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Generating sums in constant average time

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ABSTRACT.

We derive an algorithm that requires uniformly bounded time to generate the sum of n iid uniform [0,1] random variables. The expected time spent on the computation of the density of the sum per generated random variate tends to zero as $n \to \infty$.

INTRODUCTION.

Assume that we wish to generate many independent copies of the random variable $S_n = \sum_{i=1}^{n} X_i$, where the X_i's are iid random variables having a given density f. Obviously, we could always do this in time $O(n)$ by generating and summing the X_i's. For very large n, this is rather inefficient. At the same time, it is unacceptable to generate a random variate with a properly normalized limit law for S_n since approximations are not allowed. However, local central limit theorems can be helpful in the design of an exact generator. Section 14.4 of Devroye (1986) deals with precisely this problem. Basically, one could in general develop generators for distributions with known characteristic function (see also Devroye, 1986), and apply these to the situation at hand. This is promising if we know the characteristic function ϕ of f, since S_n must then have characteristic function ϕ_n. It is immediately apparent that the resulting algorithms are rather cumbersome.

The strategy we will explore in this note rests on the simple principle of solving a complex problem by solving many easier problems, i.e. on the principle of basic building blocks. In particular, we will develop and analyze a generator for the sum S_n of n iid uniform [−1,1] random variables. This generator takes $O(1)$ expected time per random variate. From this, the user can build at will. Indeed, many densities can be written as mixtures

$$f(x) = \sum_{i=1}^{\infty} p_i f_i(x) ,$$

where the f_i's are uniform densities on intervals $[a_i, b_i]$. The sum S_n of n iid random variables with density f can be generated out as follows.

The mixture method for simulating sums

Generate a multinomial $(n, p_1, p_2, \ldots)$ random sequence $N_1, N_2, \ldots$ (note that the N_i's sum to n). Let K be the index of the largest nonzero N_i.

$X \leftarrow 0$

FOR i:=1 TO K DO

Generate S, the sum of N_i iid random variables with common density f_i.

$X \leftarrow X + S$

RETURN X

The validity of the algorithm is obvious. The algorithm is put in its most general form, allowing for infinite mixtures. A multinomial random sequence is of course defined in the standard way: imagine that we have an infinite number of urns, and that n balls are independently thrown in the urns. Each ball lands with probability p_i in the i-th urn. The sequence of cardinalities of the urns is a multinomial $(n, p_1, p_2, \ldots)$ random sequence. To simulate such a sequence, note that N_1 is binomial (n, p_1), and that given N_1, N_2 is binomial $(n-N_1, p_2/(1-p_1))$, etcetera. If K is the index of the last occupied urn, then it is easy to see that the multinomial sequence can be generated in expected time $O(E(K))$.

Note that there is no special reason why the f_i's have to be uniform densities. However, uniform densities are convenient in many cases, especially when f is unimodal.

In section 14.4.6 of Devroye (1986), we pinpoint the difficulties in developing a uniformly fast generator for the sum of uniform [−1,1] random variables. These are related to the fact that the density f_n of S_n can only be computed at $\Omega(n)$ time cost. This is best seen by noting first that S_n has characteristic function $\left(\frac{\sin(t)}{t}\right)^n$. For all $n \geq 2$, the density f_n can be obtained by the inversion formula

$$f_n(x) = \frac{1}{2\pi} \int \left(\frac{\sin(t)}{t}\right)^n \cos(tx)\, dt .$$

This yields

$$f_n(x) = \frac{n}{2^n} \sum_{k=0}^{\lfloor (n-x)/2 \rfloor} \frac{(-1)^k}{k!\,(n-k)!} (n-2k-x)^{n-1} \quad (0 \leq x < n) .$$

We know that a uniform [0,1] random variable U has a binary expansion whose bits are iid Bernoulli random variables with parameter 1/2. Thus, the sum S_n of n iid uniform [0,1] random variables can be written as

$$S_n = \sum_{j=1}^{\infty} \frac{Y_j}{2^j} ,$$

where the Y_j's are iid binomial $(n, \frac{1}{2})$ random variables. There are several algorithms now available for generating such random variates in expected time uniformly bounded over n (see Devroye (1986), Fishman (1979), Ahrens and Dieter (1980), and Kachitvichyanukul (1982)). Thus, if one desires S_n with a fixed number (d) of accurate bits, it suffices basically to consider a sum truncated to its first $d + \log_2 n$ terms. This time grows with n, and the resulting S_n is only an approximation!

THE GENERAL STRATEGY.

We have seen above that the evaluation of the density f_n of S_n takes time proportional to n. Let us formalize this, and assume that the algorithm has a random integer cost associated with it, consisting of the number of uniform random variates needed before the algorithm halts, and of n times the number of evaluations of f_n (thus, reflecting the fact that each evaluation takes time proportional to n). These two components will be called R and N respectively. The algorithm we are after has the following desirable properties:

A. Uniformly over all n, $E(R) \le c < \infty$ for some constant c.

B. As $n \to \infty$, $E(N) \to 0$.

This means that the contribution from the evaluation of f_n is asymptotically negligible. In other words, one could be rather sloppy in the implementation of these evaluations, and barely notice any impact on the expected time per random variate. Furthermore, the overall expected time is uniformly bounded over n. To be able to avoid the evaluations of f_n nearly all the time means that we must in fact derive a relatively accurate expression for the actual density of S_n. The problem we are trying to solve can be tackled as suggested in exercise 14.4.6, based upon the Gram-Charlier series (see e.g. Ord, 1972, p. 26). The truncated Gram-Charlier series leads to the function

$$g_n(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \left(1 + \frac{6x^2 - 3 - x^4}{20\, n}\right) ,$$

which approximates the density $f_n(x)$ of the normalized sum $\sqrt{3/n}\, S_n$. This normalization will be assumed throughout the remainder of this note. We need to know how good the approximation is. Thus, we need something like the following Lemma:

Lemma 1.

$$\sup_x |f_n(x) - g_n(x)| \le \frac{A}{n^2} ,$$

where $A = 3.9608280445...$ is equal to

$$\frac{27\sqrt{3}}{4\pi\, e^{3/2}} + \frac{96}{5\pi \sqrt{2}\, e^{5/2}} + \frac{2^{7/2}}{\sqrt{3}\, \pi\, e^2 \log^2 2} + \frac{263503}{48000\sqrt{2\pi}} .$$

This bound leads to a uniform local limit theorem (see Petrov, 1975). For nonuniform bounds, see e.g. Maejima (1980). Our bound serves two purposes. First, we could try to use it to derive a dominating curve for use in the rejection method. Indeed, we know that $\sqrt{3/n}\, S_n$ has support on $[-\sqrt{3n}, \sqrt{3n}]$. Thus, we have on this support set, maximizing the quartic polynomial in the definition of g_n with respect to x,

$$f_n(x) \le \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \left(1 + \frac{6}{20\, n}\right) + \frac{A}{n^2} .$$

The dominating curve has integral not exceeding

$$1 + \frac{6}{20\, n} + \frac{2A\sqrt{3}}{n^{3/2}} = 1 + O\left(\frac{1}{n}\right) .$$

This is good enough for us. Next, we need to look at how Lemma 1 can be used. This is done through a squeeze step, both for rejection and acceptance. An evaluation of f_n is only necessary when both squeeze tests fail. It is known that the expected number of evaluations of f_n is the the area between the squeeze curves (Devroye, 1986, p.54). In our case, this yields a value not exceeding

$$\frac{4A\sqrt{3}}{n^{3/2}} .$$

Hence, $E(N)$, which equals n times this value, is $O(1/\sqrt{n})$. We have thus achieved our goals A and B stated at the outset of this section.

In an additional section, we will improve the latter result by an additional tail bound, to obtain $E(N) = O(\sqrt{\log n}\, / n)$.

THE ALGORITHM.

The inequalities of the previous section allow us to use the rejection method in a straightforward manner:

Rejection method.

[SET-UP]

$A \leftarrow \frac{27\sqrt{3}}{4\pi e^{3/2}} + \frac{96}{5\pi\sqrt{2}\, e^{5/2}} + \frac{2^{7/2}}{\sqrt{3}\,\pi\, e^2 \log^2} + \frac{263503}{48000\sqrt{2\pi}}$. Note that $A = 3.9608280445\ldots$. Compute $p \leftarrow 1 + \frac{6}{20\,n}$ and $q \leftarrow 2A\sqrt{3}/n^{3/2}$.

[GENERATOR.]

REPEAT

Generate a uniform [0,1] random variate U.

IF $U \le \frac{q}{p+q}$

THEN Generate a uniform $[-n,n]$ random variate X (or set $X \leftarrow -n + 2n(p+q)U/q$).

ELSE Generate a normal random variate X.

Generate a uniform [0,1] random variate V.

$T \leftarrow V\left[(1+\frac{6}{20\,n})\frac{e^{-X^2/2}}{\sqrt{2\pi}} + An^{-2}\right]$

IF $|X| > \sqrt{3n}$ THEN Accept $\leftarrow$ False

ELSE IF $T \le g_n(X) - An^{-2}$ THEN Accept $\leftarrow$ True

ELSE IF $T \ge g_n(X) + An^{-2}$ THEN Accept $\leftarrow$ False

ELSE Accept $\leftarrow [T < f_n(X)]$

UNTIL Accept.

RETURN X. Note: X is the normalized sum, $S_n\sqrt{3/n}$.

The properties of the algorithm are well understood: the expected number of iterations is $1+\frac{6}{20\,n}+\frac{2A\sqrt{3}}{n^{3/2}}$, the expected number of evaluations of f_n is not greater than $4A\sqrt{3}/n^{3/2}$, and thus $E(N) \le 4A\sqrt{3/n}$.

PROOF OF LEMMA 1.

Lemma 2.

We have the following simple identities:

$$\frac{x^2 e^{-x^2/2}}{\sqrt{2\pi}} = \frac{1}{2\pi}\int (1-t^2)e^{-t^2/2}\cos tx\; dt\ ,$$

$$\frac{x^4 e^{-x^2/2}}{\sqrt{2\pi}} = \frac{1}{2\pi}\int (3-6t^2+t^4)e^{-t^2/2}\cos tx\; dt\ .$$

Proof of Lemma 2.

If $\phi(t) = e^{-t^2/2}$ is the normal characteristic function, then it is easy to check that $\phi^{(2)} = (t^2-1)\phi$ and that $\phi^{(4)} = (3-6t^2+t^4)\phi$. Let f be the normal density. Lemma 2 now follows from the facts that

$$x^2 f(x) = -\frac{1}{2\pi}\int \phi^{(2)}(t)\cos tx\; dt$$

and

$$x^4 f(x) = \frac{1}{2\pi}\int \phi^{(4)}(t)\cos tx\; dt\ . \blacksquare$$

As mentioned above, f_n is the density of the normalized sum $\sqrt{3/n}\,S_n$, and $g_n(x)$ is the approximation of $f_n(x)$ given by $\gamma + \alpha x^2 + \beta x^4$ where $\gamma = 1-\frac{3}{20n}$, $\alpha = \frac{6}{20n}$ and $\beta = -\frac{1}{20n}$. If ϕ_n and ψ_n are the respective characteristic functions (or rather, Fourier transforms), then we have

$$\sup_x |f_n(x) - g_n(x)| \le \frac{1}{2\pi}\int |\phi_n(t) - \psi_n(t)|\; dt\ .$$

To compute this bound, we split the integral over two sets, $D = [-\sqrt{2n/3}, \sqrt{2n/3}]$, and its complement, D^c. The bound now reduces to some simple but tedious computations. We start with the easiest ones.

Lemma 3.

Assume that $n \ge 2$.

$$\frac{1}{2\pi}\int_{D^c} |\phi_n(t)|\; dt \le \frac{2^{-n/2}\sqrt{2n/3}}{\pi(n-1)}$$

and

$$\frac{1}{2\pi}\int_{D^c} |\psi_n(t)|\; dt \le \frac{1}{2\pi}\sqrt{\frac{3}{2}}\left[\frac{2\,e^{-n/3}}{\sqrt{n}} + \frac{64\,e^{-n/6}}{5e^2\,n^{3/2}}\right].$$

Proof of Lemma 3.

We recall that the characteristic function of the normalized sum is $\left[\frac{\sin(t\sqrt{3/n})}{t\sqrt{3/n}}\right]^n$. Thus,

$$\int_{D^c} |\phi_n(t)|\; dt \le 2\int_{\sqrt{2n/3}}^{\infty}\left[t\sqrt{\frac{3}{n}}\right]^{-n} dt$$

$$= 2\left[\frac{n}{3}\right]^{n/2}\left[\frac{3}{2}\right]^{(n-1)/2}\frac{n^{-(n-1)/2}}{n-1} = \frac{\sqrt{8/3}\;2^{-n/2}\sqrt{n}}{n-1}\ .$$

Also, $\psi_n(t) = e^{-t^2/2}(\gamma + \alpha(1-t^2) + \beta(3-6t^2+t^4))$, which can be simplified to $e^{-t^2/2}(1 - \frac{t^4}{20\,n})$. Thus,

$$\int_{D^c} |\psi_n(t)|\; dt \le \int_{D^c} e^{-t^2/2}\, dt + \int_{D^c} e^{-t^2/2}\frac{t^4}{20\,n}\, dt$$

$$\le 2\int_{\sqrt{2n/3}}^{\infty} e^{-t^2/2}\, dt + 2\int_{\sqrt{2n/3}}^{\infty}\frac{16\,e^{-t^2/4}}{5e^2\,n}\, dt$$

where we used the simple inequality $t^4 e^{-t^2/4} \le 64\,e^{-2}$. If g is a nonnegative function with nonnegative nondecreasing first derivative, then $\int_u^{\infty} e^{-g(t)}\, dt \le e^{-g(u)}/g'(u)$. This can be used on both integrals in the last sum. We obtain the further upper bound

$$\sqrt{\frac{3}{2}}\left[\frac{2\,e^{-n/3}}{\sqrt{n}} + \frac{64\,e^{-n/6}}{5e^2\,n^{3/2}}\right]. \blacksquare$$

The main part is treated in Lemma 4.

Lemma 4.

$$\frac{1}{2\pi}\int_D |\phi_n(t)-\psi_n(t)|\,dt \le \frac{263503}{48000\sqrt{2\pi}n^2}\,.$$

Proof of Lemma 4.

In this proof, the letters $\theta,\eta,\zeta,\rho,\chi,\lambda,\sigma$ and ξ are used to denote numbers in [0,1] which may depend upon t and/or n. They all originate from the remainder term in Taylor's series expansion. We begin by noting that, by Taylor's series expansion of $\sin u$,

$$\phi_n^{1/n}(t) = 1-\frac{t^2}{2n}+\frac{3t^4}{40\,n^2}-\frac{3\theta t^6}{560n^3}$$

and thus that

$$\phi_n^{1/n}(t)$$

$$=\exp\left[-\frac{t^2}{2n}-\frac{1}{2}\left[\frac{t^2}{2n}\right]^2-\frac{\xi}{3}\left[\frac{t^2}{2n}\right]^3\right]\left[1+\frac{\frac{3t^4}{40\,n^2}-\frac{3\theta t^6}{560n^3}}{1-\frac{t^2}{2n}}\right]$$

$$=\exp\left[-\frac{t^2}{2n}-\frac{1}{2}\left[\frac{t^2}{2n}\right]^2-\frac{\xi}{3}\left[\frac{t^2}{2n}\right]^3+\frac{\frac{3t^4}{40\,n^2}-\frac{3\theta t^6}{560n^3}}{1-\frac{t^2}{2n}}-\frac{\eta}{2}\left[\frac{\frac{3t^4}{40\,n^2}-\frac{3\theta t^6}{560n^3}}{1-\frac{t^2}{2n}}\right]^2\right]$$

where we used the fact that $t^4/40n^2 \ge t^6/560n^3$. Using the fact that $(1-t^2/2n)^{-1} = 1+2\chi t^2/2n$ when $t^2\le n$, we have

$$\log(\phi_n(t))$$

$$=-\frac{t^2}{2}-\frac{t^4}{8n}-\frac{\xi t^6}{24n^2}+\left(\frac{3t^4}{40\,n}-\frac{3\theta t^6}{560n^2}\right)\left(1+\frac{2\chi t^2}{2n}\right)-\frac{9\,\eta\,t^8\,(1+\frac{2\chi t^2}{2n})^2(1-\frac{\theta t^2}{14n})^2}{3200\,n^3}$$

$$=-\frac{t^2}{2}-\frac{4t^4}{80n}+\frac{(\frac{6\chi}{80}-\frac{3\theta}{560}-\frac{\xi}{24})t^6}{n^2}-\frac{3\chi\theta t^8}{560n^3}-\frac{9\,\eta\,t^8\,(1+\frac{\chi t^2}{n})^2(1-\frac{\theta t^2}{14n})^2}{3200\,n^3}$$

$$=-\frac{t^2}{2}-\frac{4t^4}{80n}+\frac{(\frac{6\chi}{80}-\frac{79\rho}{1680})t^6}{n^2}-\frac{3\chi\theta t^8}{560n^3}-\frac{9\,\zeta\,t^8}{800\,n^3}$$

$$=-\frac{t^2}{2}-\frac{t^4}{20n}+\frac{(\frac{6\chi}{80}-\frac{79\rho}{1680})t^6}{n^2}-\frac{93\sigma t^8}{5600n^3}$$

$$\stackrel{\Delta}{=}-\frac{t^2}{2}-u_n(t)\,.$$

Since $u_n(t)\ge 0$ on D (which needs the fact that $t^2\le 2n/3$), we see that

$$\phi_n(t)=e^{-t^2/2}\,(1-u_n(t)+\frac{\lambda}{2}u_n^2(t))$$

$$=\psi_n(t)-e^{-t^2/2}\left[\frac{(\frac{6\chi}{80}-\frac{79\rho}{1680})t^6}{n^2}-\frac{93\sigma t^8}{5600n^3}\right]+\frac{1}{2}\,\lambda\,e^{-t^2/2}\,u_n^2(t)\,.$$

We conclude from this that

$$\frac{1}{2\pi}\int_D|\phi_n(t)-\psi_n(t)|\,dt$$

$$\le\frac{1}{4\pi}\int_D e^{-t^2/2}\,u_n^2(t)\,dt+\frac{1}{2\pi}\int_D e^{-t^2/2}\,\frac{6t^6}{80n^2}\,dt$$

$$\le\frac{1}{4\pi}\int_D e^{-t^2/2}\left[\frac{t^4}{20n}+\frac{79t^6}{1680n^2}+\frac{93t^8}{5600n^3}\right]^2dt+\frac{9}{8\sqrt{2\pi}n^2}$$

$$\le\frac{\left[\frac{1}{20}+\frac{2}{3}\frac{79}{1680}+\frac{4}{9}\frac{93}{5600}\right]^2}{4\pi\,n^2}\int_D t^8\,e^{-t^2/2}\,dt+\frac{9}{8\sqrt{2\pi}n^2}$$

$$\le\frac{(173/600)^2\,105\,\sqrt{2\pi}}{4\pi n^2}+\frac{9}{8\sqrt{2\pi}n^2}$$

$$=\frac{209503}{48000\sqrt{2\pi}n^2}+\frac{9}{8\sqrt{2\pi}n^2}=\frac{263503}{48000\sqrt{2\pi}n^2}\,.\ \blacksquare$$

Finally, we are in a position to combine all the bounds into one, and prove Lemma 1.

Proof of Lemma 1.

We will repeatedly use the fact that $x^c e^{-x} \le (c/e)^c$ for all positive c and x. The bounds of Lemmas 3 and 4 can be added together to get a general bound, provided that we successfully bound the exponential terms. Indeed, the fact that $n\ge 3$ implies that the first bound of Lemma 3 can be estimated from above as follows:

$$\sqrt{\frac{2}{3}}\,\frac{\sqrt{n}\;2^{-n/2}}{\pi(n-1)}\le\sqrt{\frac{2}{3}}\,\frac{n^2e^{-n(\log 2)/2}}{\pi\,n^2}$$

$$\le\sqrt{\frac{2}{3}}\,(\frac{2}{e})^2\,(\frac{2}{\log 2})^2\,\frac{1}{\pi\,n^2}\,.$$

Next, the value of the second bound of Lemma 3 is

$$\frac{1}{2\pi}\sqrt{\frac{3}{2}}\left[\frac{2\,e^{-n/3}}{\sqrt{n}}+\frac{64\,e^{-n/6}}{5e^2\,n^{3/2}}\right]$$

$$=\frac{1}{2\pi}\sqrt{\frac{3}{2}}\left[\frac{2 3^{3/2}\,(n/3)^{3/2}\,e^{-n/3}}{n^2}+\frac{64\sqrt{6}\,\sqrt{n/6}\,e^{-n/6}}{5e^2\,n^2}\right]$$

$$\le\frac{1}{2\pi\,n^2}\sqrt{\frac{3}{2}}\left[2 3^{3/2}\,(3/2e)^{3/2}+\frac{64\sqrt{6}\,\sqrt{1/2e}}{5e^2}\right]$$

$$= \frac{1}{2\pi\, n^2} \sqrt{\frac{3}{2}} \left[\frac{27}{\sqrt{2}\, e^{3/2}} + \frac{64\sqrt{3}}{5\, e^{5/2}} \right] .$$

All these bounds, together with Lemma 4, prove Lemma 1. ■

We can obtain a messier but better inequality if the estimates from Lemmas 3 and 4 are directly used in Lemma 1.

However, doing so would make the presentation too heavy, and in any case, not much is lost in the simplified version.

AN IMPROVEMENT VIA TAIL BOUNDS.

Improvements in $E(N)$ can be obtained in several manners, such as (i) approximations based upon Gram-Charlier or Edgeworth series with extra terms added in; (ii) bounds in the local central limit theorem that are a function of x and n (see Devroye (1986, pp. 720-731) for a worked out example); (iii) additional quick rejection steps that are effective in the tails of the distribution. The first two approaches are straightforward but very tedious and space-consuming. Interestingly, the third approach is both simple and effective. Note however that by introducing an extra squeeze step, we don't change the expected number of iterations in the rejection algorithm. The only quantity that is affected is $E(N)$. Suppose for example that we can show that

$$f_n(x) \le h_n(x)\ ,\ |x| \ge a_n\ ,$$

for some symmetric bounding function h_n and some sequence of constants a_n. Then, obviously,

$$E(N) \le n \left[\frac{4A\sqrt{3}\, a_n}{n^2} + 2 \int_{a_n}^{\infty} h_n(x)\, dx \right] .$$

We will see that we can take a_n proportional to $\sqrt{\log n}$, and find a function h_n such that, in fact, $E(N) = O(\frac{\sqrt{\log n}}{n})$, a considerable improvement over the $O(1/\sqrt{n})$ rate obtained without the modification.

The modification in the algorithm.

After the quick rejection step, introduce another rejection step: "ELSE IF $|X| > a_n$ AND $T \ge h_n(X)$ THEN Accept ← False".

We will see that we can take $a_n = 1$ and $h_n(x)$ as defined in Lemma 7 below. The derivation of useful bounds rests upon the combination of two techniques, a monotonicity argument extending the methodology of chapter VII.3 of Devroye (1986), and Chernoff's exponential bounding technique for sums of independent random variables (see Chernoff (1952) or Petrov (1975)).

Lemma 5.

Let Y be a random variable with symmetric unimodal density f. Then, for all $t>0$,

$$f(y) \le \frac{t\, E(e^{tY})}{e^{t|y|} - 1} .$$

For $t \ge 1/|y|$, we have

$$f(y) \le \frac{e}{e-1}\, t\, E(e^{tY})\, e^{-t|y|} .$$

Proof of Lemma 5.

Observe that

$$\int e^{tz} f(z)\, dz \ge \int_0^y e^{tz} f(z)\, dz \ge f(y)\, \frac{e^{ty} - 1}{t}\ ,\ y > 0\ .\ ■$$

Lemma 6.

Let S_n be the sum of n iid uniform [−1,1] random variables. Then

$$P(S_n \ge x\sqrt{n}) \le e^{-tx\sqrt{n}}\, E(e^{tS_n})\ ,$$

where $t>0$ is arbitrary, and

$$E(e^{tS_n}) \le e^{\frac{1}{6} n t^2 e^{t^2/20}} .$$

Proof of Lemma 6.

The first inequality is the cornerstone of Chernoff's bounding method. The second inequality can be obtained as follows: we have

$$E(e^{tS_n}) = E^n(e^{tX_1}) = \left[\frac{1}{2} \int_{-1}^{1} e^{ty}\, dy \right]^n = \left[\frac{e^t - e^{-t}}{2t} \right]^n .$$

The Lemma follows from this and the inequality $1+u \le e^u$, provided that we can show that $(e^t - e^{-t})/(2t) \le 1 + t^2 e^{t^2/20}/6$. This is most easily achieved by expanding both sides into their Taylor series, and comparing all terms pairwise: the expansions are

$$\frac{e^t - e^{-t}}{2t} = \sum_{j=0}^{\infty} \frac{t^{2j}}{(2j+1)!}\ ,$$

and

$$1 + \frac{1}{6} t^2\, e^{t^2/20} = 1 + \sum_{j=0}^{\infty} \frac{t^{2j+2}}{6\ 20^j\ j!}\ .$$

The ratio of the first over the second coefficient of t^{2j} $(j \ge 1)$ is

$$\frac{6\ 20^{j-1}\ (j-1)!}{(2j+1)!}\ .$$

For $j=1$, we have equality. Increasing j by one makes the numerator jump by a factor of $20j$, while the denominator jumps by a factor of $(2j+3)(2j+2)$. The latter jump equals $6+10j+4j^2$, which is $\geq 20j$ for all integer j. This concludes the proof of Lemma 6. ■

Lemma 7.

The density f_n of $\sqrt{3/n}S_n$ satisfies the inequality

$$f_n(x) \leq h_n(x) \overset{\Delta}{=} \frac{|x|e}{e-1} \exp\left[-x^2 + \frac{x^2}{2} e^{3x^2/(20n)}\right] \ (\ |x| \geq 1\ ) .$$

Furthermore, defining $h_n(x)=0$ for $|x|>\sqrt{3n}$, we have for all $z \geq 1$,

$$\int_{|x|\geq z} h_n(x)\, dx \leq \frac{2e}{(e-1)(2-e^{9/20})} e^{-\frac{2-e^{9/20}}{2} z^2} .$$

Proof of Lemma 7.

From Lemmas 5 and 6, we see that the density of S_n, evaluated at y, does not exceed

$$\frac{e}{e-1} t\, e^{\frac{1}{6}nt^2\exp(t^2/20)} e^{-t|y|} ,\ t \geq 1/|y| .$$

In this bound we can choose t. The nearly optimal (but convenient) choice is $t=3|y|/n$, which forces us to require that $y^2 \geq n/3$. Resubstitution of this value of t, together with the transformation $x=y\sqrt{3/n}$ shows that the density f_n of $\sqrt{3/n}S_n$ can be bounded as follows:

$$f_n(x) \leq \frac{e|x|}{e-1} \exp(-x^2+\frac{x^2}{2}e^{3x^2/(20n)}) \overset{\Delta}{=} h_n(x)$$

$$\leq \frac{e|x|}{e-1} \exp(-\frac{x^2}{2}(2-e^{9/20})) ,\ |x| \geq 1 .$$

From this, we see that for $z \geq 1$,

$$\int_{|x|\geq z} h_n(x)\, dx \leq \frac{2e}{e-1} \int_z^\infty x \exp(-\frac{x^2}{2}(2-e^{9/20}))\, dx$$

$$= \frac{2e}{e-1}(2-e^{9/20})^{-1} \exp(-\frac{z^2}{2}(2-e^{9/20})) .\ ■$$

Assume that the additional quick rejection is inserted, i.e. $|X| \geq 1$ and $T \geq h_n(X)$ together imply rejection, then we note that for any $z \geq 1$,

$$E(N) \leq \frac{4A\sqrt{3}z}{n} + 2n\int_z^\infty h_n(x)\, dx$$

$$\leq \frac{4A\sqrt{3}z}{n} + \frac{2en}{e-1}(2-e^{9/20})^{-1} \exp(-\frac{z^2}{2}(2-e^{9/20})) .$$

This expression is of the form $Bz+Ce^{-Dz^2}$, and is approximately minimal when z is chosen equal to $\max(1, \sqrt{\frac{1}{D}\log(2CD/B)})$. With this choice, z grows as a constant times $\sqrt{\log n}$, and

$$E(N) \leq (1+o(1)) \frac{8A\sqrt{3}}{\sqrt{2-e^{9/20}}} \frac{\sqrt{\log n}}{n} .$$

This is what we had to show.

OTHER POSSIBLE METHODS.

The evaluation of f_n, however rare, may cause some numerical worries for large values of n. Here it would help if we could avoid evaluating f_n altogether, and replace the rejection algorithm by a series-type rejection method based upon a converging series of f_n. Such series can be obtained in many ways. Firstly, we could mimick the development found on pages 698-700 of Devroye (1986), which applies to all densities f with symmetric, absolutely integrable, nonnegative characteristic function ϕ (which is the case for S_{2n} for all n). Then $f_n(x)$ is sandwiched between consecutive partial sums in the series

$$f_n(0)-\frac{x^2}{2!}f_n'(0)+\frac{x^4}{4!}f_n''(0)- \cdots .$$

This can be seen as follows: since $\cos(tx)$ is sandwiched between consecutive partial sums in its Taylor series expansion, and since

$$f_n(x) = \frac{1}{2\pi}\int \phi_n(t)\cos(tx)\, dt ,$$

where ϕ_n is the characteristic function for f_n, we see that by our assumptions on ϕ_n, $f_n(x)$ is sandwiched between consecutive partial sums in

$$\nu_{0,n}-\frac{x^2}{2!}\nu_{2,n}+\frac{x^4}{4!}\nu_{4,n}- \cdots ,$$

where

$$\nu_{2j,n}=\frac{1}{2\pi}\int t^{2j}\phi_n(t)\, dt .$$

There exists a simple recursive formula for the matrix of coefficients $\nu_{2j,n}$, so that all the coefficients can be computed on-line provided that the column of coefficients $\nu_{0,n}$ is known.

In a second approach, we could follow section XIV.3 of Devroye (1986), in which better and better approximations for f_n are obtained by finer and finer numerical inversions of the characteristic function ϕ_n.

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