$ where $\mu$ is a given measure, and $\theta > 0$ is a parameter. When we compute the mean and the variance, and force the variance to be a quadratic function of the mean as $\theta$ is varied, the number of families becomes severely restricted. Morris (1982) showed that there are in fact only six natural exponential families with this property: the binomial, Poisson, gamma, exponential, negative binomial and NEF–GHS families, where GHS is an abbreviation for generalized hyperbolic secant. The NEF–GHS distribution with parameters $\rho > 0$ and $\lambda \in \mathbb{R}$ has density

$$f(x) = (1+\lambda^2)^{-\rho/2}\exp(x\arctan\lambda)f_\rho(x),$$

where $f_\rho$ is the density of the generalized hyperbolic secant (GHS) distribution with parameter $\rho$. While the other five families play crucial roles in statistics, the NEF–GHS distribution has received very little attention, undoubtedly because of its unwieldy analytic form. Indeed, $f_\rho$ is not explicitly known in any standard way. Its shortest description is as a product of two gamma functions with imaginary arguments,

$$f_\rho(x) = \frac{2^{\rho-2}}{\pi\Gamma(\rho)}\Gamma\left(\frac{\rho+ix}{2}\right)\Gamma\left(\frac{\rho-ix}{2}\right)$$

(Harkness and Harkness, 1968). One of the difficulties when running a simulation is random variate generation. The method developed below seems to be the first one in which an efficient random variate generator is obtained based upon complex-valued function representations. The purpose of this note is to discuss random variate generation for the families mentioned above. This is done in three stages: first we recall the well-known hyperbolic secant distribution. Then we move on to the GHS distribution. Finally, we give a generator for the NEF–GHS distribution. There are, of course, two things we would like to see in such generators:

(i) The generators have to be theoretically exact; no approximation of any kind is allowed.
(ii) The expected time per random variate should be uniformly bounded over the parameters (such as $\rho > 0$).

## 2. The hyperbolic secant distribution

The hyperbolic secant (HS) distribution has density

$$f_1(x) = \frac{1}{2}\operatorname{sech}\frac{\pi x}{2} = \frac{1}{\exp(\pi x/2)+\exp(-\pi x/2)}$$

and distribution function

$$F(x) = 1 - \frac{2}{\pi}\arctan(\exp(-\pi x/2))$$

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(Baten, 1934, and Talacko, 1951). Random variate generation was discussed in some detail in Devroye (1986a). Here is a partial listing of possible methods:

(i) Generate $X$ as $\log|C|$, where $C$ is a Cauchy random variable. Equivalently, generate it as $\log|N_1/N_2|$, where $N_1, N_2$ are independent standard normal random variables.
(ii) Obtain $X$ as $(2/\pi)\log\tan(\pi U/2)$ where $U$ is uniform[0, 1]. This is the inversion method.
(iii) Since $f$ is log-concave, the general rejection method for log-concave densities given in Devroye (1984) can be applied.
(iv) Since

$$\varphi(t)=\prod_{j=1}^{\infty}\left(\frac{1}{1+\frac{4t^2}{(2j-1)^2\pi^2}}\right),$$

(see e.g. Laha and Lukacs, 1960), we can obtain a HS variate as $\sum_{j=1}^{\infty} 2L_j/(\pi(2j-1))$, where the $L_j$'s are i.i.d. Laplace random variables. This property is obviously only of theoretical interest.

## 3. The GHS distribution

The GHS distribution has characteristic function

$$\varphi(t)=(\operatorname{sech} t)^{\rho}=\left(\frac{2}{\exp(t)+\exp(-t)}\right)^{\rho},$$

where $\rho>0$ is a parameter. The GHS distribution is symmetric about 0, has mean 0 and variance $\rho$, and is unimodal with mode at 0. It possesses exponentially decaying tails. For the case $\rho=1$, we obtain the HS distribution. For integer $\rho$, a GHS random variate $X$ is distributed as

$$X_1+\cdots+X_{\rho},$$

where the $X_i$'s are i.i.d. HS random variables (Harkness and Harkness, 1968). The time taken by the naive method that exploits this property grows linearly with $\rho$.

Of course, our aim is to be able to deal with non-integer $\rho$ as well, and to obtain uniformly bounded expected time. This will be done in the next three sections.

**Remark: special cases**
For the sake of completeness, we mention three special cases in which explicit forms of $f_{\rho}$ are known (see Harkness and Harkness, 1968);

$$f_2(x)=\frac{x}{2}\operatorname{cosech}\frac{\pi x}{2}=\frac{x}{\exp(\pi x/2)-\exp(-\pi x/2)};$$

$$f_{2n+1}(x)=\frac{2^{2n-1}}{(2n)!}\operatorname{sech}\frac{\pi x}{2}\prod_{j=1}^{n}\left(\frac{x^2}{4}+\left(\frac{2j-1}{2}\right)^2\right);$$

$$f_{2n}(x)=\frac{4^{n-1}x}{2(2n-1)!}\operatorname{cosech}\frac{\pi x}{2}\prod_{j=1}^{n-1}\left(\frac{x^2}{4}+j^2\right).$$

The last two expressions show that there are simple recursive relations between the densities of the GHS distribution whose parameters $\rho$ differ by 2.

Instead of relying on the representations of the previous remark, we will use the gamma function representation. It will also allow us to deal with non-integer $\rho$.

## 4. The rejection method for the GHS distribution

Assuming the unimodality of $f$ with mode at 0 (to be shown below), we have from Devroye (1986a, pp. 313–316; see Theorem VII.3.3)

$$f(x)\le\min\left(f(0),\frac{3\sigma^2}{2|x|^3}\right),$$

where $\sigma^2$ is the variance of $f$. The area under the bounding curve is easily seen to be

$$(81/2)^{1/3}(f(0)\sigma)^{2/3}.$$

This is equal to the expected number of iterations required if the rejection method is employed. We note that $\sigma^2=\rho$, so that the given area is

$$\left(\frac{2^{\rho-5/2}9\sqrt{\rho}\,\Gamma^2(\rho/2)}{\pi\Gamma(\rho)}\right)^{1/3}$$

As $\rho\to\infty$, we have $f(0)\sim 1/\sqrt{2\pi\rho}$ by Stirling's approximation for the $\Gamma$ function. The area under the bounding curve is asymptotically $(81/(4\pi))^{1/3}$. It is uniformly bounded over all $\rho\ge 1$. Unfortunately, as $\rho\downarrow 0$, the area tends to $\infty$, so another design is required in that case.

**Rejection method for the GHS distribution**

repeat
  generate a uniform $[-1,1]$ random variate $U$.
  generate i.i.d. uniform $[0,1]$ random variates $V, W$.
  $X\leftarrow\left(\frac{3\sigma^2}{2f(0)}\right)^{1/3}\frac{U}{V^2}$
until $W\min(f(0),3\sigma^2/(2|X|^3))<f(X)$
return $X$

The main obstacle here is that $f$ is only available as a product of two complex-valued functions. At issue here is whether we should assume that the $\Gamma$ function with complex argument is available to the general user. If it is not, the rejection method needs to be replaced by the series method, which is based upon quickly converging approximations for such functions. The decision $Y < f(X)$ can in fact be made correctly without ever having to evaluate $f$ directly. To do so, we first deduce a further representation of $f$ in Lemma G2.

## 5. Properties of $\Gamma$

**Lemma G1** (Whittaker and Watson, 1980, pp. 249–251) For all complex $z$ with $\Re(z) > 0$,

$$\Gamma(z) = J(z)\exp(R(z)),$$

where

$$J(z) \stackrel{\text{def}}{=} \left(\frac{z}{e}\right)^z \sqrt{\frac{2\pi}{z}},$$

and

$$R(z) \stackrel{\text{def}}{=} 2\int_0^\infty \frac{\arctan(t/z)}{\exp(2\pi t)-1}\,dt = \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{\exp(t)-1}\right)\frac{\exp(-tz)}{t}\,dt.$$

**Lemma G2**

$$f(x) = \frac{(\rho/e)^\rho}{\Gamma(\rho+1)}\left(1+\frac{x^2}{\rho^2}\right)^{(\rho-1)/2}\exp(-x\arctan(x/\rho)) \times \exp\left(\int_0^\infty \psi(t)2\exp(-t\rho/2)\cos\frac{tz}{2}\,dt\right),$$

where

$$\psi(t) \stackrel{\text{def}}{=} \frac{1}{t}\left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}\right).$$

**Lemma G3** The function $\psi(t)$ is monotonically decreasing in $t \geq 0$, with $\psi(0) = 1/12$ and $\lim_{t\to\infty}\psi(t) = 0$.

**Lemma G4** Define

$$C = \sqrt{2\pi\rho}\,\frac{(\rho/e)^\rho}{\Gamma(\rho+1)}\exp\left(\frac{1}{3\rho}\right)$$

and

$$g(x) = \frac{1}{\sqrt{2\pi\rho}}\left(1+\frac{x^2}{\rho^2}\right)^{(\rho-1)/2}\exp(-x\arctan(x/\rho)).$$

Then

$$1 \leq \frac{f(x)}{Cg(x)} \leq \exp\left(\frac{\rho}{3(\rho^2+x^2)}\right) \leq \exp\left(\frac{1}{3\rho}\right).$$

Furthermore, $C \leq 1$ and $C \to 1$ as $\rho \to \infty$.

## 6. Rejection with a perfect asymptotic fit

Rejection with a rejection constant tending to one as $\rho \to \infty$, forces one to choose as a dominating curve the limiting density in the family, in this case the normal density.

**Lemma R1** Define $\xi = \rho^{5/8}$.

$$f(x) \leq \begin{cases} \dfrac{C}{\sqrt{2\pi\rho}}\exp\left(-\dfrac{x^2}{2\rho}\right)\exp(1/(3\sqrt{\rho})) & (|x| \leq \xi) \\ Cg(\xi)\exp(g'(\xi)(|x|-\xi)g(\xi)) & (|x| \geq \xi). \end{cases}$$

The rejection algorithm with rejection from the normal distribution in the main body and rejection from exponential tails can be sped up via quick rejection and acceptance steps by using the inequalities of Lemma G4. We summarize the algorithm as follows:

**Rejection method for the GHS distribution,** $\rho \geq 1$

[Set-up.]
$t \leftarrow \rho^{5/8}$
$s \leftarrow \exp(1/(3\sqrt{\rho}))$
$C \leftarrow \sqrt{2\pi\rho}\,\dfrac{(\rho/e)^\rho}{\Gamma(\rho+1)}$
$p_n \leftarrow Cs$
$g_t \leftarrow g(t)$ where $g(x) \stackrel{\text{def}}{=} (2\pi\rho)^{-1/2}\left(1+\dfrac{x^2}{\rho^2}\right)^{(\rho-1)/2}$
  $\exp(-x\arctan(x/\rho))$
$\lambda \leftarrow g(t)/|g'(t)|$ (i.e., $\lambda \leftarrow (t/(\rho^2+t^2)+\arctan(t/\rho))^{-1}$)
$p_t \leftarrow 2Cg_t\lambda$
[Generator.]
repeat
  generate i.i.d. uniform $[0,1]$ random variates $U, V$.
  if $U < \dfrac{p_n}{p_n+p_t}$
   then generate a standard normal random variate $N$
    set $X \leftarrow N\sqrt{\rho}$
    if $|X| > t$ then Accept $\leftarrow$ False
       else $W \leftarrow Vp_n(2\pi\rho)^{-1/2}\exp(-X^2/(2\rho))$
        Accept $\leftarrow [W < Cg(X)]$
        if not Accept
          then Accept $\leftarrow [W < Csg(X)]$
            if Accept then
            Accept $\leftarrow [W < f(X)]$
   else generate an exponential random variate $E$
    set $X \leftarrow t + \lambda E$
    $W \leftarrow VCg_t\exp(-E)$
    Accept $\leftarrow [W < f(X)]$
    if Accept, then with probability 1/2, set $X \leftarrow -X$
until Accept
return $X$

**Lemma R2** The rejection method given above is asymptotically optimal, i.e. the expected number of iterations tends to

1 as $\rho \to \infty$. Also, the expected number of iterations is uniformly bounded over all $\rho \geq 1$.

The proof of Lemma R2 reveals that the expected number of decisions $[W < f(X)]$ is $O(1)$. The evaluation of $f$ can therefore be programmed in a less critical manner. This is done in the next section.

## 7. Evaluation of $f$

In the algorithm shown above, we need to make a decision whether $W < f(X)$. From Lemma G2, we recall the representation

$$f(x) = Cg(x)\exp\left(\int_0^\infty I(t)\,dt\right),$$

where

$$I(t) = \psi(t) 2\exp(-t\rho/2)\cos(tx/2)$$
$$= \psi(t)(\exp(-tz) + \exp(-t\bar{z})),$$

and $z = (\rho + ix)/2$. Let $k$ and $n$ be positive integers to be picked further on. The integral can be computed to any desired accuracy by the approximation

$$\int_0^\infty \psi(t)(\exp(-tz) + \exp(-t\bar{z}))dt$$
$$= \sum_{j=0}^{kn-1} \psi(j/n) \int_{j/n}^{(j+1)/n} (\exp(tz) + \exp(-t\bar{z}))dt + \Delta_n$$
$$= \sum_{j=0}^{kn-1} \psi(j/n) R(\rho, j, x, n) + \Delta_n$$

where

$$R(\rho, j, x, n) \stackrel{\text{def}}{=} 2\mathrm{e}^{(j+1)\rho/2n}$$
$$\times \frac{\left(\frac{\rho}{2}\right)\cos\left(\frac{(j+1)x}{2n}\right) + \left(\frac{x}{2}\right)\sin\left(\frac{(j+1)x}{2n}\right)}{(\rho/2)^2 + (x/2)^2}$$
$$- 2\exp\left(\frac{j\rho}{2n}\right)\frac{\left(\frac{\rho}{2}\right)\cos\left(\frac{jx}{2n}\right) + \left(\frac{x}{2}\right)\sin\left(\frac{jx}{2n}\right)}{(\rho/2)^2 + (x/2)^2},$$

and

$$|\Delta_n| \leq \int_0^\infty \psi(t)|\exp(-tz) + \exp(-t\bar{z})|dt + \frac{\|\psi'\|_\infty}{n}$$
$$\times \int_0^k |\exp(-tz) + \exp(-t\bar{z})|dt$$
$$\leq \int_k^\infty \frac{1}{2t}(2\exp(-t\rho/2))dt + \frac{\|\psi'\|_\infty}{n}\int_0^k 2\exp(-t\rho/2)dt$$
$$\leq \frac{2\exp(-k\rho/2)}{k\rho} + \frac{4\|\psi'\|_\infty}{n\rho}$$
$$\leq \frac{2\exp(-k\rho/2)}{k\rho} + \frac{8}{5n\rho}. \tag{1}$$

Here we used the facts that $\psi(t) \leq 2/t$ and that $\|\psi'\|_\infty \leq 2/5$. These properties are proved in Lemma E1 below.

**Lemma E1** $\psi(t) \leq 2/t$ and $\|\psi'\|_\infty \leq 2/5$.

We can use the approximation as follows. To decide whether $W < f(X)$ we argue as follows:

```
ε ← 1/(10ρ)
repeat
  ε ← ε/2
  n ← ⌈16/(5ερ)⌉
  k ← ⌈(2/ρ) log(4/ε)⌉
  Z ← Σ_{j=0}^{kn-1} ψ(j/n)R(ρ,j,X,n)
  Accept ← [W < Cg(X) exp(Z − ε)]
  Reject ← [W > Cg(X) exp(Z + ε)]
until Accept or Reject
if Accept then decide W < f(X) else decide W ≥ f(X)
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The procedure shown above reduces the accuracy $\epsilon$ by a factor of 2 at each iteration, so that indeed the method halts with probability 1. Observe that the choice of $k$ and $n$ ensures that for fixed $\epsilon$, the approximation $Z$ is such that

$$\left|Z - \int_0^\infty I(t)dt\right| < \epsilon$$

(see (1)). However, using this piece of code destroys the uniform boundedness of the expected time.

## 8. Generalizations of the GHS distribution

There are several generalizations of the GHS distribution that introduce asymmetry. The Laha–Lukacs distribution is defined via the characteristic function

$$\varphi(t) = (\cosh t - i\lambda \sinh t)^{-\rho},$$

where $\rho > 0, \lambda \in \mathbb{R}$ (Laha and Lukacs, 1960).

Morris's NEF–GHS distribution with parameters $\rho > 0$ and $\lambda \in \mathbb{R}$ is given in the introduction. The mean is $\rho\lambda$ and the variance is $\rho(1 + \lambda^2)$. By computing the mean and variance of the Laha–Lukacs distribution given above, it is easy to see that the variance function is indeed quadratic, and that it coincides with the NEF–GHS family, a fact perhaps overlooked in the literature thus far. Very little is known about the NEF–GHS distribution, other than what is given in Morris (1982). It is interesting to note in this respect that for integer $\rho$,

$$X_1 + \cdots + X_\rho$$

is NEF–GHS $(\rho, \lambda)$ if the $X_i$'s are i.i.d. NEF–GHS $(1, \lambda)$.

The latter random variates can each be obtained as

$$\frac{1}{\pi}\log\frac{B}{1-B},$$

where $B$ is a beta$(\frac{1}{2}+(\tan\lambda/\pi),\frac{1}{2}-(\tan\lambda/\pi))$ random variable. This shortcut is only useful if $\rho$ is a small integer.

**Remark: cubic variance functions.** Letac and Mora (1990) studied all the natural exponential families with cubic variance functions. These include the GHS distribution and several other families of distributions, some of which require special care when random variate generators are needed. Several of the distributions given by them have representations not unlike those for the NEF–GHS family, and it is hoped that random variate generators might be based upon principles close to those developed in this paper.

Let $f$ denote the NEF–GHS density. Good rejection algorithms are either based upon some properties of $f$, or on some properties of a function that bounds $f$ from above. In view of the unusual representations of $f$, it is easier to work with bounds for $f$. To this end, we use the estimate from Lemma G4: if

$$C=\sqrt{2\pi\rho}\,\frac{(\rho/e)^{\rho}}{\Gamma(\rho+1)}\exp\left(\frac{1}{3\rho}\right)$$

and

$$g(x)=\frac{C}{\sqrt{2\pi\rho}}(1+\lambda^2)^{-\rho/2}\left(1+\frac{x^2}{\rho^2}\right)^{(\rho-1)/2}\times\exp(x\arctan\lambda-x\arctan(x/\rho)),$$

then

$$\exp(-1/(3\rho))\leq\frac{f(x)}{g(x)}\leq 1.$$

But even the bounding $g$ is rather elusive — for example, it is difficult to find where its mode lies even though we can approximate it. However, $g$ has one saving feature: it is log-concave when $\rho\geq 1$. It is easy to see that the derivative of $\log g$ is

$$\frac{g'(x)}{g(x)}=-\frac{x}{\rho^2+x^2}+\arctan\lambda-\arctan(x/\rho),$$

and that the second derivative is non-positive for $\rho\geq 1$. In the remainder of this section, we therefore assume that $\rho\geq 1$. The log-concavity implies that at every $x,y$,

$$g(y)\leq g(x)\exp((y-x)g'(x)/g(x)).$$

This provides us with an infinite number of possible bounds for $g$ since we can choose $x$ at will. At least two $x$'s are needed to obtain two bounds whose minimum yields an integrable curve. We find it convenient to use three $x$'s. Ideally, one would like to have a flat bound at the mode of $g$ and two exponentially decaying tails. Since the mode of $g$ is not available in explicit analytical form, we will settle for a nearly flat center bound at the mean $x=\lambda\rho$. The bound is applied to all $x\in(\lambda\rho-\delta,\lambda\rho+\delta)$, where $\delta$ is of the order of magnitude of the standard deviation:

$$\delta=D\sqrt{\rho}\sqrt{1+\lambda^2},$$

with $D\geq 4$. To be precise, we assume without loss of generality that $\lambda>0$ (since an NEF–GHS $(\rho,\lambda)$ variable is distributed as minus an NEF–GHS $(\rho,-\lambda)$ random variable). Next, define the following constants:

$$(t_l,t_m,t_r)=(\lambda\rho-\delta,\lambda\rho,\lambda\rho+\delta);$$

$$(\lambda_l,\lambda_m,\lambda_r)=\left(\frac{g'(t_l)}{g(t_l)},\frac{g'(t_m)}{g(t_m)},\frac{g'(t_r)}{g(t_r)}\right).$$

Of the last three, $\lambda_l>0$, while $\lambda_m\leq 0$ and $\lambda_r<0$; this will be established further on. The bound we will use in the rejection algorithm then is

$$f(x)\leq g(x)\leq\begin{cases}g(t_l)\exp(\lambda_l(x-t_l)) & (x\leq t_l)\\ g(t_m)\exp(\lambda_m(x-t_m)) & (t_l<x<t_r)\\ g(t_r)\exp(\lambda_r(x-t_r)) & (x\geq t_r).\end{cases}$$

The areas under the left, middle, and right pieces are denoted by $p_l$, $p_m$ and $p_r$, where

$$(p_l,p_m,p_r)=\left(\frac{g(t_l)\exp(-\delta\lambda_l)}{\lambda_l},\frac{g(t_m)|\exp(\delta\lambda_m)-\exp(-\delta\lambda_m)|}{|\lambda_m|},\frac{g(t_r)\exp(\delta\lambda_r)}{|\lambda_r|}\right).$$

If $\lambda_m=0$ (which happens when $\lambda=0$), the middle term should be replaced by its limit, $2\delta g(t_m)$. The areas just given should be used as weights when picking a piece in the rejection algorithm. We summarize the algorithm:

**Generator for the NEF–GHS distribution,** $\rho\geq 1,\lambda>0$

repeat
  generate $U$, $V$ i.d.d. uniformly on $[0,1]$
  if $U<p_l/(p_l+p_m+p_r)$ then generate $E$ exponential
    $X\leftarrow t_l-E/\lambda_l$
    $T\leftarrow Vg(t_l)\exp(-E)$
  else if $U>(p_l+p_m)/(p_l+p_m+p_r)$ then generate $E$ exponential
    $X\leftarrow t_r-E/\lambda_r$
    $T\leftarrow Vg(t_r)\exp(-E)$
  else generate $W$ uniformly on $[0,1]$
    $X\leftarrow t_l+(1/\lambda_m)\log(1-W(1-\exp(2\delta\lambda_m)))$
    (if $\lambda_m=0$, set $X\leftarrow t_l+2\delta W$)
    $T\leftarrow Vg(t_m)\exp(\lambda_m(X-t_m))$
    (if $\lambda_m=0$, set $T\leftarrow Vp_m/(2\delta)$)
  Accept $\leftarrow[T<g(X)\exp(-1/(3\rho))]$ ('quick accept')
  if not Accept

then Accept ← $[T < g(X)]$ ('quick reject')
if Accept then Accept ← $[T < f(X)]$
until Accept
return $X$

One can easily verify that the middle $X$ obtained from the random variable $W$ is exponentially distributed on $(t_l, t_r)$ via the probability integral transform (the inversion method). We require once again the evaluation of $f$ if the quick acceptance and rejection steps fail. This can be done on the basis of the algorithm of the previous section. The expected number of iterations before the algorithm halts is $p_l + p_m + p_r$. The main result of this paper is that this quantity is uniformly bounded.

**Theorem 1** If $D \geq 4$, the quantity $p_l + p_m + p_r$ is uniformly bounded over all $\rho \geq 1$ and $\lambda \geq 0$.

The proof of this theorem is given in the next section. The condition $D \geq 4$ can probably be relaxed somewhat. We recommend that in practice, the value $D = 4$ be selected. We finally note that the expected number of accesses of the statement 'Accept ← $[T < f(X)]$' is $O(1)$.

## 9. Proofs

**Proof of Lemma G2** Let $z = (\rho + ix)/2$, and let $\bar{z}$ denote the complex conjugate of $z$. From Lemma C1 and Lemma G1, we see that

$$f(x) = \frac{2^{\rho-2}}{\pi\Gamma(\rho)} J(z)J(\bar{z}) \exp(R(z) + R(\bar{z})),$$

in the notation of those lemmas. Now, if $z = |z|\exp(i\theta)$, we have

$$\begin{aligned} J(z)J(\bar{z}) &= (|z|\exp(i\theta - 1))^z \sqrt{2\pi/z}(|z|\exp(-i\theta - 1))^{\bar{z}}\sqrt{2\pi/\bar{z}} \\ &= (2\pi/|z|)|z|^{z+\bar{z}} \exp(-(z+\bar{z}))\exp(i\theta(z-\bar{z})) \\ &= (2\pi/|z|)(|z|/e)^{2\Re z}\exp(-2\theta\Im z) \end{aligned}$$

We note that $\Re z = \rho/2$, $\Im z = x/2$, $|z| = \sqrt{\rho^2 + x^2}/2$, and $\theta = \arctan(x/\rho)$. Resubstitution in the formula shows that

$$\begin{aligned} J(z)J(\bar{z}) &= \left(\frac{\sqrt{\rho^2+x^2}}{2e}\right)^{\rho} \exp(-x\arctan(x/\rho)) \\ &\quad \times \frac{4\pi}{\sqrt{\rho^2+x^2}} \\ &= 4\pi\rho^{\rho}\left(1 + \frac{x^2}{\rho^2}\right)^{(\rho-1)/2} (2e)^{-\rho}\exp(-x\arctan(x/\rho)). \end{aligned}$$

Collecting these identities, we see that we are only left with the proof of the equation

$$R(z) + R(\bar{z}) = \int_0^\infty \psi(t) 2\exp(-t\rho/2)\cos\frac{tx}{2}\,dt,$$

From the definition of $R(z)$,

$$\begin{aligned} R(z) + R(\bar{z}) &= \int_0^\infty \psi(t)\exp(-tz)dt + \int_0^\infty \psi(t)\exp(-t\bar{z})dt \\ &= \int_0^\infty \psi(t)(\exp(-tz) + \exp(-t\bar{z}))dt \\ &= \int_0^\infty \psi(t)2\exp(-t\Re z)\cos(t\Im z)dt \\ &= \int_0^\infty \psi(t)2\exp(-t\rho/2)\cos(tx/2)dt. \end{aligned}$$

**Proof of Lemma G3.** We have

$$\psi(t) = \frac{1}{2t} - \frac{1}{t^2} + \frac{1}{t(\exp(t)-1)},$$

$$\psi'(t) = \frac{1}{2t^2} + \frac{2}{t^3} - \frac{1}{t^2(\exp(t)-1)} - \frac{\exp(t)}{t(\exp(t)-1)^2}.$$

We see that $\psi'(t) \leq 0$ for $t \geq 4$. Also, by L'Hôpital's rule, $\psi(0) = 1/12$, and $\psi'(0) = 0$. Firstly, for $t \geq 4$, we have $\psi'(t) \leq 0$ since $2/t^3 \leq -1/(2t^2)$. Define

$$S(t) = \frac{\exp(t) - (1 + t + t^2/2! + t^3/3! + t^4/4!)}{t^5/5!}$$

Observe that $S(0) = 1$ and that $S$ is ↑ in $t$. Furthermore, for $t > 0$,

$$S(t) = 1 + \frac{t}{6} + \frac{t^2}{7{\cdot}6} + \cdots > 1 + \frac{t}{6}.$$

Also, for $t < 8$,

$$\begin{aligned} S(t) &\leq 1 + \frac{t}{6} + \frac{t^2}{42}\left(1 + \frac{t}{8} + \left(\frac{t}{8}\right)^2 + \cdots\right) \\ &= 1 + \frac{t}{6} + \frac{t^2}{42(1 - t/8)}. \end{aligned}$$

Taking into account that $e^t - 1 \geq t$, we have for $8 > t > 0$,

$$\psi'(t) = \frac{S(2t)(32/30 - 8t/30) - S(t)(2/30 + t^2/16) - 1 - t/12}{2((e^t-1)/t)^2}.$$

The denominator is positive. The numerator is at most equal to

$$\frac{S(2t)(32/30 - 8t/30) - S(t)(2/30 + t^2/16) - 1 - t/12}{2}$$

$$\leq \frac{(1+2t/6+32t^2/(42(8-2t)))(32/30-8t/30)-(1+t/6)(2/30+t^2/16)-1-t/12}{2}$$

$$= \frac{-t/180-251t^2/5040-t^3/96}{2}$$

$$< 0.$$

This completes the proof of Lemma G3.

**Proof of Lemma G4** Combine Lemmas G2 and G3, and note that

$$\int_0^\infty \exp(-t\rho/2)\cos\frac{tx}{2}\,dt = \frac{2\rho}{\rho^2+x^2}.$$

By Stirling's formula, we can verify that $C \to 1$ and $\rho \to \infty$, and that $C \leq 1$.

**Proof of Lemma R1** For $|x| \leq \rho^{5/8}$,

$$f(x) \leq Cg(x) \leq \frac{C}{\sqrt{2\pi\rho}}\left(1+\frac{x^2}{\rho^2}\right)^{(\rho-1)/2}\exp(-x\arctan(x/\rho))$$

$$\leq \frac{C}{\sqrt{2\pi\rho}}\exp\left(\frac{x^2(\rho-1)}{2\rho^2}\right)\exp(-x^2/\rho+x^4/(3\rho^3))$$

$$= \frac{C}{\sqrt{2\pi\rho}}\exp\left(-\frac{x^2}{2\rho}-\frac{x^2}{2\rho^2}+\frac{x^4}{3\rho^3}\right)$$

$$\leq \frac{C}{\sqrt{2\pi\rho}}\exp\left(-\frac{x^2}{2\rho}\right)\exp(1/(3\sqrt{\rho}))$$

where we used the fact that for $u > 0$, $\arctan u \geq u - u^3/3$. Consider next $|x| > \rho^{5/8}$. It is easy to verify that $g$ is log-concave when $\rho \geq 1$. Thus,

$$\frac{g'(x)}{g(x)} = -\frac{x}{\rho^2+x^2} - \arctan\frac{x}{\rho}.$$

Furthermore,

$$(\log g)'' = \frac{x^2-\rho^2-\rho^3-\rho x^2}{(\rho^2+x^2)^2} < 0.$$

For log-concave functions, we always have the bound

$$g(x) \leq g(t)\exp(g'(t)(x-t)/g(t))(x \geq t)$$

(Devroye, 1986a, p. 308). Collecting this,

$$f(x) \leq Cg(\xi)\exp(g'(\xi)(x-\xi)/g(\xi))(x \geq \xi).$$

**Proof of Lemma R2** The integral of the bound of Lemma R1 between $t = \rho^{5/8}$ and $\infty$ is, for $t > 0$,

$$\frac{g^2(t)}{|g'(t)|}.$$

The rejection algorithm thus has a rejection constant equal to

$$C\exp(1/(3\sqrt{\rho})) + \frac{2Cg^2(t)}{|g'(t)|}.$$

We recall from Lemma G4 that $C \to 1$ as $\rho \to \infty$, and that $C \leq 1$ in all cases. Also, $|g'(t)|/g(t) \sim \rho^{-3/8}$ as $\rho \to \infty$, so that

$$\frac{2Cg^2(t)}{|g'(t)|} \sim 2g(t)\rho^{3/8}$$

$$= 2(2\pi\rho)^{-1/2}\rho^{3/8}(1+\rho^{-6/8})^{(\rho-1)/2}$$

$$\times \exp(-\rho^{5/8}\arctan(\rho^{-3/8}))$$

$$\leq \rho^{-1/8}\exp(\rho^{2/8}/2)\exp(-\rho^{5/8}(\rho^{-3/8}-\rho^{-9/8}/3))$$

$$\leq \rho^{-1/8}\exp(-\rho^{2/8}/2+\rho^{-4/8}/3)$$

$$\to 0$$

as $\rho \to \infty$.

**Proof of Lemma E1** The first inequality is immediate from the definition of $\psi$. The second part follows by a combination of two bounds. Clearly,

$$\psi'(t) = -\frac{1}{2t^2}+\frac{2}{t^3}-\frac{1}{t^2(e^t-1)}-\frac{e^t}{t(e^t-1)^2}.$$

We have already seen that $\psi'(t) < 0$ for all $t > 0$. Using $e^t - 1 \geq t$, we note that for all $t > 0$,

$$\psi' \geq -\frac{1}{2t^2}+\frac{2}{t^3}-\frac{1}{t^3}-\frac{1}{t^2}-\frac{1}{t^3} = -\frac{3}{2t^2}.$$

Note that for $t \geq 2$, $\psi'(t) \geq -3/8$. So assume $t < 2$. Recall that in the notation of Lemma G3,

$$\psi'(t) =$$

$$\frac{S(2t)(32/30-8t/30)-S(t)(2/30+t^2/16)-1-t/12}{2((e^t-1)/t)^2}$$

$$\geq \frac{(1+t/3)(32/30-8t/30)-(1+t/6+t^2/21)(2/30+t^2/16)-1-t/12}{2((e^t-1)/t)^2}$$
$$= \frac{-t/180-4t^2/45-t^2/16-t^3/96-(t^2/21)(2/30+t^2/16)}{2((e^t-1)/t)^2}$$
$$\geq \frac{-4/5}{2((e^t-1)/t)^2}$$
$$\geq -2/5.$$

**Proof of Theorem 1** We need only show that $p_l+p_m+p_r$ is uniformly bounded for $\rho \geq 1$ and $\lambda \geq 0$. We begin with $p_m$. Clearly,

$$\lambda_m = -\frac{\lambda}{\rho(1+\lambda^2)} \leq 0.$$

Also,

$$g(\rho\lambda) = \frac{C}{\sqrt{2\pi\rho}\sqrt{1+\lambda^2}}.$$

Note that $g(t_m)\delta = DC/\sqrt{2\pi}$. Also, $\delta|\lambda_m| \leq D/\sqrt{\rho} \leq D$. Thus, using $|\exp(-u)-\exp(u)| \leq |u|(\exp(-u)+\exp(u))$,

$$\begin{aligned} p_m &= \frac{g(t_m)|\exp(\delta\lambda_m)-\exp(-\delta\lambda_m)|}{|\lambda_m|} \\ &\leq g(t_m)\delta(\exp(\delta\lambda_m)+\exp(-\delta\lambda_m)) \\ &\leq DC((2\pi)^{-1/2}(\exp(D)+\exp(-D)). \end{aligned}$$

The bound remains formally valid if $\lambda = 0$. In all cases, one should try to make $D$ as small as possible. This will have to be offset against a requirement that $D$ be made large to make $p_l+p_r$ small.

Next, we consider $p_r$. The following inequality will be needed twice:

$$\arctan(\lambda+\delta/\rho) - \arctan\lambda = \int_\lambda^{\lambda+\delta/\rho} \frac{1}{1+x^2}\,dx \geq \frac{\delta/\rho}{1+(\lambda+\delta/\rho)^2}.$$

A bound for $\lambda_r$ is obtained as follows:

$$\lambda_r = \arctan\lambda - \arctan(\lambda+\delta/\rho) - \frac{\lambda_\rho+\delta}{\rho^2+(\lambda_\rho+\delta)^2} \leq -\frac{\delta/\rho}{1+(\lambda+\delta/\rho)^2}.$$

Furthermore,

$$\begin{aligned} g(t_r) &= \frac{C}{\sqrt{2\pi\rho(1+(\lambda+\delta/\rho)^2)}}\left(\frac{1+(\lambda+\delta/\rho)^2}{1+\lambda^2}\right)^{\rho/2} \\ &\quad \times \exp((\lambda\rho+\delta)(\arctan\lambda-\arctan(\lambda+\delta/\rho))) \\ &\leq \frac{C}{\sqrt{2\pi\rho(1+(\lambda+\delta\rho)^2)}}\exp\left(\frac{\lambda\delta}{1+\lambda^2}+\frac{\delta^2}{2\rho(1+\lambda^2)} - \frac{(\lambda\rho+\delta)\delta/\rho}{1+(\lambda+\delta/\rho)^2}\right). \end{aligned}$$

Combining all this yields the following estimate:

$$\begin{aligned} p_r &= \frac{g(t_r)\exp(\delta\lambda_r)}{|\lambda_r|} \\ &\leq \frac{C(1+(\lambda+\delta/\rho)^2)}{(\delta/\rho)\sqrt{2\pi\rho(1+(\lambda+\delta/\rho)^2)}}\exp\left(\frac{\lambda\delta}{1+\lambda^2}+\frac{\delta^2}{2\rho(1+\lambda^2)}\right) \\ &\quad \times \exp\left(-\frac{(\lambda\rho+\delta)\delta/\rho}{1+(\lambda+\delta/\rho)^2}\right)\exp\left(-\frac{\delta^2/\rho}{1+(\lambda+\delta/\rho)^2}\right) \\ &\leq \frac{C\sqrt{1+(\lambda+\delta/\rho)^2}}{(D\sqrt{1+\lambda^2})\sqrt{2\pi}}\exp\left(\frac{\lambda\delta}{1+\lambda^2}+\frac{D^2}{2} - \frac{\lambda\delta+2D^2(1+\lambda^2)}{1+(\lambda+\delta/\rho)^2}\right) \\ &\leq \frac{C\sqrt{1+2\lambda^2+2\delta^2/\rho^2}}{(D\sqrt{1+\lambda^2})\sqrt{2\pi}}\exp\left(\frac{D^2}{2}+\frac{\lambda\delta D^2/\rho-2D^2}{1+(\lambda+\delta/\rho)^2}\right) \\ &\leq \frac{C\sqrt{2+2D^2}}{D\sqrt{2\pi}}\exp\left(\frac{D^2}{2}\,\frac{\lambda^2+\delta^2/\rho^2+4\lambda\delta/\rho-3}{1+(\lambda+\delta/\rho)^2}\right) \\ &\leq \frac{C\sqrt{1+1/D^2}}{\sqrt{\pi}}. \end{aligned}$$

We finally turn to $p_l$, which is by far the most difficult case to handle. We first turn to $\lambda_l$. We have

$$\lambda_l = \arctan\lambda - \arctan(\lambda-\delta/\rho) - \frac{(\lambda-\delta/\rho)(1/\rho)}{1+(\lambda-\delta/\rho)^2} \stackrel{\text{def}}{=} \text{I}+\text{II}.$$

Using the inequality $1+u \leq \exp(u)$ and

$$\frac{\delta^2}{2\rho(1+\lambda^2)} \leq \frac{D^2}{2},$$

we obtain

$$g(t_l) = \frac{C}{\sqrt{2\pi\rho(1+(\lambda-\delta/\rho)^2)}}\left(\frac{1+(\lambda-\delta/\rho)^2}{1+\lambda^2}\right)^{\rho/2}$$
$$\times \exp((\lambda\rho-\delta)(\arctan\lambda - \arctan(\lambda-\delta/\rho)))$$
$$\leq \frac{C}{\sqrt{2\pi\rho(1+(\lambda-\delta/\rho)^2)}}\exp\left(-\frac{\lambda\delta}{1+\lambda^2}+\frac{D^2}{2}\right)$$
$$\times \exp((\lambda\rho-\delta)(\arctan\lambda - \arctan(\lambda-\delta/\rho))).$$

We distinguish between several cases.

*Case 1*: $\lambda - \delta/\rho \leq 0$. We note that $\mathrm{II} \geq 0$, while

$$\mathrm{I} \geq \frac{\pi}{4}\max\left(\min(1,\lambda), \min\left(1, \frac{\delta}{\rho}-\lambda\right)\right) \geq \frac{\pi}{4}\min\left(1,\frac{\delta}{2\rho}\right).$$

This follows from the fact that arctan is concave on $(0,\infty)$, so that $\arctan(u) \geq \pi u/4$ for $0 \leq u \leq 1$ and $\arctan(u) \geq \pi/4$ when $u \geq 1$. Clearly, $\lambda_l \geq 0$, as was required. Also, if we define

$$\mathrm{III} = -\delta\lambda_l - \frac{\lambda\delta}{1+\lambda^2}+\frac{D^2}{2}$$
$$+(\lambda\rho-\delta)(\arctan\lambda - \arctan(\lambda-\delta/\rho)),$$

then

$$p_l = \frac{g(t_l)\exp(-\delta\lambda_l)}{\lambda_l}$$
$$\leq \frac{C}{\lambda_l\sqrt{2\pi\rho(1+(\lambda-\delta/\rho)^2)}}\exp(\mathrm{III})$$
$$\leq \frac{4C\exp(D^2/2)}{\pi\min(1,\delta/(2\rho))\sqrt{2\pi\rho}}$$
$$\leq \max\left(\frac{4C\exp(D^2/2)}{\pi\sqrt{2\pi}}, \frac{8C\exp(D^2/2)}{\pi D\sqrt{2\pi(1+\lambda^2)}}\right)$$
$$\leq \max(1, 2/D)\frac{4C\exp(D^2/2)}{\pi\sqrt{2\pi}}.$$

This is a gross overestimate, but it will do for now.

*Case 2*: $\lambda - \delta/\rho \geq 0$. Define

$$\mathrm{IV} = \frac{C}{\lambda_l\sqrt{2\pi\rho(1+(\lambda-\delta/\rho)^2)}}$$

and

$$\mathrm{V} = -\delta\lambda_l - \frac{\lambda\delta}{1+\lambda^2}+\frac{D^2}{2}$$
$$+(\lambda\rho-\delta)(\arctan\lambda - \arctan(\lambda-\delta/\rho)).$$

We have

$$p_l = \frac{g(t_l)\exp(-\delta\lambda_l)}{\lambda_l} \leq \mathrm{IV}\exp(\mathrm{V}).$$

Also,

$$\mathrm{V} = -\delta\lambda_l - \frac{\lambda\delta}{1+\lambda^2}+\frac{D^2}{2}+(\lambda\rho-\delta)\left(\lambda_l + \frac{(\lambda-\delta/\rho)(1/\rho)}{1+(\lambda-\delta/\rho)^2}\right)$$
$$\leq (\lambda\rho - 2\delta)\lambda_l - \frac{\lambda\delta}{1+\lambda^2}+\frac{D^2}{2}+\frac{(\lambda-\delta/\rho)^2}{1+(\lambda-\delta/\rho)^2}$$
$$\leq (\lambda\rho - 2\delta)\lambda_l - \frac{\lambda\delta}{1+\lambda^2}+\frac{D^2}{2}+1.$$

When $\lambda\rho - 2\delta \leq 0$, and $\lambda_l \geq 0$, the exponent in the upper bound is not larger than $D^2/2+1$. Otherwise, we note that

$$\lambda_l \leq \int_{\lambda-\delta/\rho}^{\lambda} x^{-2}dx = \frac{\delta/\rho}{\lambda(\lambda-\delta/\rho)},$$

so that

$$\mathrm{V} \leq \frac{(\lambda-2\delta/\rho)\delta}{\lambda(\lambda-\delta/\rho)} - \frac{\lambda\delta}{1+\lambda^2}+\frac{D^2}{2}+1$$
$$\leq \frac{(\lambda-\delta/\rho)\delta}{\lambda^2} - \frac{\lambda\delta}{1+\lambda^2}+\frac{D^2}{2}+1$$
$$\leq \frac{(\delta-\delta^2)\lambda}{1+\lambda^2}+\frac{D^2}{2}+1$$
$$\leq \frac{D^2}{2}+1.$$

We conclude that in all cases,

$$p_l \leq \frac{C\exp(1+D^2/2)}{\lambda_l\sqrt{2\pi\rho(1+(\lambda-\delta/\rho)^2)}}.$$

Thus, we need only bound the denominator uniformly from below away from zero. This is done by considering three subcases.

*Subcase A*: $0 < \lambda - \delta/\rho \leq 1, \lambda \leq 2$.

$$\lambda_l \geq \int_{\lambda-\delta/\rho}^{\lambda}\frac{1}{1+x^2}dx - \frac{\lambda-\delta/\rho}{\rho} \geq \left(\frac{1}{\rho}\right)\left(\frac{\delta}{5}-1\right) \geq \frac{\delta}{30\rho}$$

if $\delta \geq 6$. A sufficient condition for this is that $D \geq 6/\sqrt{2\pi}$. Thus,

$$\lambda_l\sqrt{2\pi\rho(1+(\lambda-\delta/\rho)^2)} \geq \frac{\delta\sqrt{2\pi}}{30\sqrt{\rho}} \geq \frac{D\sqrt{2\pi}}{30}.$$

*Subcase B*: $0 < \lambda - d/\rho \leq 1, \lambda > 2$.

$$\lambda_l \geq \arctan(2) - \arctan(1) - \frac{\lambda-\delta/\rho}{\rho} \geq 0.3217\ldots$$
$$-\frac{1}{\rho} \geq 0.3217\ldots - \frac{4}{5D^2} \geq 0.2$$

if $D > \sqrt{8/3}$. Here we used the fact that $r \leq (5/4)D^2$. To see this, note that

$$r < \delta/\lambda = D\sqrt{\rho}\sqrt{1+\lambda^2}/\lambda$$

and thus

$$\sqrt{\rho} \leq D\sqrt{1+1/\lambda^2} \leq D\sqrt{5/4}.$$

Therefore,

$$\lambda_l\sqrt{2\pi\rho(1+(\lambda-\delta/\rho)^2)} \geq 0.2\sqrt{2\pi}.$$

*Subcase C*: $\lambda - \delta/\rho \geq 1$. By the convexity of $1/(1+x^2)$ for $x > 1$, we see that

$$\lambda_l \geq \frac{\delta/(2\rho)}{1+\lambda^2} + \frac{\delta/(2\rho)}{1+(\lambda-\delta/\rho)^2} - \frac{(1/\rho)(\lambda-\delta/\rho)}{1+(\lambda-\delta/\rho)^2}$$
$$\geq \frac{\delta/(2\rho) - \lambda/\rho}{1+(\lambda-\delta/\rho)^2} \geq \frac{\delta/(4\rho)}{1+(\lambda-\delta/\rho)^2}$$

provided that $\delta \geq 4\lambda$. A sufficient condition for this is that $D \geq 4$. Resubstitution shows that

$$\lambda_l\sqrt{2\pi\rho(1+(\lambda-\delta/\rho)^2)} \geq \frac{D\sqrt{2\pi}\sqrt{1+\lambda^2}}{4\sqrt{1+(\lambda-\delta/\rho)^2}} \geq \frac{D\sqrt{2\pi}}{4}.$$

This concludes the proof of Theorem 1.

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