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Moment Inequalities for Random Variables in Computational Geometry

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Abstract — Zusammenfassung

Moment Inequalities for Random Variables in Computational Geometry. Let $X_1, \ldots, X_n$ be independent identically distributed R^d-valued random vectors, and let $A_n = A(X_1, \ldots, X_n)$ be a subset of $\{X_1, \ldots, X_n\}$, invariant under permutations of the data, and possessing the inclusion property ($X_1 \in A_n$ implies $X_1 \in A_i$ for all $i \leq n$). For example, the convex hull, the collection of all maximal vectors, the set of isolated points and other structures satisfy these conditions.

Let N_n be the cardinality of A_n. We show that for all $p \geq 1$, there exists a universal constant $C_p > 0$ such that $E(N_n^p) \leq C_p \max(1, E^p(N_{\underline{n/q}}))$ where $q = \overline{p}$. This complements Jensen's lower bound for the p-th moment: $E(N_n^p) \geq E^p(N_n)$.

The inequality is applied to the expected time analysis of algorithms in computational geometry. We also give necessary and sufficient conditions on $E(N_n)$ for linear expected time behavior of divide-and-conquer methods for finding A_n.

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Momentenungleichungen für Zufallsvariable bei geometrischen Berechnungsverfahren. $X_1, \ldots, X_n$ seien unabhängige und gleichartig verteilte Zufallsvektoren im R^d, ferner sei $A_n = A(X_1, \ldots, X_n)$ eine Teilmenge von $\{X_1, \ldots, X_n\}$, die invariant ist gegenüber einer Permutation der Daten und die die Inklusionseigenschaft ($X_1 \in A_n \Rightarrow X_1 \in A_i$ für $i \leq n$) besitzt. Beispielsweise erfüllen die konvexe Hülle, die Menge der Maximal-Vektoren, die Menge der isolierten Punkte und andere Strukturen diese Bedingungen.

Sei N_n die Kardinalzahl von A_n. Wir zeigen, daß es für jedes $p \geq 1$ eine universelle Konstante C_p gibt, so daß $E(N_n^p) \leq C_p \max(1, E^p(N_{\underline{n/q}}))$ gilt, mit $q = \overline{p}$. Dies ist das Gegenstück zur unteren Schranke in Jensen für das p-te Moment: $E(N_n^p) \geq E^p(N_n)$.

Die Ungleichung wird zur Analyse der erwarteten Laufzeit von Algorithmen für geometrische Berechnungen verwendet. Ferner werden notwendige und hinreichende Bedingungen bezüglich $E(N_n)$ angegeben, damit ein lineares Laufzeitverhalten bei Divide-and-Conquer-Methoden zur Berechnung von A_n zu erwarten ist.

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1. Introduction

Let $X_1, \ldots, X_n$ be independent identically distributed R^d-valued random vectors and let $A_n = A(X_1, \ldots, X_n)$ be a subset of $X_1, \ldots, X_n$ such as, for example, the set of all X_i's that belong to the convex hull of $X_1, \ldots, X_n$. In general, we assume that A satisfies:

$$\begin{aligned} &\text{(i)} \quad A(x_1, \ldots, x_n) = A(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \text{ for all } x_1, \ldots, x_n \in R^d, \\ &\qquad \text{and all permutations } \sigma(1), \ldots, \sigma(n) \text{ of } 1, \ldots, n. \\ &\text{(ii)} \quad x_1 \in A(x_1, \ldots, x_n) \text{ implies } x_1 \in A(x_1, \ldots, x_i), \text{ all } i \le n. \end{aligned} \tag{1}$$

Let $N = \sum_{i=1}^{n} I_{[X_i \in A_n]}$ where I is the indicator function. In this note, we are interested in inequalities linking $E(N^p)$ to $E^p(N)$, and in the application of these inequalities in the study of the average complexity of various algorithms in computational geometry.

From Jensen's inequality, we know that

$$E(N^p) \ge E^p(N), \text{ all } p \ge 1. \tag{2}$$

Regardless of (1), we always have the partial converse

$$E(N^p) \le n^{p-1} E(N), \text{ all } p \ge 1. \tag{3}$$

But (3) is too weak for most applications. If we exploit the structure of A given in (1), stronger converses of (2) are obtainable. Our main result is the following theorem:

Theorem 1: *Assume that* (1) *holds, and that $p \ge 1$ is fixed. Let $q = \overline{p}$, and let N_n be defined as N, to make the dependence upon n explicit. Then there exist universal positive constants C and D only depending upon p such that*

$$E(N_n^p) \le \max\left(C, D E^p(N_{\underline{n/q}})\right). \tag{4}$$

We can always take $C = (2q)^p (e-1)^{p/q}$ and $D = (2q^2)^p (e-1)^{p/q}$.

The proof of theorem 1 is given in section 2. Some direct applications of it are outlined in section 3. In section 4, we derive some results about the average complexity of divide-and-conquer algorithms that use inequality (4) in crucial places.

Remark 1: If $E(N_n)$ is nondecreasing in n, then we have a converse of (2):

$$E(N_n^p) \le \max\left(C, D E^p(N_n)\right) \tag{5}$$

for some universal positive constants C, D only depending upon p. The monotonicity condition for $E(N_n)$ is often hard to check. The most useful form of (4) is the following: if $E(N_n) \le a_n$ and a_n is nondecreasing, then

$$E(N_n^p) \le \max(C, D a_n^p). \tag{6}$$

Remark 2: An important notion in computer science is that of *comparable sequences*: two sequences $a_n > 0$ and $b_n > 0$ are said to be *comparable* (written $a_n = \theta(b_n)$) when

$$0 < \liminf \frac{a_n}{b_n} \le \limsup \frac{a_n}{b_n} < \infty. \tag{7}$$

For example, (2) and (5) imply that $E(N_n^p)$ and $E^p(N_n)$ are comparable when (1) holds, $E(N_n)\to\infty$, and $E(N_n)$ is nondecreasing in n. The same remains true when $E(N_n)\sim a_n$ where $a_n\to\infty$ and a_n is nondecreasing in n.

Remark 3: In most applications we know that $E(N_n)\sim a_n$ for some nondecreasing sequence a_n, and thus remark 2 applies. In some rare instances, $E(N_n)$ oscillates. When the oscillations are slight, theorem 1 is still powerful enough to imply that $E(N_n^p)=\theta(E^p(N_n))$: for example, it suffices that $E(N_n)$ is regularly varying or that $E(N_n)\sim a_n$ where a_n is regularly varying, and that $E(N_n)\to\infty$ (a sequence a_n is said to be regularly varying if for some finite number r, $\lim a_{\lfloor cn\rfloor}/a_n=c^r$ for all $c>0$). This follows from (2) and (4) after noting that

$$E(N_{\lfloor n/q\rfloor})\sim q^{-r}E(N_n).$$

Remark 4: Theorem 1 gives us information about polynomial moments. It can also be used to obtain upper bounds for other moments, as we will now illustrate on one important example. Let C, D be the constants of theorem 1 for $p=2$. Then,

$$E(N_n\log(N_n+e))\le\sqrt{E(N^2)\,E(\log^2(N_n+e))}\quad\text{(Cauchy's inequality)}$$

$$\le\sqrt{\max(C,DE^2(N_{\lfloor n/2\rfloor}))}\log(E(N_n+e))\quad\text{(concavity of }\log^2\text{, Jensen's inequality, and theorem 1)}$$

$$\le\max(\sqrt{C},\sqrt{D}\,E(N_{\lfloor n/2\rfloor}))\log(E(N_n+e)).$$

By the convexity of $u\log(u+e)$, we also have

$$E(N_n\log(N_n+e))\ge E(N_n)\,\log(E(N_n+e)).$$

Assume thus that (1) holds and that $E(N_n)\to\infty$. Then

$$E(N_n\log(N_n+e))=\theta(a_n\log(a_n))$$

when $E(N_n)\sim a_n$ for a_n nondecreasing or regularly varying.

2. Proof of Theorem 1

Assume first that p is integer, $p\ge 2$, and that n is a multiple of p. Define $B_1,\dots,B_p$ by

$$B_i=A(X_i,X_{i+p},X_{i+2p},\dots,X_{n+i-p}),\ 1\le i\le p,\ i\le n.$$

By the independence of the X_i's and (1),

$$\begin{aligned}E(N_n^p)&=E\left(\left(\sum_{i=1}^{n}{}_{[X_i\in A_n]}\right)^p\right)\\&\le\sum_{i=1}^{p}i^p\binom{n}{i}P(X_1,\dots,X_i\in A_n)\\&\le\sum_{i=1}^{p}i^p\binom{n}{i}\prod_{j=1}^{i}P(X_j\in B_j)\\&\le\sum_{i=1}^{p}p^p[nP(X_1\in B_1)]^i/i!.\end{aligned}\tag{8}$$

Let $a=p^p(e-1)$.

Now, if $nP(X_1\in B_1)\leq 1$, (8) is less than a. If $nP(X_1\in B_1)\geq 1$, it is less than or equal to $(nP(X_1\in B_1))^p a$. Thus, we have shown that

$$E(N_n^p)\leq a\max(1, E^p(N_{n/p})\,p^p). \tag{9}$$

If p is integer but n is not a multiple of p, then let $m=p\cdot\lfloor n/p\rfloor$. Note that $n-p\leq m\leq n$, and that, by (1), $N_n\leq N_m+p$. Thus, applying (9),

$$\begin{aligned} E(N_n^p)&\leq E((N_m+p)^p)\leq 2^p\max(p^p, E(N_m^p))\leq 2^p a\max(1, p^p E^p(N_{m/p}))\\ &=\max(C_p, D_p E^p(N_{m/p})) \end{aligned} \tag{10}$$

where $C_p=2^p a=(2p)^p(e-1)$ and $D_p=(2p)^p a=(2p^2)^p(e-1)$.

When p is not integer, we let $q=\lceil p\rceil$, and apply Jensen's inequality:

$$E(N_n^p)\leq (E(N_n^q))^{p/q}\leq \max(C_q^{p/q}, D_q^{p/q} E^p(N_m))$$

where $m=\lfloor n/q\rfloor$. This concludes the proof of theorem 1.

3. Applications

Inequalities for the Binomial Distribution

Let X_i be $\{0,1\}$-valued with $P(X_i=1)=1-P(X_i=0)=q\in(0,1)$, and let A_n be the collection of X_i's taking the value 1. By the independence of the X_i's, N is binomial (n,p) and $E(N)=np$. Clearly, (1) holds and remark 1 applies. In particular, (5) holds:

$$E(N^p)\leq\max(C, D(nq)^p).$$

The Number of Convex Hull Points

We say that $X_i\in R^d$ is *isolated* $(X_i\in A_n)$ if the closed sphere of radius r centered at X_i contains no X_j, $1\leq j\leq n$, $j\neq i$. Here too, (1) holds. Let N be the total number of isolated points among $X_1,\ldots,X_n$. Often $E(N)=nP(X_1\in A_n)$ is easy to compute or bound. The moments $E(N^p)$ can be bounded by (2) and (4).

The Number of Convex Hull points

When A_n is the convex hull of $X_1,\ldots,X_n$, the distribution of N is generally hard to find. For many distributions, the asymptotical behavior of $E(N)$ is known. In these cases, theorem 1 can be used to get upper bounds for $E(N^p)$, $p\geq 1$. Among the known results, we cite:

1. $E(N)=o(n)$ whenever X_1 has a density (Devroye, 1981).
2. For the normal distribution in R^d, $E(N)=O((\log n)^{(d-1)/2})$ (Raynaud, 1970). For $d=2$, it is known that $E(N)\sim 2\sqrt{2\pi\log n}$ (Renyi and Sulanke, 1963/1964). Remark 1 applies in the former case, and remark 2 in the latter.
3. When X_1 is uniformly distributed in the unit hypersphere of R^d, then $E(N)=O(n^{(d-1)/(d+1)})$ (Raynaud, 1970).

4. When X_1 is uniformly distributed on a polygon of R^2 with k vertices, then $E(N) \sim \frac{2k}{3} \log n$ (Renyi and Sulanke, 1963, 1964, 1968). Once again, remark 2 applies.

5. The behavior of $E(N)$ for radial distributions on R^2 is quite exhaustively treated by Carnal (1970). For example, if $P(\| X_1 \| > u) = u^{-r} L(u)$ for some $r \geq 0$, where L is slowly varying (i.e., $L(cx)/L(x) \to 1$ as $x \to \infty$ for all $c > 0$), then $E(N) \to c(r) > 0$. In another example, let $P(\| X_1 \| > u) \sim c(1-u)^r$ for some $c > 0$, $r \geq 0$, when $u \uparrow 1$, and let it be 0 for $u > 1$. Then $E(N) \sim c(r) n^{1/(2r+1)}$ for some constant $c(r) > 0$. The uniform distribution on the unit circle satisfies the said condition with $c = 2$, $r = 1$. In all these examples, remark 2 applies.

Minimum Covering Spheres and Ellipsoids

The minimal covering ellipsoid (sphere) is the ellipsoid (sphere) of minimal volume that covers $X_1, \ldots, X_n$. It can be found by first finding the convex hull A_n of $X_1, \ldots, X_n$ and then performing some operations on the convex hull points, at least when $d = 2$. For example, it is known that the minimal covering circle has either three points of A_n on its perimeter, or two points (in which case they define the diagonal of the circle). Thus, given A_n, the most naive algorithm to find the minimal covering circle takes time proportional to N^4. The average time of the entire algorithm is equal to the average time of the convex hull algorithm plus a constant times $E(N^4)$. By (6), $E(N^4) = O(n)$ whenever $E(N) = O(n^{-1/4})$. The latter condition is satisfied for most distributions cited in the previous paragraph. Of course, we could also use the $O(N^2)$ algorithm of Elzinga and Hearn (1972) (see also Francis (1974)) or the $O(N \log(N))$ algorithms of Shamos (1978) or Preparata (1977). By (6) and remark 4, the construction of the minimum covering sphere from A_n takes on average time $O(n)$ when $E(N) = O(\sqrt{n})$ and $E(N) = O(n/\log(n))$ respectively. To find A_n in average time $O(n)$, see section 4 below and the survey paper of Devroye and Toussaint (1980).

Silverman and Titterington (1980) find the minimal covering ellipse in R^2 from A_n in time bounded by cN^6. Thus, their algorithm has linear expected time if A_n can be found in linear expected time and if $E(N) = O(n^{1/6})$ (by Theorem 1).

The Diameter of a Set of Points

The diameter $D = D(X_1, \ldots, X_n)$ of $X_1, \ldots, X_n$ is the maximal distance between any two points X_i and X_j. Since both X_i and X_j that are furthest apart must belong to A_n, one can find D by first finding A_n and then comparing all $\binom{N}{2}$ distances between points belonging to A_n (see Bhattacharya (1980) for a development of this algorithm and a comparison with other algorithms for finding D). By theorem 1, it is clear that the total average complexity is $O(n)$ when A_n can be found in average time $O(n)$, and when $E(N) = O(\sqrt{n})$. Notice that the latter condition is satisfied for all dimensions d when X_1 is normally distributed, or when X_1 is uniformly distributed in the unit cube of R^d.

The Number of Maximal Vectors

Let A_n be the collection of maximal vectors of $X_1, \ldots, X_n$, that is, $X_i \in A_n$ if and only if no other X_j dominates X_i in all its components. One can easily check that (1) is valid. Also, whenever X_1 has a density and its components are independent, $E(N)$ is monotone (Devroye, 1980). In fact,

$$E(N) \sim (\log n)^{d-1}/(d-1)!$$

(Barndorff-Nielsen and Sobel, 1966; Devroye, 1980). Thus, by remark 2, $E(N^p)$ and $(\log n)^{p(d-1)}$ are comparable for all $p \geq 1$.

The Throw-Away Principle

The convex hull of $X_1, \ldots, X_n$ can be found very rapidly by finding the extremes in the directions $d_1, \ldots, d_s$, throwing away all the X_i's that are strictly interior to the polyhedron formed by these extremes, and then finding the convex hull of all the remaining points via a simple convex hull algorithm (see for example, Jarvis (1973) for an $O(n^2)$ convex hull algorithm in R^2, and Graham (1972) for an $O(n \log n)$ algorithm in R^2; and see Devroye (1981) and Devroye and Toussaint (1981) for the throw-away principle). It is essential that one has a good upper bound for $E(N^2)$ or $E(N \log_+ N)$ where $\log_+ N = \max(\log N, O)$, and N is the number of points not thrown away (and collected in A_n). It is easy to check that A_n satisfies (1). Thus, by theorem 1, Jarvis' algorithm will yield $O(n)$ average time when $E(N) = O(\sqrt{n})$. By remark 4, Graham's algorithm will do the same when $E(N) \log_+ E(N) = O(n)$. In essence, one must only find the asymptotical behavior of $E(N)$ to study the average complexity of these throw-away algorithms. For some results along this line, see Devroye (1981) and Devroye and Toussaint (1981).

4. Divide and Conquer Methods

Because of property (1)(ii), A_n can be found very elegantly by divide-and-conquer methods. Assume for simplicity that $n = 2^k$ for some integer $k \geq 1$, and consider the following algorithm:

(i) Set $i \leftarrow 2$. Let $A_{1j} = A(X_j)$, $1 \leq j \leq n$.
(ii) Let $A_{ij} = A(A_{i-1\,2j-1}, A_{i-1\,2j})$, $1 \leq j \leq n/2^{i-1}$. (Thus, merge the solutions $A_{i-1\,2j-1}$ and $A_{i-1\,2j}$.)
(iii) If $i = k$, $A_n \leftarrow A_{k1}$ and exit. Otherwise, $i \leftarrow i+1$, go to (ii).

The crucial observation here is that if N_n is the cardinality of A_n, then each A_{ij} has on the average

$$E\left(\sum_{m=1}^{2^i} I_{[X_m \in A_{2^i}]}\right) = E(N_{2^i})$$

elements.

Theorem 2: *Assume that two A-sets of sizes k_1 and k_2 can be merged and edited in time bounded by $c(f(k_1)+f(k_2))$ for some constant c and some nondecreasing function f, and assume that $E(f(N))\le b_n$ for some nondecreasing sequence b_n. Then the divide and conquer algorithm given above finds A_n in average time*

$$O\left(n\sum_{i=1}^{2n} b_i/i^2\right),$$

which is $O(n)$ if

$$\sum_{n=1}^{\infty} b_n/n^2<\infty. \tag{11}$$

If the merging and editing takes time bounded from below by $c'(f(k_1)+f(k_2))$ and $E(f(N))\ge c'' b_n$, all n large enough (c', c'' are positive constants; b_n and f are nondecreasing), then condition (11) is necessary as well for $O(n)$ average time behavior of the given divide and conquer algorithm.

Proof of Theorem 2:

The average time for the entire algorithm does not exceed, for $n=2^k$, $k\ge 1$,

$$\begin{aligned} c\sum_{i=0}^{k} n2^{-i}b_{2^i} &\le cn\sum_{i=0}^{k}2^{-2i}\sum_{j=2^i}^{2^{i+1}-1} b_j\\ &\le 4cn\sum_{i=0}^{k}\sum_{j=2^i}^{2^{i+1}-1} b_j/j^2\\ &\le 4cn\sum_{j=1}^{2n} b_j/j^2,\end{aligned}$$

from which the sufficiency of (11) follows. The necessity follows by a similar argument since the average time of the algorithm is bounded from below by

$$\begin{aligned} c'c''\sum_{i=0}^{k} n2^{-i}b_{2^i} &\ge c'c''n\sum_{i=1}^{k}2^{-(2i-1)}\sum_{j=2^{i-1}+1}^{2^i} b_j\\ &\ge \frac{c'c''n}{2}\sum_{i=1}^{k}\sum_{j=2^{i-1}+1}^{2^i} b_j/j^2\\ &= \frac{c'c''n}{2}\sum_{j=2}^{n} b_j/j^2,\end{aligned}$$

so the average time cannot be bounded by Kn for any $K>0$, if (11) diverges.

Example 1: *Finding the maximal vectors.*

Let A_n be the set of maximal vectors among $X_1,\dots,X_n$. Merging and editing in the divide and conquer algorithm is accomplished by the brute force method: (i) merge the sets; (ii) by pairwise comparisons, find all the maximal vectors in the merged set, and delete the other X_i's from it. Theorem 2 applies with $f(n)=n^2$ for both the upper and lower time bound for the merging and editing. Assume that we know that $E(N)\sim a_n$ for some nondecreasing function $a_n\to\infty$. Then, by theorem 2, the divide and conquer algorithm runs in linear average time *if and only if*

$$\sum_{n=1}^{\infty} a_n^2/n^2<\infty. \tag{12}$$

Here we also needed remark 2. For example, when X_1 has a density and its components are independent, then $a_n = (\log n)^{d-1}/(d-1)!$. Clearly, (12) holds for *any* d. For such distributions, the convex hull can be found in average time $O(n)$ as well since $a_n^{d+1} = O(n)$: just notice that the convex hull is a subset of A_n that can be obtained from A_n in time $O(N^{d+1})$, and that $E(N^{d+1}) = O(a_n^{d+1}) = O(n)$.

Example 2: *Convex hulls in* R^2.

Two convex hulls with angularly ordered elements in the plane can be merged in time proportional to the total number of elements involved, and the result is a new convex hull with angular ordering (Shamos, 1978). Theorem 2 applies with $f(n) = n$ if a divide and conquer method is used to find the convex hull of $X_1, \ldots, X_n$. Thus, if $E(N) = O(a_n)$, and a_n is nondecreasing, then

$$\sum_{n=1}^{\infty} a_n/n^2 < \infty \tag{13}$$

is sufficient for linear average time behavior of the algorithm. If $\liminf E(N)/a_n > 0$, then (13) is necessary too. This improves a result by Bentley and Shamos (1978) who required that $E(N) = O(n^{1-\delta})$ for some $\delta > 0$ for linear average time of their divide and conquer convex hull algorithm. Notice that (13) follows when $a_n = O(n/\log^{1+\delta} n)$ or $a_n = O\big(n/(\log n \log^{1+\delta} \log n)\big)$ for some $\delta > 0$. *All* the planar distributions of section 3 satisfy these requirements.

Example 3: *Convex hulls in* R^d.

Let A_n be the convex hull of $X_1, \ldots, X_n$, and let us merge and edit in step (ii) in the most trivial possible way: merge to the two sets, consider all d-tuples of elements, and check if all the remaining elements fall on the same side of the halfspace determined by the d-tuple. Such an algorithm takes time

$$O\big((k_1 + k_2)^{d+1}\big) = O(k_1^{d+1} + k_2^{d+1})$$

when the two sets involved have k_1 and k_2 elements, respectively. For average linear time of the divide and conquer algorithm it is sufficient that $E(N) = O(a_n)$ for some nondecreasing function a_n, and that

$$\sum_{n=1}^{\infty} a_n^{d+1}/n^2 < \infty \,. \tag{14}$$

(Just combine theorem 2 and remark 1.) Condition (14) is satisfied for *all* d for the normal distribution, and for the uniform distribution on the unit cube of R^d. Because two convex hulls of sizes k_1 and k_2 can be merged in time

$$O\big((k_1 + k_2)^{\lfloor (d+1)/2 \rfloor} + (k_1 + k_2) \log (k_1 + k_2)\big)$$

(Seidel, 1981), condition (14) can be replaced by

$$\sum_{n=1}^{\infty} (a_n^{\lfloor (d+1)/2 \rfloor} + a_n \log a_n)/n^2 < \infty \tag{15}$$

whenever $E(N) = O(a_n)$ for some nondecreasing function a_n.

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