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HOW TO REDUCE THE AVERAGE COMPLEXITY OF CONVEX HULL FINDING ALGORITHMS

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Abstract—Let $X_1 \ldots, X_n$ be a sequence of independent R^d-valued random vectors with a common density f. The following class of convex hull finding algorithms is considered: find the extrema in a finite number of carefully chosen directions; eliminate the X_i's that belong to the interior of the polyhedron formed by these extrema; apply an $0(\Delta(n))$ worst-case complexity algorithm to find the convex hull of the remaining points.

We give weak sufficient conditions that imply that the overall average complexity is $0(\Delta(n))$. We also show that for the standard normal density, the average complexity is $0(n)$ whenever $\Delta(n) = n \log n$.

1. INTRODUCTION

In this paper we will prove some general theorems about the average complexity of convex hull finding algorithms that use the throw-away principle [1].

Let $\{x_1, \ldots, x_n\}$ be a collection of points from R^d, let S be the unit sphere of $R^d (S = \{x \| \|x\| = 1\})$, let $A \subseteq S$, and let $x'y$ denote the inner product of x and y, two points from R^d.

Definition

The *extremal polyhedron P* of $\{x_1, \ldots, x_n\}$ with respect to A is the polyhedron whose vertices v are the extremal points of $\{x_1, \ldots, x_n\}$ with respect to A. A point $v \in \{x_1, \ldots, x_n\}$ is an *extremal point* with respect to A if $v'y \geq x_i'y$ for all i and some $y \in A$. The *comvex hull* of $\{x_1, \ldots, x_n\}$ is the set of extremal points of $\{x_1, \ldots, x_n\}$ with respect to S.

We note here that the convex hull of an extremal polyhedron P is the set of vertices of P. Also, if card $(A) = k$, then P cannot have more than k vertices. Extremal polyhedra of $\{x_1, \ldots, x_n\}$ can be found in time $0(n)$ whenever A is a finite set. The members of A can be considered as "*directions*" in which extrema are found. Akl and Toussaint [1] and Toussaint *et al.* [2] have shown that extremal polyhedra are very useful in the development of fast convex hull finding algorithms. Consider for example the following class of algorithms:

ALGORITHM CH

(i) Find the extremal polyhedron P of $\{x_1, \ldots, x_n\}$ with respect to a finite $A \subseteq S$.

(ii) Eliminate from $\{x_1, \ldots, x_n\}$ all x_i's that belong to interior (P).

(iii) Find the convex hull of the remaining points. Use an algorithm of your choice.

Step (ii) will be called the *throw-away* step. If the points $\{x_1, \ldots, x_n\}$ all belong to S, then no points are eliminated in the throw-away step. However, (ii) becomes effective when the x_i's are sufficiently smoothly distributed. What we mean by "sufficiently smoothly distributed" will be clarified further on. From now on we will only consider random vectors $X_1, \ldots, X_n$ from R^d that are *independent* and have a common *density f*. Let N be the number of elements of the convex hull of $\{X_1, \ldots, X_n\}$. For particular choices of f, the properties of N as $n \to \infty$ are well-known (see Refs. [3–7]). Theorem 1, in contrast, is valid for *all* densities f. It shows that whenever the X_i's have a density f, then only an asymptotically negligible fraction of them can belong to the convex hull.

2. THE NUMBER OF POINTS ON THE CONVEX HULL

THEOREM 1

For *any* density on R^d, we have

$$E(N) = o(n) \tag{1}$$

and

$$\frac{N}{n} \to 0 \text{ a.s. as } n \to \infty. \tag{2}$$

Proof of Theorem 1. Let $N_{(k,l]}$ be the number of elements of the convex hull of $\{X_{k+1},\ldots,X_l\}$. Clearly, $0 \le N_{(k,s]} \le N_{(k,l]} + N_{(l,s]}$, all $1 \le k < l < s$. Thus, by the subadditive ergodic theorem ([8, 9]) there exists a constant $c \ge 0$ such that $N/n \to c$ a.s. as $n \to \infty$. Also, $\lim_{n\to\infty} E(N)/n$ exists and equals c. Thus, Theorem 1 follows if we can show that $E(N) = o(n)$. Since $E(N) = np_n$ where $p_n = P(X_1$ belongs to the convex hull), it is clear that we need only establish that $p_n \to 0$ as $n \to \infty$. Let p_{nx} be the probability that x belongs to the convex hull of $\{x, X_2,\ldots,X_n\}$. Then

$$p_n = \int p_{nx} f(x)\,\mathrm{d}x.$$

Thus, by the Lebesgue dominated convergence theorem, $p_n \to 0$ if $p_{nx} \to 0$ as $n \to \infty$ for almost all $x(f)$. This can be proven by using a special version of the Lebesgue density theorem. If $x = (x^1,\ldots,x^d) \in R^d$, then there are 2^d d-fold products of intervals of the form $(-\infty, x^i]$ or (x^i, ∞). Each of these sets of R^d will be called a *quadrant* at x, and will be denoted by Q_x. We let $S_{x,r}$ be the closed ball of R^d with center at x and radius $r > 0$. Then there exists a set B of R^d for which

(i) $f(x) > 0, \quad x \in B,$

(ii) $\lim_{r\downarrow 0} \sup_{Q_x} r^{-d} \int_{S_{x,r}\cap Q_x} |f(y) - f(x)|\,\mathrm{d}y = 0, \; x \in B,$

(iii) $\int_B f(x)\,\mathrm{d}x = 1.$

(see for example, Ref. [10]). In particular, if $V = \pi^{d/2}/\Gamma((d/2)+1)$ is the volume of S, then for $x \in B$ and all quadrants Q_x, we have

$$\int_{S_{x,r}\cap Q_x} f(y)\,\mathrm{d}y \sim f(x) V 2^{-d} r^d \text{ as } r \downarrow 0.$$

Now, for all $r > 0$, and fixed $x \in B$,

$$p_{nx} \le \sum_{\text{all quadrants } Q_x} P\Big(\bigcap_{i=2}^{n} [X_i \notin S_{x,r} \cap Q_x]\Big) \tag{3}$$

which for r small enough is not greater than

$$(1 - f(x) V r^d / 2^{d+1})^{n-1} \to 0 \text{ as } n \to \infty.$$

This concludes the proof of Theorem 1.

Remark 1

We say that x_1 is a *maximal vector* of $\{x_1,\ldots,x_n\}$ when at least one of the quadrants Q_{x_1} at x_1 does not contain any x_i, $i \ne 1$. Let N^* be the number of maximal vectors of $\{X_1,\ldots,X_n\}$ where the X_i's are independent R^d-valued random vectors with common density f. Clearly, $N \le N^* \le n$ because every convex hull point of $\{X_1,\ldots,X_n\}$ is also a maximal vector of $\{X_1,\ldots,X_n\}$. In Theorem 1, we have in fact shown that $E(N^*)/n \to 0$ and $N^*/n \to 0$ a.s. as $n \to \infty$. Without additional assumptions on f, very little additional information can be obtained about N^*. We just menion here that if X_1 has a density f, and if all the (d) components of X_1 are independent, then

$$E(N^*) \sim 2^d (\log n)^{d-1}/(d-1)! \text{ as } n \to \infty.$$

(see Refs. [11, 12]

3. RADIAL DENSITIES

Let M be the number of X_i's among $X_1, \ldots, X_n$ that do not belong to the interior of P, the extremal polyhedron of $\{X_1, \ldots, X_n\}$ with respect to A. When A is finite, $E(M)/n$ is not necessarily small even when $E(N)/n$ is. For example, if f is the uniform density on S, then it is necessarily true that

$$E(M)/n \geq c > 0 \tag{4}$$

for some constant $c = c(A)$, although Renyi and Sulanke[3] for $d = 2$ and Raynaud[6] for $d \geq 2$ have shown that

$$E(N)/n = 0(n^{-2/(d+1)}).$$

Thus, in view of equation (4), the effectiveness of the throw-away step is limited. Nevertheless, for some classes of densities we will have $E(M)/n \to 0$ as $n \to \infty$. For example, Toussaint *et al.*[2] have shown that when f is uniformly distributed on a rectangle of R^2 and A consists of 4 points of the form $(\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$, then

$$E(M^2) = 0(n).$$

However, unless the support of f has a special shape, there seems to be very little hope for obtaining small values for $E(M)$ when f has a compact support. Of the class of densities with infinite support, the radial densities are undoubtedly the most important ones.

Definition

A density f on R^d is called *radial* when it is of the form

$$f(x) = f_0(r) \tag{5}$$

where $r = \|x\|$ is the usual Euclidean norm of x.

The properties of radial densities are well explained in Kelker[22]. For example, when equation (5) holds, then the random variable $R = \|X\|$ has density

$$g(r) = Vd\, r^{d-1} f_0(r), \quad r > 0, \tag{6}$$

whenever X has density f. We recall here that V is the volume of S. We will also use

$$G(r) = P(R \geq r) = \int_r^\infty g(u)\, \mathrm{d}u. \tag{7}$$

Definition

A function L on $[0, \infty)$ is *slowly varying* when $L(t) > 0$ for all $t > 0$ small enough, and

$$\lim_{t \downarrow 0} \frac{L(tu)}{L(t)} = 1, \quad \text{all} \quad u > 0.$$

Definition

A density f on R^d is called *slowly varying radial* (*s.v.r.*) when it is radial, and when the function G^{-1} determined by

$$G^{-1}(u) = \inf\{t | G(t) = u\}$$

from G (see equation (7)) is slowly varying and $G^{-1}(u) \to \infty$ as $u \downarrow 0$.

LEMMA 1

For all $a > 0$,

$$\int_r^\infty u^{a-1}e^{-u}du \sim r^{a-1}e^{-r} \text{ as } r \to \infty, \tag{8}$$

and

$$\int_r^\infty u^{a-1}e^{-u^2/2}du \sim r^{a-2}e^{-r^2/2} \text{ as } r \to \infty. \tag{9}$$

Proof. For (8), see Tricomi [14]. Property (9) follows from (8) by using the transformation $u = t^2/2$.

Examples of s.v.r. densities

When f is standard normal, then

$$g(r) = Vd(2\pi)^{-d/2}r^{d-1}\,e^{-r^2/2}$$

and, by (9),

$$G(r) \sim Vd(2\pi)^{-d/2}r^{d-2}\,e^{-r^2/2} \text{ as } r \to \infty.$$

It is not hard to establish that f is s.v.r. from the last expression. Similarly, if $f_0(r) = e^{-r}/(Vd!)$, then $g(r) = r^{d-1}\,e^{-r}/(d-1)!$ is the gamma density and $G(r) \sim r^{d-1}\,e^{-r}/(d-1)!$ as $r \to \infty$. Once again, f is s.v.r.

Definition

A *cone* $C = C(x, y, \theta)$ or R^d is determined by its *top* $x \epsilon R^d$, its central *direction* y (where $y \in S$) and its *angle* $\theta > 0$. It is the *open* set of all points z of R^d that satisfy

$$\text{angle } (y, (z - x)) < \theta.$$

Definition

A collection $\mathscr{C}$ of cones $C(0, y, (\pi/2))$, $y \in A \subseteq S$, with the property that

$$\bigcup_{y \in A} C\left(0, y, \frac{\pi}{2}\right)$$

covers R^d (except possibly the origin), is called a *simple cone cover* of R^d. In that case, we say that A *generates a simple cone cover of* R^d.

Any $d + 1$ points of S that form a regular simplex define a set A that generates a simple cone cover of R^d. On the other hand, the minimal number of elements in A in order that $\mathscr{C}$ be a simple cone cover of R^d is $d + 1$. Besides the notion of a simple cone cover, we will also require an interesting property of all s.v.r. densities in R^d:

LEMMA 2

Let $\mathscr{B}_0$ be a partition of S into a finite number of measureable sets B_0. For each B_0, let $B = \{x | x = cy \text{ for some } c > 0,\ y \in B_0\}$ be the star set generated by B_0, and let each set B have infinite Lebesgue measure. Let $\mathscr{B} = \{B | B_0 \in \mathscr{B}_0\}$.

Assume that $X_1, \ldots, X_n$ are independent random vectors from R^d with common s.v.r. density f, and define

$$X(B) = \begin{cases} X_i & \text{if } X_i \in B \text{ and } \|X_i\| = \max_{j:\, X_j \in A} \|X_j\|, \\ 0 & \text{if no } X_j \text{ belongs to } B. \end{cases}$$

Then

(i)

$$\max_{B\in\mathscr{B}} \|X(B)\|/\min_{B\in\mathscr{B}}\|X(B)\| \to 1 \text{ in probability as } n\to\infty,$$

and

(ii)

$$\max_{B\in\mathscr{B}}\|X(B)\| = \max_{1\le i\le n}\|X_i\| \to \infty \text{ a.s. as } n\to\infty.$$

Proof of Lemma 2. Let $p = \inf_{B\in\mathscr{B}} P(X_1 \in B) > 0$, and let $N(B)$ the number of X_i's in B. By the strong law of large numbers, $N(B)/n \to P(X_1 \in B)$ a.s. as $n\to\infty$. If we let m be the largest integer in $np/2$, then (i) follows if we can show that

$$\max_{i\le m}\|X_i\|/G^{-1}\left(\frac{1}{n}\right) \to 1 \text{ probability as } n\to\infty, \tag{10}$$

and

$$\max_{i\le n}\|X_i\|/G^{-1}\left(\frac{1}{n}\right) \to 1 \text{ in probability as } n\to\infty. \tag{11}$$

It is known that $\max_{i\le n}\|X_i\|/a(n) \to 1$ in probability as $n\to\infty$ for some sequence of numbers $a(n)$ if and only if

$$\lim_{r\to\infty}\frac{G(ru)}{G(r)} = 0, \quad \text{all } u > 1 \tag{12}$$

see Refs. [26, 15, 16], and that in such a case we may take $a(n) = G^{-1}(1/n)$[17]. If (12) is valid, then also

$$\max_{i\le m}\|X_i\| \sim G^{-1}\left(\frac{1}{n}\right) \text{ in probability as } n\to\infty.$$

Here we used the fact that G^{-1} is slowly varying. We merely have to check equation (12). But equation (12) is implied by the fact that G^{-1} is slowly varying and that G is continuous. Finally, (ii) is a straightforward consequence of $G^{-1}(u)\to\infty$ as $u\downarrow 0$. This concludes the proof of Lemma 2.

4. THE AVERAGE COMPLEXITY OF ALGORITHM CH

Consider algorithm CH with a given finite set A. For a given set $\{x_1,\ldots,x_n\}$, we will let C_i be the complexity of the ith step in the algorithm. By our assumptions, it is clear that

$$C_1 + C_2 = 0(n), \quad \text{uniformly over all } \{x_1,\ldots,x_n\}. \tag{13}$$

The convex hull finding algorithm in step 3 operates on $M \le n$ points. We are not specifying which algorithm will be used here, but we do assume the following: if the convex hull finding algorithm of step 3 is fed a sequence $\{x_1,\ldots,x_n\}$, then its complexity is uniformly bounded (over all such sequences) by

$$\Delta(n). \tag{14}$$

In R^2, we can consider that $\Delta(n) = 0(n \log n)$ for the algorithms of Graham[18], Preparata and Hong[19], Shamos[20], Toussaint *et al.*[2] and Bentley and Shamos[21], and that $\Delta(n) = 0(n^2)$

for the algorithms of Eddy[28] and Jarvis[13]. In R^3, Preparata and Hong[19] have proposed an algorithm with $\Delta(n) = 0(n \log n)$. We mention here that Avis[23] and Yao[24] have essentially established that $\Delta(n) \geq cn \log n$ for some $c > 0$ when $d = 2$.

Assume now that we present algorithm CH with a sequence $X_1, \ldots, X_n$, and that $C = C(X_1, \ldots, X_n)$ is its complexity. With assumptions (13) and (14) it is clear that

$$E(C) = 0(n) + 0(E(\Delta(M))) \tag{15}$$

where M is the number of the X_i's not eliminated in the throw-away step. Thus, the average complexity of algorithm CH is small when M is small. We can now present our main theorems.

THEOREM 2

If A generates a simple cone cover of R^d, and if f is s.v.r., then

$$E(M) = o(n) \text{ as } n \to \infty. \tag{16}$$

THEOREM 3

If A is finite and generates a simple cone cover of R^d, if f is s.v.r. and if $\Delta(n)/n \uparrow \infty$, then the complexity C of algorithm CH satisfies

$$E(C) = o(\Delta(n)). \tag{17}$$

Proof of Theorems 2 *and* 3. Theorem 3 follows easily from Theorem 2: by equations (13)–(15) we have

$$\begin{aligned} E(C) &= 0(n) + 0(E(\Delta(M))) \\ &= 0(n) + 0\left(E\left(\frac{\Delta(M)}{M}\frac{M}{n}n\right)\right) \\ &= 0(n) + 0(\Delta(n)E(M)/n) \\ &= 0(n) + o(\Delta(n)) \\ &= o(\Delta(n)). \end{aligned} \tag{18}$$

Next, note that if $P(\{x_1, \ldots, x_n\})$ denotes the extremal polyhedron of $\{x_1, \ldots, x_n\}$ with respect to A, then

$$\begin{aligned} E(M) &= nP(X_1 \notin \text{int}(P(\{X_1, \ldots, X_n\}))) \\ &\leq nP(X_1 \notin \text{int}(P(\{X_2, \ldots, X_n\}))) \\ &= np_{n-1}. \end{aligned}$$

It suffices to show that $p_n \to 0$ as $n \to \infty$. We may always assume that A is a finite set, because if it is not, we can find a finite subset of A such that this finite subset generates a simple cone cover of R^d (by the Heine–Borel theorem), and because $M = M(A, X_1, \ldots, X_n) \leq M(A', X_1, \ldots, X_n)$ whenever $A' \subseteq A$.

Consider thus a finite set A with cardinality K, and let B_n be the radius of the largest sphere with center at the origin that is entirely contained in $P(\{X_1, \ldots, X_n\})$. It is clear that $B_n \to \infty$ in probability as $n \to \infty$ implies $p_n \to 0$ as $n \to \infty$.

Let $\rho = \rho(A)$ be the radius of the largest sphere that is entirely contained in the polyhedron formed by the elements $y_1, \ldots, y_K$ of A, and let A' be another set of K points of S, $y_1', \ldots, y_K'$. The distance between A and A' is

$$\mathrm{d}(A, A') = \max_i \|y_i - y_i'\|.$$

For every $\epsilon > 0$, there exists $\xi = \xi(\epsilon) > 0$ such that $d(A, A') < \xi$ implies that $|\rho(A) - \rho(A')| < \epsilon$ because ρ is a continuous function of $y_1, \ldots, y_K$. From here on, we let

$$\xi = \xi\left(\frac{\rho}{2}\right),$$

and define

$$\theta = \frac{1}{2}\left(\frac{\cos(\xi/2)}{\cos(\xi)} - 1\right).$$

Consider all cones $C_i = C(0, y_i, (\xi/2))$ and $C'_i = C(0, y_i, \xi)$, and form the differences $C''_i = C'_i - C_i$. Let $C_0 = R^d - UC'_i$, and let $X(C_i)$ and $X(C_i)$ and $X(C''_i)$ be defined from $X_1, \ldots, X_n$ as in the proof of Lemma 2. Define further

$$W = \max_{0 \le i \le K} \|X(C_i)\| \vee \max_{1 \le i \le K} \|X(C''_i)\|$$

and

$$W' = \min_{0 \le i \le K} \|X(C_i)\| \wedge \min_{1 \le i \le K} \|X(C''_i)\|.$$

By Lemma 2, when $\xi > 0$ is small enough, $W/W' \to 1$ and $W' \to \infty$ in probability as $n \to \infty$. Notice further that

$$(1 + \theta)\cos\xi = \frac{1}{2}\left(\cos\frac{\xi}{2} + \cos\xi\right) < \cos\frac{\xi}{2}.$$

Therefore, by a purely geometrical argument, $(W/W') \le 1 + \theta$ implies $B_n \ge (\rho/2)W'$. Thus, for all constants c, however large,

$$\begin{aligned} P(B_n < c) &\le P\left(B_n < \frac{\rho}{2}W'\right) + P\left(\frac{\rho}{2}W' < c\right) \\ &\le P\left(\frac{W}{W'} > 1 + \theta\right) + P\left(W' < \frac{2c}{\rho}\right) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

This concludes the proof of Theorem 2.

Remark 2

Theorem 3 can be considered as a validation of algorithm CH in view of its generality. In essence, for all s.v.r. densities, we can construct an $E(C) = o(n \log n)$ convex hull finding algorithm in R^2 merely by the use of a throw-away step. It suffices to take for example a set A with the directions $(1, 0)$, $(-1, 0)$, $(0, 1)$ and $(0, -1)$; but the set $\{(1, 0), (-1/\sqrt{2}), (1/\sqrt{2})), (-(1/\sqrt{2}), -(1/\sqrt{2}))\}$ will also do. In R^d, the d unit vectors and their opposites always generate a simple cone cover of R^d.

Remark 3

Let X have an s.v.r. density f, and let A be a given nonsingular dxd matrix, then AX has an elliptically symmetric slowly varying density. Theorems 2 and 3 remain valid for such densities.

5. THE NORMAL DENSITY

We wish to conclude with a more specific result announced in Toussaint *et al.*[2] for normal densities. Since the normal density is s.v.r., we have $E(C) = o(\Delta(n))$. Thus, since $\Delta(n) \ge cn \log n$ for some $c > 0$ by the Avis–Yao result[23, 24], at best Theorem 3 will allow us to conclude that $E(C) = o(n \log n)$. For many densities, such as the normal density, this can be considerably improved (see Theorems 4 and 5 below). The following Lemma will be useful.

LEMMA 3

Let $c > 0$, and let $n' \sim cn$ be a sequence of integers. Let $X_1, \ldots, X_{n'}$ be a sequence of independent random variables with common density

$$c' x^{d-1} e^{-x^2/2}, \ x > 0,$$

where $c' = Vd/(2\pi)^{d/2}$ is a normalization constant. (Note that if Y is standard normal in R^d, then $\|Y\|$ is distributed as X_1.) For all $\epsilon > 0$,

$$P(\max_{i \le n'} X_i > \sqrt{(2(1+\epsilon)\log n)}) = 0((\log n)^{(d/2)-1} n^{-\epsilon}) \tag{19}$$

and

$$P(\max_{i \le n'} X_i < \sqrt{(2(1-\epsilon)\log n)}) = 0(e^{-n^{\epsilon}}) \tag{20}$$

as $n \to \infty$.

Proof of Lemma 3. Let $F = 1 - G$ be the distribution function of X_1, and recall from Lemma 1 that for any sequence $a_n \to \infty$, $G(a_n) \sim c' a_n^{d-2} \exp(-a_n^2/2)$. Thus,

$$P(\max_{i \le n'} X_i < a_n) = F^{n'}(a_n) = (1 - G(a_n))^{n'}$$

$$\le \exp(-n' G(a_n))$$

$$= \exp(-cc'(1 + o(1)) n a_n^{d-2} e^{-a_n^2/2}).$$

With $a_n = (2(1-\epsilon)\log n)^{1/2}$, the exponent becomes

$$-(c'' - o(1)) (\log n)^{(d/2)-1} n^{\epsilon}$$

for some constant $c'' > 0$. Formula (20) follows trivially. Also,

$$P(\max_{i \le n'} X_i > a_n) \le n' G(a_n)$$

$$= cc'(1 + o(1)) n^{-\epsilon} (2(1+\epsilon)\log n)^{(d/2)-1}$$

when $a_n = (2(1+\epsilon)\log n)^{1/2}$. This concludes the proof of Lemma 3.

THEOREM 4

If A generates a simple cone cover of R^d, and if f is the standard normal density, then there exists an $\epsilon = \epsilon(A, d) > 0$ such that

$$E(M) = 0(n^{1-\epsilon}). \tag{21}$$

THEOREM 5

If A is finite and generates a simple cone cover of R^d, if f is the standard normal density, and if $\Delta(n) = 0(n \log n)$, then algorithm CH satisfies:

$$E(C) = 0(n). \tag{22}$$

Proof of Theorems 4 and 5. Theorem 5 follows from Theorem 4 in view of

$$\begin{aligned} E(C) &\le 0(n) + 0(E(M \log M)) \\ &\le 0(n) + 0(E(M) \log n) \\ &\le 0(n) + o(n^{1-\epsilon} \log n) \\ &= 0(n). \end{aligned}$$

We inherit the notation of the proof of Theorem 2, and note that it suffices to show that $p_n = o(n^{-\epsilon})$ for some $\epsilon = \epsilon(A, d) > 0$. If $(.)^c$ denotes the complement of a set, then for some sequence $a_n \to \infty$,

$$\begin{aligned} p_n &\le E\left(\int_{S^c_{\delta,B_n}} f(x)\,\mathrm{d}x \right) \\ &\le P(B_n < a_n) + P(\|X_1\| > a_n). \end{aligned} \tag{22}$$

By Lemmas 1 and 3, $P(\|X_1\| > a_n) \sim c' a_n^{d-2} \exp(-a_n^2/2)$ where c' is defined in Lemma 3. Choose ρ, ξ and θ as in the proof of Theorem 2, and note that they only depend upon A and d. Let us take $a_n = (\rho/2)\surd(\log n) = (\rho/2)\surd(2(1 - [1/2]) \log n$. Clearly,

$$P(\|X_1\| > a_n) = 0((\log n)^{(d/2)-1} n^{-\rho^2/8}). \tag{23}$$

Since we can always assume that $\theta < 1$, we have in particular $(1 + \theta/2)/(1 - \theta/2) < (1 + \theta)^2$. Hence,

$$\begin{aligned} P(B_n < a_n) &\le P(B_n < (\rho/2)W') + P(W' < (2/\rho)a_n) \\ &\le P\left(\frac{W}{W'} > 1 + \theta\right) + P(W' < \surd(\log n)) \\ &\le P\left(W > \sqrt{\left(2\left(1 + \frac{\theta}{2}\right) \log n\right)} + 2P\left(W' < \sqrt{\left(2\left(1 - \frac{\theta}{2}\right) \log n\right)}\right.\right. . \end{aligned} \tag{24}$$

Since $W \le \max_{i \le n} \|X_i\|$, Lemma 3 shows that the former term of equation (24) is

$$0((\log n)^{(d/2)-1} n^{-\theta/2}). \tag{25}$$

Let $\mathscr{C}$ be the collection of sets $\{C_0, C_1, \ldots, C_K, C_1'', \ldots, C_K''\}$ defined in the proof of Theorem 2. For any $C \in \mathscr{C}$, let $p(C) = P(X_1 \in C)$ and let $N(C) = \sum_{i=1}^{n} I_{[X_i \in C]}$ where I is the indicator function. Note that $\inf p(C) = p > 0$. Let m be the largest integer in $pn/2$. Then we have

$$\begin{aligned} &P\left(W' < \sqrt{\left(2\left(1 - \frac{\theta}{2}\right) \log n\right)}\right) \\ &\le (2K + 1)P\left(\max_{i \le m} \|X_i\| < \sqrt{\left(2\left(1 - \frac{\theta}{2}\right) \log n\right)}\right) \\ &\quad + \sum_{c \in \mathscr{C}} P(N(C) \le np(C)/2). \end{aligned} \tag{26}$$

The former term on the right-hand-side of (26) $o(\exp(-n^{\theta/2}))$ Lemma 3. By Chebyshev's inequality [25], the latter term of (26) is $0(n^{-1})$ (in fact, one can show that it is $0(\exp(-np^2/2))$ by employing Hoeffding's exponential inequality). Combining (22), (23), (24),

(25) and (26) shows that Theorem 4 is valid with

$$0 < \epsilon < \min\left(\frac{\rho^2}{8}, \frac{\theta}{2}\right).$$

REFERENCES

1. S. G. Akl and G. T. Toussaint, A fast convex hull algorithm. *Information Processing Letters*, **7**, 219–222 (1978).
2. G. T. Toussaint, S. G. Akl and L. Devroye: Efficient convex hull algorithms for points in two and more dimensions, *Tech. Rep. SOCS* 78.5. School of Computer Science, McGill University, Montreal (1978).
3. A. Renyi and R. Sulanke, Uber die konvexe Hülle von n zufällig gewählten Punkten I, *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **2**, 75–84 (1963).
4. A. Renyi and R. Sulanke, Uber die konvexe Hülle von n zufällig gewählten Punkten II. *Zeitschrift für Wahrscheinlichkeitsthearie und verwandte Gebiete* **3**, 138–147 (1964).
5. B. Efron, The convex hull of a random set of points. *Biometrika* **52**, 331–343 (1965).
6. H. Raynaud, Sur le comportement asymptotique de l'enveloppe convex d'un nuage de points tirés au hasard dans R^n. *Compt. Rend. Acad. Sci. Paris* **261**, 627–629 (1965).
7. H. Carnal, Die knovexe Hülle von n rotationssymmetrische verteilten Punkten. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* **15**, 168–176 (1970).
8. J. F. C. Kingman, The ergodic theory of subadditive stochastic processes. *J. Roy. Stat. Soc. Ser. B.* **30**, 499–510 (1968).
9. J. F. C. Kingman, Subadditive ergodic theory. *Ann. Prob.* **1**, 883–909 (1973).
10. R. L. Wheeden and A. Zygmund, *Measure and Integral*, pp. 108–109. Marcel Dekker, New York (1977).
11. O. Barndorff-Nielsen: On the limit behaviour of extreme order statistics. *Ann. Math. Stat.* **34**, 992–1002 (1963). *Theory of Probability and its Applications*, **11**, 249–269 (1966).
12. L. Devroye, A note on finding convex hulls via maximal vectors. *Information Processing Letters* **11**, 53–56 (1980).
13. R. A. Jarvis, On the identification of the convex hull of a finite set of points in the plane. *Information Processing Letters* **2**, 18–21 (1973).
14. F. G. Tricomi, *Funzione ipergeometriche confluenti* p. 174. Edizione Cremonese, Rome (1954).
15. B. V. Gnedenko, Sur la distribution limite du terme maximum d'un série aléatoire. *Ann. Math.* **44**, 423–453 (1943).
16. J. Geffroy, Contributions à la théorie des valeurs extrêmes. *Publications de l'institut de Statistique de l'Université de Paris*, **7**, 37–121 (1958).
17. W. Vervaat, *Limit theorems for partial maxima and records.* Mathematisch Centrum, Amsterdam (1978).
18. R. L. Graham, An efficient algorithm for determining the convex hull of a planar set. *Information Processing Letters* **1**, 132–133 (1972).
19. F. P. Preparata and S. J. Hong, Convex hulls of finite sets of points in two and three dimensions. *Commun. ACM* **20**, 87–93 (1977).
20. M. I. Shamos, Geometric complexity. *Proc. 7th Annual ACM Symp. on Automata and Computability Theory* pp. 224–233 (1977).
21. J. L. Bentley, H. T. Kung, M. Schkolnick and C. D. Thompson, On the average number of maxima in a set of vectors and applications. *J. ACM*, **25**, 536–543 (1978).
22. D. Kelker, Distribution theory of spherical distributions and a location-scale parameter generalization. *Sankhya Series A* **32**, 419–430 (1970).
23. D. Avis, On the complexity of finding the convex hull of a set of points. *Tech. Rep. SOCS* 79.2. School of Computer Science, McGill University, Montreal (1979).
24. A. C. Yao, A lower bound to finding convex hulls. *Tech. Rep. STAN-CS-79-733.* Stanford University (1979).
25. W. Feller, *An Introduction to Probability Theory and its Applications.* 2nd Edn, Vol. 2. Wiley, New York (1971).
26. O. Barndorff-Nielsen and M. Sobel, On the distribution of the number of admissible points in a vector random sample. *Theory of Probability and its Applications*, **11**, 249–269 (1966).
27. J. L. Bentley and M. I. Shamos, Divide and conquer for linear expected time. *Information Processing Letters* **7**, 87–91 (1978).
28. W. F. Eddy, A new convex hull algorithm for planar sets. *ACM Trans. Math. Software* **3**, 398–403, 411–412 (1977).